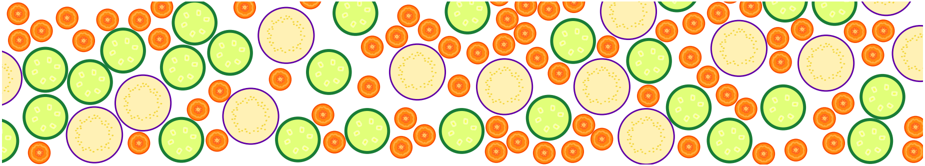
Triangulated ternary disc packings that maximize the density

Daria Pchelina

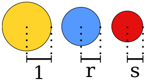
supervised by

Thomas Fernique

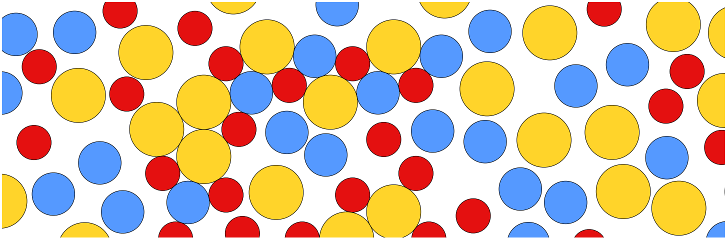
September 29, 2020



Discs:



Packing P:
(in R^2)



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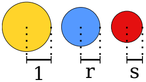

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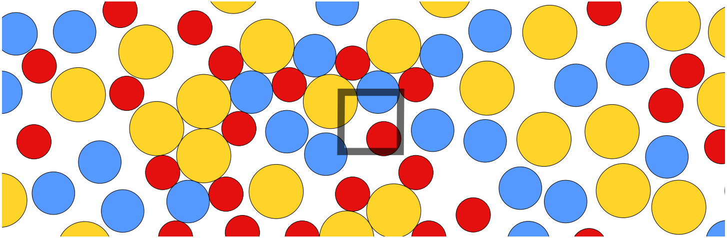

Density:

$$\delta(P) = \limsup_{n\to\infty} \frac{area([-n,n]^2 \cap P)}{area([-n,n]^2)}$$

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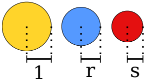

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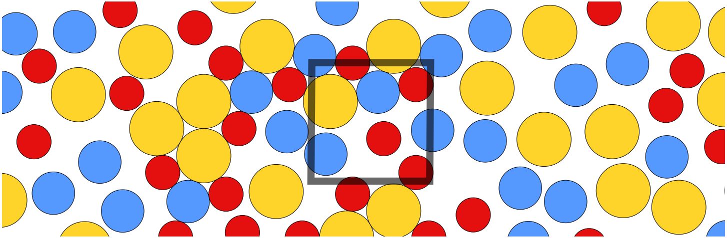

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What is a packing?

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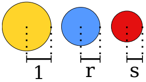

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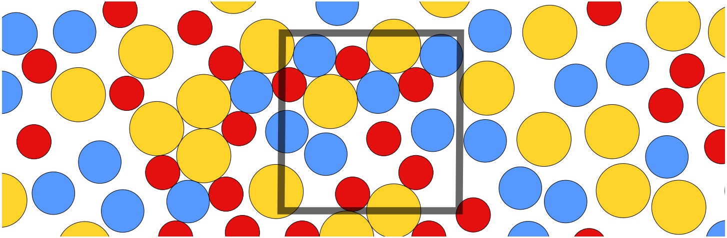

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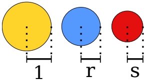

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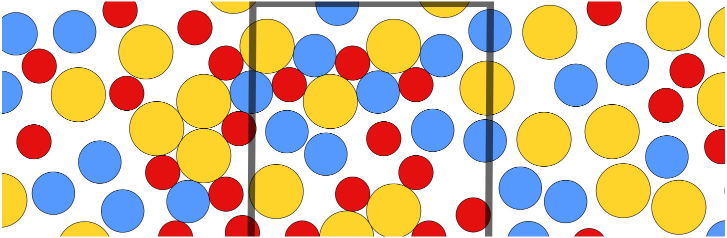

