Counting with tiles

Igor Pak, UCLA

Journée-séminaire de combinatoire, Paris Nord

June 2, 2020

OEIS

OEIS now has over 300,000 sequences!

> *Our policy has been to include all interesting sequences, no matter how obscure the reference.* [N.J.A. Sloane and S. Plouffe, EIS, 1995]

> [The EIS contains] *the unrelenting cascade of numbers,* [..] *lists Hard, Disallowed and Silly sequences.* [Richard Guy, 1997]

Question 1: What makes an integer sequence ***combinatorial***?

Question 2: What makes a combinatorial sequence ***nice***?

Traditional Answers:

(1) A sequence is *combinatorial* if it counts combinatorial objects.

(2) Combinatorial sequence is *nice* if it is given by a nice formula.

(2′) The nicer the formula the nicer the sequence.

(2″) Nice formulas can be efficiently computed.

Our Answers:

(1) A sequence is *combinatorial* if it counts combinatorial objects.

(1′) Objects are *combinatorial* if they can be verified by an algorithm.

(2) Combinatorial sequence is *nice* if the corresponding algorithm is efficient.

(2′) The algorithm *efficient* if it requires $Const$ memory space.

More Precisely:

(3) A sequence $\{a_n\}$ is *combinatorial* and *nice* if there exists a finite set T of Wang tiles, so that $a_n = \#$ tilings of an n-rectangle.

Note: Here *nice* $=$ algorithmically efficient.

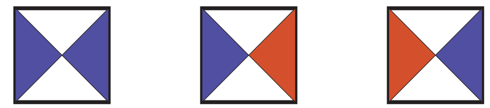
Efficient means restrictions on the model of computation.

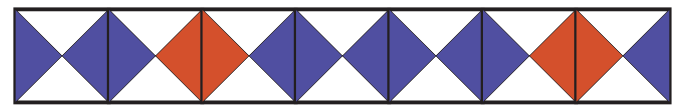
Motivation: Think of this as a special *combinatorial interpretation.*

When such an interpretation is found, it in itself can lead to better understanding AND new algorithmic solutions.

Counting with Wang tiles

Fibonacci numbers:

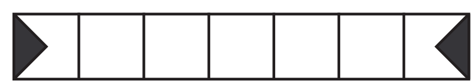




12112

More generally: Wang tilings of a rectangle

T



n

Let $a_n(T)$ = the number of tilings of $[1 \times n]$ with T.

Transfer matrix method:

$$\mathcal{A}(t) \, = \, \sum_{n=0}^{\infty} a_n t^n \, = \, \frac{P(t)}{Q(t)}$$

$\mathbb{N}$-Rational Functions $\mathcal{R}_1$

Definition: Let $\mathcal{R}_1$ be the smallest set of functions $F(x)$ which satisfies

(1) $0, x \in \mathcal{R}_1$,

(2) $F, G \in \mathcal{R}_1 \implies F + G, F \cdot G \in \mathcal{R}_k$,

(3) $F \in \mathcal{R}_1,\ F(0) = 0 \implies 1/(1-F) \in \mathcal{R}_1$.

Note that all $F \in \mathcal{R}_1$ satisfy: $F \in \mathbb{N}[[x]]$, and $F = P/Q$, for some $P, Q \in \mathbb{Z}[x]$.

For example,

$$\frac{1}{1-x-x^2} \quad \text{and} \quad \frac{x^3}{(1-x)^4} \ \in \mathcal{R}_1\,.$$

Theorem [Schützenberger + folklore]

For every finite set T of Wang tiles, we have $\mathcal{A}_T(x) \in \mathcal{R}_1$.

Conversely, for every $F(x) \in \mathcal{R}_1$ there is a *rational* T, s.t. $F(x) = \mathcal{A}_T(x)$.

$\mathbb{N}$-rational functions of one variable:

Word of caution: $\mathcal{R}_1$ is already quite complicated, see [Gessel, 2003].