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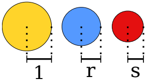

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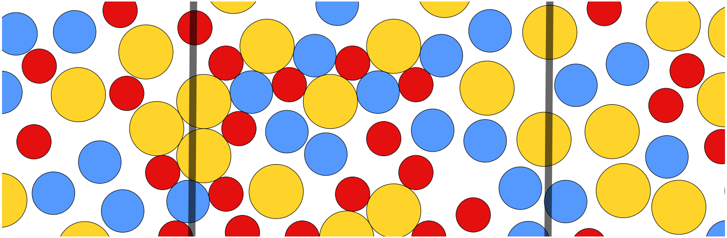

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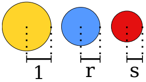

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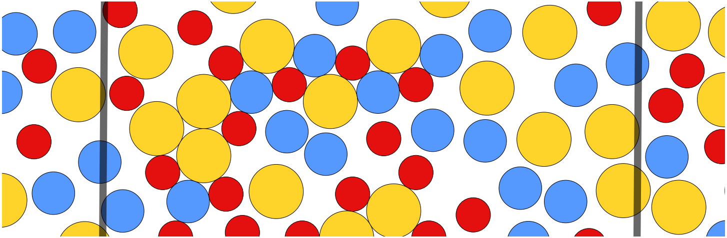

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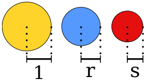

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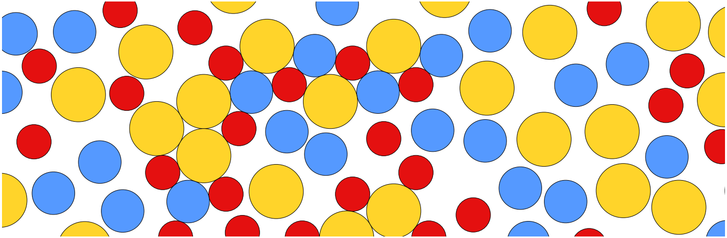

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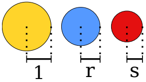

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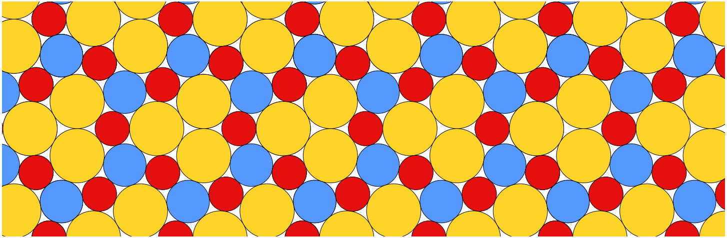

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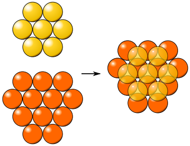
Why do we study packings?

- To pack fruits



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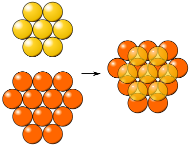
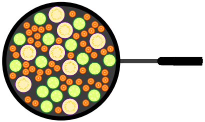


- and vegetables



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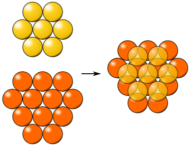

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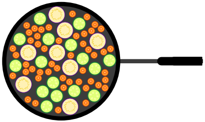
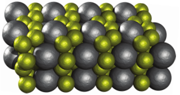
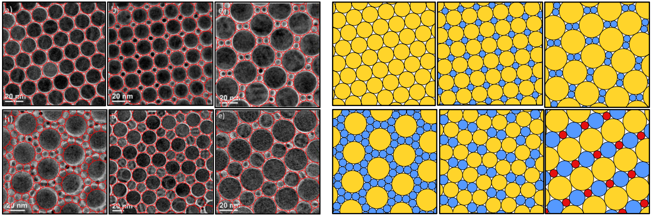


- To make compact materials

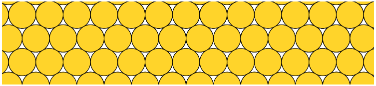




Binary and ternary superlattices self-assembled from colloidal nanodisks and nanorods. Journal of the American Chemical Society, 137(20):6662–6669, 2015.