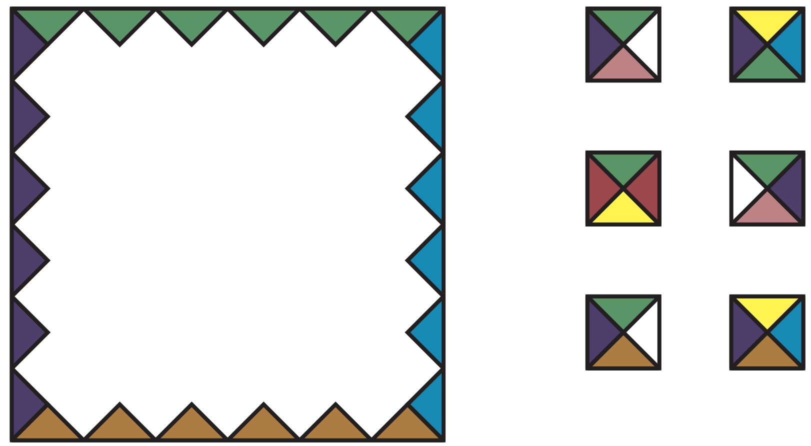
For example, take the following $F, G \in \mathbb{N}[[t]]$:

$$F(t) = \frac{t + 5t^2}{1 + t - 5t^2 - 125t^3}, \qquad G(t) = \frac{1 + t}{1 + t - 2t^2 - 3t^3}.$$

Then $F \notin \mathcal{R}_1$ and $G \in \mathcal{R}_1$; neither of these are obvious.

The proof follows from results in [Berstel, 1971] and [Soittola, 1976], who completely characterized class $\mathcal{R}_1$, see also [Katayama–Okamoto–Enomoto, 1978].

Wang tilings of a square



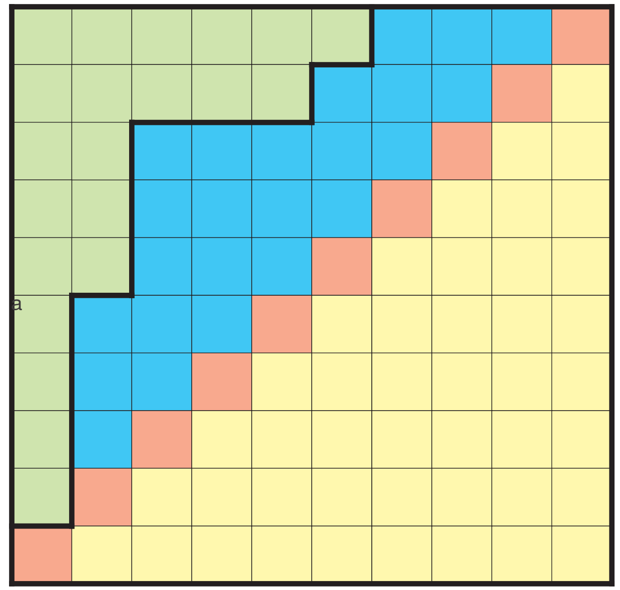
Let $a_n(T)$ = the number of tilings of $[n \times n]$ with T.

Theorem [Mennen–P., 2018+]

Number of tilings $a_n(T)$ is #EXP-complete.

In other words, essentially any function can be the number of tilings.

Catalan numbers



An example Catalan number matrix, and the corresponding lattice path.

Note: Can be implemented with (at most) 169 Wang tiles.

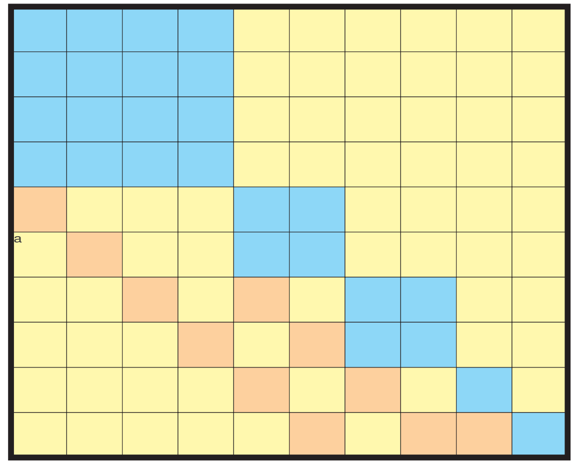
Note: Permutations and alternating permutations can be implemented with at most 405 and 146410 Wang tiles, respectively.

Theorem (Garrabrant, P.)

The following functions count Wang Tilings of a square:

(1) The number of integer partitions of n,
(2) The number of set partitions of an n element set (ordered Bell numbers),
(3) The Catalan number C_n,
(4) The Motzkin number M_n.
(5) The number of Gessel walks of length n,
(6) $n!$,
(7) The number of alternating permutations $Alt(n)$ of length n,
(8) The number of permutations of length n whose assents and descents follow a given periodic sequence,
(9) The number $D(n)$ of derangements of length n,
(10) The ménage numbers A_n,
(11) The Menger number $L(k, n)$ of n by k Latin squares for any fixed k,
(12) The number $Pat_k(n)$ of permutations of length n with no increasing subsequence of length k,
(13) The number $B(n)$ of Baxter permutations of length n,
(14) The number $Alt(n)$ of alternating sign matrices of size n,
(15) The number $G(n)$ of labeled connected graphs on n vertices.

Integer Partitions:



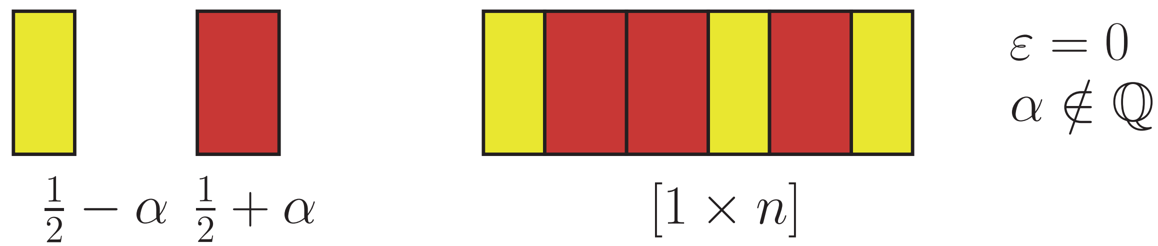
The matrix corresponding to the partition 42211.

Irrational Tilings of $[1 \times n]$ rectangles

Fix $\varepsilon \geq 0$ and a finite set $T = \{\tau_1, \ldots, \tau_k\}$ of *irrational tiles* of height 1.

Let $a_n = a_n(T, \varepsilon)$ the number of tilings of $[1 \times (n + \varepsilon)]$ with T.

Observe: we can get *algebraic* g.f.'s $\mathcal{A}_T(t)$.



Here $a_n = \binom{2n}{n}$, $\mathcal{A}(t) = \frac{1}{\sqrt{1-4t}}$.

Question: What else can we get?

Diagonals of Rational Functions

Let $G \in \mathbb{Z}[[x_1, \ldots, x_k]]$. A *diagonal* is a g.f. $\mathcal{B}(t) = \sum_n b_n t^n$, where

$$b_n = \left[x_1^n \cdots x_k^n\right] G(x_1, \ldots, x_k).$$

Theorem: *Every $\mathcal{A}_T(t) \in \mathcal{F}$ is a diagonal of a rational function P/Q, for some polynomials $P, Q \in \mathbb{Z}[x_1, \ldots, x_k]$.*

For example,

$$\binom{2n}{n} = [x^n y^n] \frac{1}{1 - x - y}.$$

Proof idea: Say, $\tau_i = [1 \times \alpha_i]$, $\alpha_i \in \mathbb{R}$. Let $V = \mathbb{Q}\langle \alpha_1, \ldots, \alpha_k \rangle$, $d = \dim(V)$.