Context and

2D hexagonal -packing:



$\delta = \frac{\pi}{2\sqrt{3}}$

Lagrange, 1772

Hexagonal packing maximize the density among lattice packings.

Thue, 1910 (Toth, 1940)

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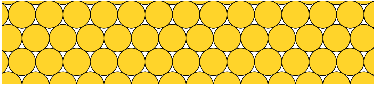

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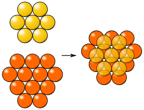


$\delta = \frac{\pi}{3\sqrt{2}}$

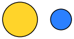
Gauss, 1831

Hexagonal packing maximize the density among lattice packings.

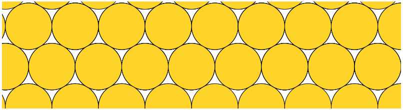
Hales, Ferguson, 1998–2014 (Conjectured by Kepler, 1611)

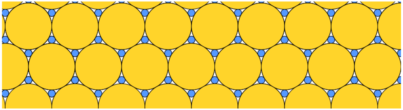
Hexagonal packing maximize the density.

Two discs of radii 1 and r:



Lower bound on the density: $\frac{\pi}{2\sqrt{3}}$ (hexagonal packing with only 1 disc used)





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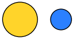

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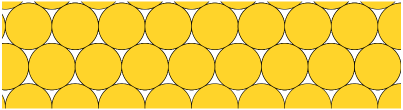
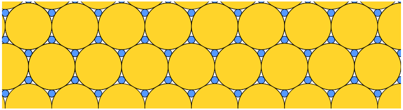
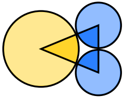




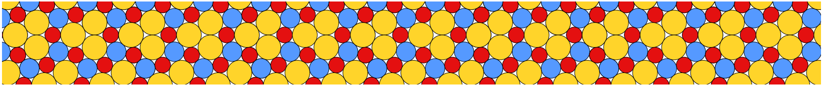
Upper bound on the density:

Florian, 1960

The density of a packing never exceeds the density in the following triangle:

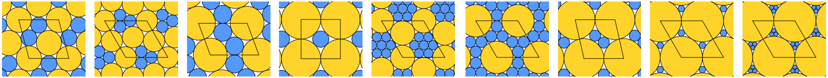


A packing is called **triangulated** if each "hole" is bounded by three tangent discs.



Kennedy, 2006

There are 9 values of r allowing triangulated packings.



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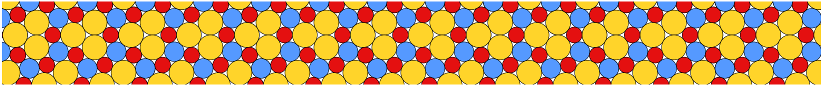

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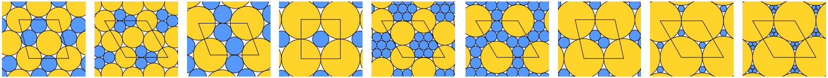
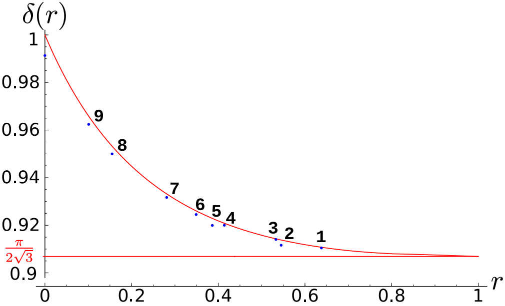