We have natural maps $\varepsilon \mapsto (c_1, \ldots, c_d)$, $\alpha_i \mapsto v_i \in \mathbb{Z}^d \subset V$.

Interpret irrational tilings as walks $O \to (n + c_1, \ldots, n + c_d)$ with steps $\{v_1, \ldots, v_k\}$.

Properties of Diagonals of Rational Functions

(1) must be *D-finite*, see [Stanley, 1980], [Gessel, 1981].

(2) when $k = 2$, must be *algebraic*, and

(2′) every algebraic $\mathcal{B}(t)$ is a diagonal of $P(x,y)/Q(x,y)$, see [Furstenberg, 1967].

No surprise now that Catalan g.f. $C(t)$, $tC(t)^2 - C(t) + 1 = 0$, is a diagonal:

$$C_n = [x^n y^n] \, \frac{y(1-2xy-2xy^2)}{1-x-2xy-xy^2}, \qquad C_n = [x^n y^n] \, \frac{1-x/y}{1-x-y}.$$

For the first formula, see [Rowland–Yassawi, 2014].

$\mathbb{N}$-Rational Functions in many variables

Definition: Let $\mathcal{R}_k$ be the smallest set of functions $F(x_1, \ldots, x_k)$ which satisfies

(1) $0, x_1, \ldots, x_k \in \mathcal{R}_k$,

(2) $F, G \in \mathcal{R}_k \implies F + G, F \cdot G \in \mathcal{R}_k$,

(3) $F \in \mathcal{R}_k,\ F(0) = 0 \implies 1/(1 - F) \in \mathcal{R}_k$.

Note that all $F \in \mathcal{R}_k$ satisfy: $F \in \mathbb{N}[[x_1, \ldots, x_k]]$, and $F = P/Q$, for some $P, Q \in \mathbb{Z}[x_1, \ldots, x_k]$.

Let $\mathcal{N}$ be a class of diagonals of $F \in \mathcal{R}_k$, for some $k \geq 1$. For example,

$$\sum_n \binom{2n}{n} t^n \in \mathcal{N} \qquad \text{because} \qquad \frac{1}{1 - x - y} \in \mathcal{R}_2 \,.$$

Main Theorem: $\mathcal{F} = \mathcal{N}$ [Garrabrant, P., 2014]

Here $\mathcal{F}$ denote the class of g.f. $\mathcal{A}_T(t)$ enumerating irrational tilings.

In other words, every tile counting function $\mathcal{A}_T \in \mathcal{F}$ is a diagonal

of an $\mathbb{N}$-rational function $F \in \mathcal{R}_k$, $k \geq 1$, and vice versa.

Key Lemma:

Both $\mathcal{F}$ and $\mathcal{N}$ coincide with a class $\mathcal{B}$ of g.f. $F(t) = \sum_n f(n)t^n$,

where $f : \mathbb{N} \to \mathbb{N}$ is given as finite sums $f = \sum g_j$, and each g_j is of the form

$$g_j(m) = \begin{cases} \sum_{v \in \mathbb{Z}^{d_j}} \prod_{i=1}^{r_j} \binom{\alpha_{ij}(v,n)}{\beta_{ij}(v,n)} & \text{if} \quad m = p_j n + k_j, \\ 0 & \text{otherwise,} \end{cases}$$

for some $\alpha_{ij} = a_{ij}v + a'_{ij}n + a''_{ij}$, $\beta_{ij} = b_{ij}v + b'_{ij}n + b''_{ij}$, and $p_j, k_j, r_j, d_j \in \mathbb{N}$.