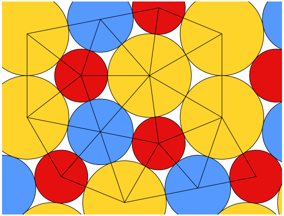
Heppes 2000,2003
Kennedy 2004
Bedaride, Fernique, 2019:

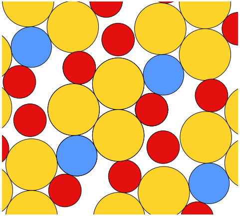
All these 9 packings
maximize the density

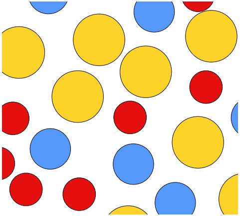


Conjecture (Connelly, 2018)

If a finite set of discs allows a **saturated** triangulated packing then the density is maximized on a saturated triangulated packing.







True for  and .

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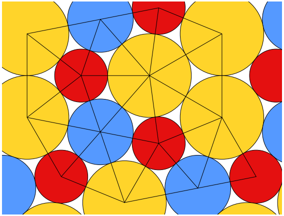
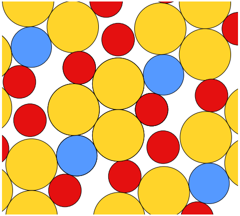
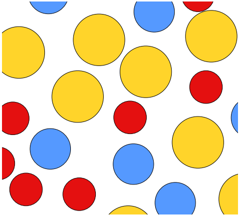


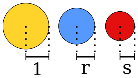




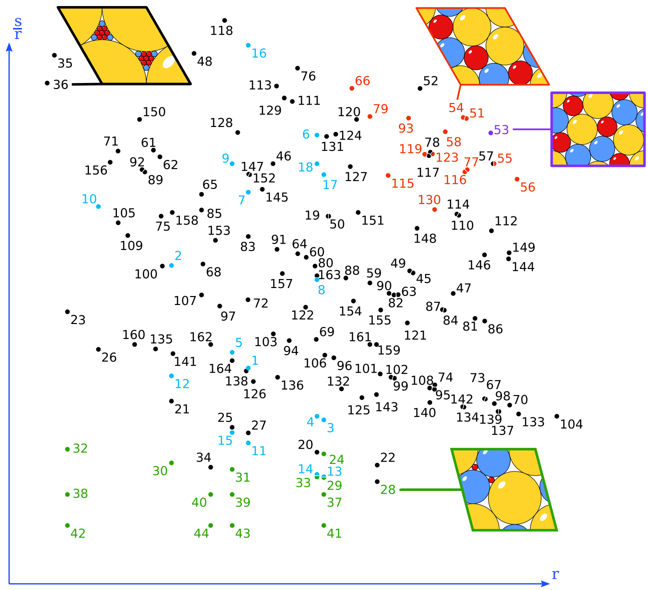
True for and .

What happens with ?

3 discs



- 164 (r, s) with triangulated packings: (Fernique, Hashemi, Sizova 2019)
- 15 non saturated
- Case 53 is proved (Fernique 2019)
- 14 more cases (the internship)



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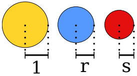

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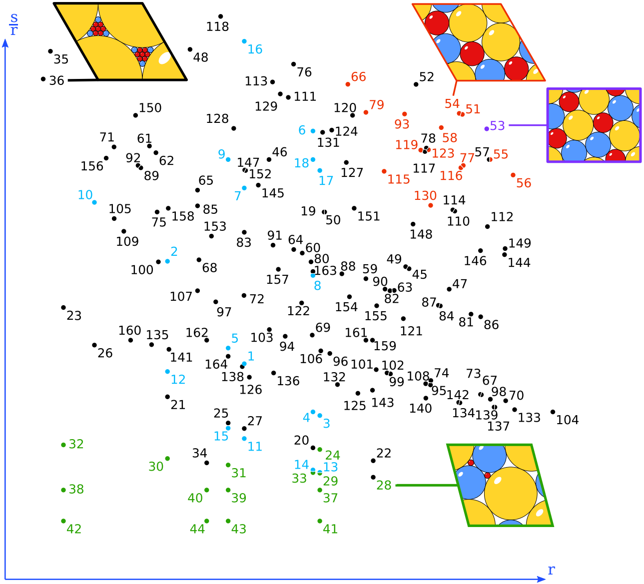
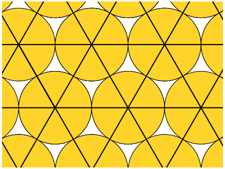


Idea of the proof for ●

A **Delaunay triangulation** of a packing: no points inside a circumscribed circle



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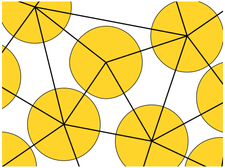

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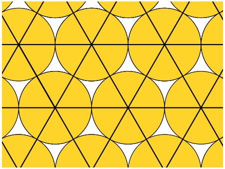

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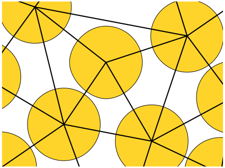

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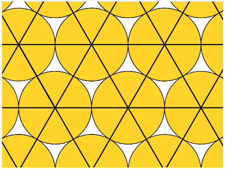

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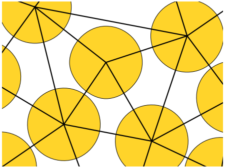

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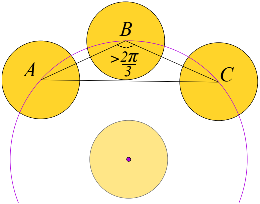

- The largest angle of any $\triangle$ is between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$

$R = \frac{|AC|}{2\sin \hat{B}} \geq \frac{1}{\sin \hat{B}}$

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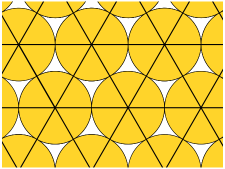

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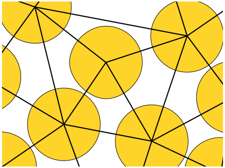

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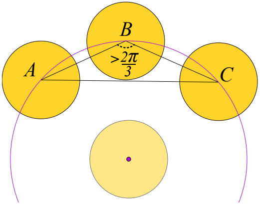

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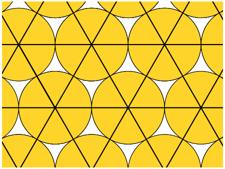
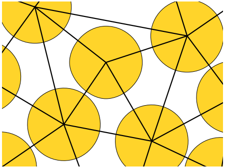
- The density of a triangle $\triangle$: $\delta_{\triangle} = \frac{\pi/2}{area(\triangle)}$

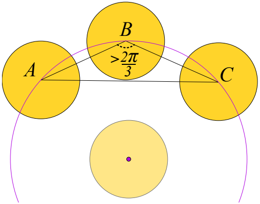
A **Delaunay triangulation** of a packing: no points inside a circumscribed circle



$\delta^* = \delta_{\triangle^*} = \frac{\pi}{2\sqrt{3}}$



$\forall \triangle, \ \ \delta_{\triangle} \leq \delta_{\triangle^*} = \delta^*$



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Idea of the proof for ●

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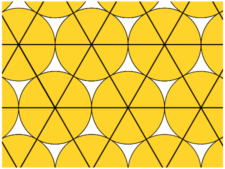

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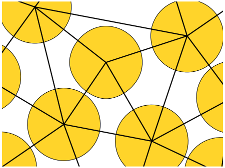

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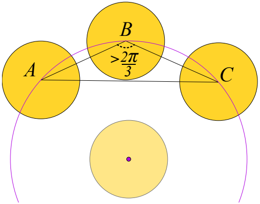