Asymptotic applications

Corollary: There exist $\sum_n f_n, \sum_n g_n \in \mathcal{F}$, s.t.

$$f_n \sim \frac{\sqrt{\pi}}{\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{7}{8}\right)} 128^n, \qquad g_n \sim \frac{\Gamma\left(\frac{3}{4}\right)^3}{\sqrt[3]{2}\,\pi^{5/2}} n^{-3/2}\, 384^n$$

Proof idea: Take

$$f_n := \sum_{k=0}^{n} 128^{n-k} \binom{4k}{k}\binom{3k}{k}.$$

Note: We have $b_n \sim K\, n^\beta \gamma^n$, where $\beta \in \mathbb{N}$, and $K, \gamma \in \overline{\mathbb{Q}}$, for all $\sum_n b_n t^n = P/Q$.

Conjecture: *For every* $\sum_n f_n \in \mathcal{F}$, *we have* $f_n \sim K n^\beta \gamma^n$, *where* $\beta \in \mathbb{Z}/2, \gamma \in \overline{\mathbb{Q}}$, *and* K *is a* generalized period, *see.* [Kontsevich–Zagier, 2001].

Curious Conjecture on Catalan numbers:

We have:

$$C(t) \notin \mathcal{F}\,, \qquad \textit{where} \qquad C(t) = \frac{1-\sqrt{1-4t}}{2t}\,.$$

In other words, there is no set T of irrational tiles and $\varepsilon \geq 0$, s.t.

$$a_n(T,\varepsilon) = C_n \quad \text{for all} \;\; n \geq 1, \quad \text{where} \quad C_n = \frac{1}{n+1}\binom{2n}{n}.$$

More on Catalan numbers

Recall

$$C_n \ \sim \ \frac{1}{\sqrt{\pi}}\, n^{-3/2}\, 4^n.$$

Corollary: (from Main theorem) *There exists*

$$\sum_n f_n t^n \in \mathcal{F} \qquad \text{such that} \qquad f_n \sim \frac{3\sqrt{3}}{\pi} C_n.$$

Furthermore, $\forall \epsilon > 0$*, there exists*

$$\sum_n f_n t^n \in \mathcal{F} \qquad \text{such that} \qquad f_n \sim \lambda C_n$$

for some $\lambda \in [1-\epsilon, 1+\epsilon]$.

Moral: Curious Conjecture cannot be proved via rough asymptotics.

Conjecture: *There is no* $\sum_n f_n t^n \in \mathcal{F}$*, s.t.* $f_n \sim C_n$.

Warning: *This conjecture probably involves deep number theory.*

More applications

Proposition: *For every* $m \geq 2$*, there is* $\sum_n f_n t^n \in \mathcal{F}$*, s.t.*

$$f_n = C_n \mod m, \quad \textit{for all} \;\; n \geq 1.$$

Proposition *For every prime* $p \geq 2$*, there is* $\sum_n g_n t^n \in \mathcal{F}$*, s.t.*

$$\operatorname{ord}_p(g_n) = \operatorname{ord}_p(C_n), \quad \textit{for all} \;\; n \geq 1,$$

where $\operatorname{ord}_p(N)$ *is the largest power of* p *which divides* N*.*

Moral: Elementary number theory does not help to prove the Curious Conjecture.

Note: For $\operatorname{ord}_p(C_n)$, see [Kummer, 1852], [Deutsch–Sagan, 2006].

Proof idea: Take

$$f_n = \binom{2n}{n} + (m-1)\binom{2n}{n-1}.$$

Schützenberger's principle

There is a general metamathematical principle that goes back to M.-P. Schützenberger and that states the following: whenever a rational series in one variable counts a class of objects, then the series is $\mathbb{N}$*-rational. This phenomenon has been observed on a large number of examples: generating series and zeta functions in combinatorics, Hilbert series of graded or filtered algebras, growth series of monoids or of groups.*

[Berstel, Reutenauer; 2008]

Open Problem: Suppose $F \in \mathcal{F}$ is rational. Does this imply that $F \in \mathcal{R}_1$?

If NO, this implies that Schützenberger's principle is FALSE, i.e. there is a set of *irrational tiles* which gives a combinatorial interpretation to a non-negative rational functions, which nonetheless is not $\mathbb{N}$-rational.

Thank you!