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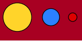
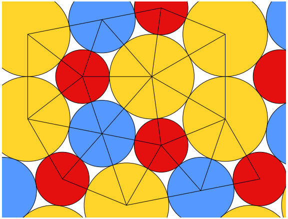
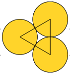
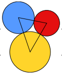
- Thus the density of ABC is less or equal to $\frac{\pi/2}{\sqrt{3}}$

Idea of the proof for 

Delaunay triangulation $\rightarrow$ weighted by the disc radii

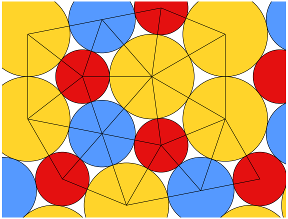


Triangles have different densities:

$\delta($  $) \neq \delta($  $)$

What to do?

Delaunay triangulation → weighted by the disc radii



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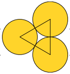
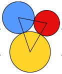
$\delta($  $) \neq \delta($  $)$

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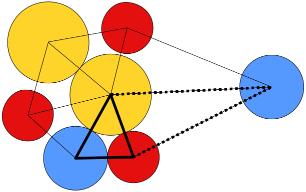
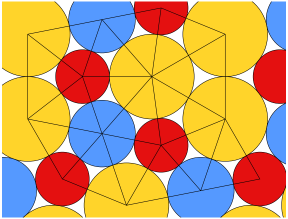
Redistribution of the densities:



Delaunay triangulation → weighted by the disc radii



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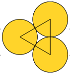
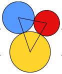
$\delta($  $) \neq \delta($  $)$

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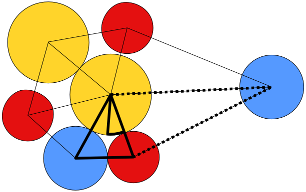
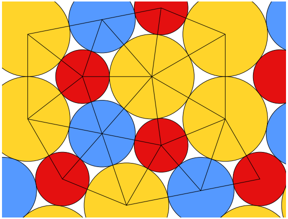


Some triangles "share their density" with neighbors

Idea of the proof for

Delaunay triangulation → weighted by the disc radii



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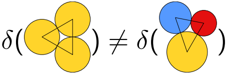

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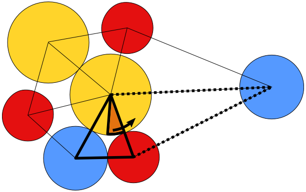

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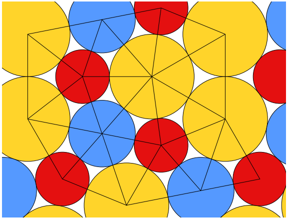

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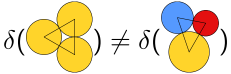

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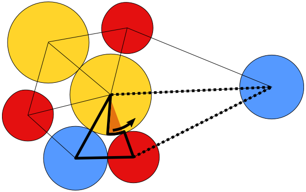

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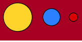
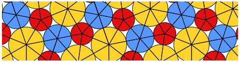
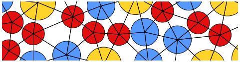
Proof for 

$\mathcal{T}^*$ – saturated triangulated packing of density δ



$\mathcal{T}$ – any other saturated packing with the same discs



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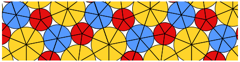

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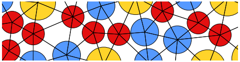

The **sparsity** of a triangle $\triangle \in \mathcal{T}$: $S(\triangle) = \delta \times area(\triangle) - cov(\triangle)$

$S(\triangle) > 0$ iff the density of covering of $\triangle$ is less than δ

$S(\triangle) < 0$ iff the density of covering of $\triangle$ is greater than δ

To prove that $\mathcal{T}$ is no denser than $\mathcal{T}^*$, we show that $\sum_{\mathcal{T}} S(\triangle) \geq 0$

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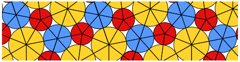

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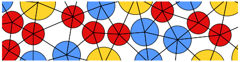

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To prove that $\mathcal{T}$ is no denser than $\mathcal{T}^*$, we show that $\sum_{\mathcal{T}} S(\triangle) \geq 0$

1: Introduce a **potential** U such that for any triangle $\triangle \in \mathcal{T}$,

$$S(\triangle) \geq U(\triangle) \qquad (\triangle)$$

and

$$\sum_{\triangle \in \mathcal{T}} U(\triangle) \geq 0 \qquad (U)$$

2: Instead of proving a **global** inequality

$$\sum_{\triangle \in \mathcal{T}} U(\triangle) \geq 0 \qquad (U)$$

we define the vertex potential: for a triangle $\triangle$ with vertices A, B and C,

$$U(\triangle) = \dot{U}^A_\triangle + \dot{U}^B_\triangle + \dot{U}^C_\triangle$$

and prove a **local** inequality for each vertex $v \in \mathcal{T}$:

$$\sum_{\triangle \in \mathcal{T} | v \in \triangle} \dot{U}^v_\triangle \geq 0 \qquad (\bullet)$$

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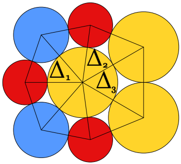
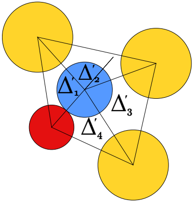


$$4\dot{U}^{v}_{\triangle_1} + 2\dot{U}^{v}_{\triangle_2} + \dot{U}^{v}_{\triangle_3} = 0$$



$$\dot{U}^{v'}_{\triangle'_1} + \dot{U}^{v'}_{\triangle'_2} + \dot{U}^{v'}_{\triangle'_3} + \dot{U}^{v'}_{\triangle'_4} > 0$$

Delaunay triangulation properties → finite number of cases → verification by computer

Proving an inequality with interval arithmetic

To store and perform computations on transcendental numbers (like π), we use intervals.

A representation of a number x is an interval I whose endpoints are exact values representable in a computer memory and such that $x \in I$.

```
sage: x = RIF(0,1)                                        # Interval [0,1]
sage: (x+x).endpoints()
(0.0, 2.0)                                                   # [0,1]+[0,1]
sage: x < 2
True                                                  # ∀t ∈ [0,1], t < 2
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sage: x < 2
True                                                 # ∀t ∈ [0,1], t < 2
```

```
sage: Ipi = RIF(pi)                                  # Interval for π
(3.14159265358979, 3.14159265358980)
sage: sin(Ipi).endpoints()                           # Interval for sin(π)
(-3.21624529935328e-16, 1.22464679914736e-16)
sage: sin(Ipi) >= 0
False                                 # Interval for sin(π) contains 0
```

Defining U, we try to make it as small as possible keeping it locally positive around any vertrex ($\bullet$).

3: How to check

$$S(\triangle) \geq U(\triangle) \tag{$\triangle$}$$

on each triangle $\triangle$? (There is a continuum of them).

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Interval arithmetic!

Delaunay triangulation properties $\rightarrow$ uniform bound on edge length:

Verify $S(\triangle_{e_1,e_2,e_3}) \geq U(\triangle_{e_1,e_2,e_3})$ where

$$e_1 = [r_a+r_b, r_a+r_b+2s] \; e_2 = [r_c+r_b, r_c+r_b+2s] \; e_3 = [r_a+r_c, r_a+r_c+2s]$$

Not precise enough $\rightarrow$ dichotomy

Conclusion

What was done and what will be done...

- 14 cases proved

various techniques: computer-assisted proofs, interval arithmetic, optimisation, combinatorics, discrete geometry

- 133 cases to prove (Connelly's conjecture)

for this: good comprehension of the density redistribution, more optimisation

- maximal density for other disc sizes (which do not allow triangulated packings)

deformations of triangulated packings keep the density high $\rightarrow$ good lower bound on the maximal density