Asymptotics for the probability of labelled objects to be irreducible

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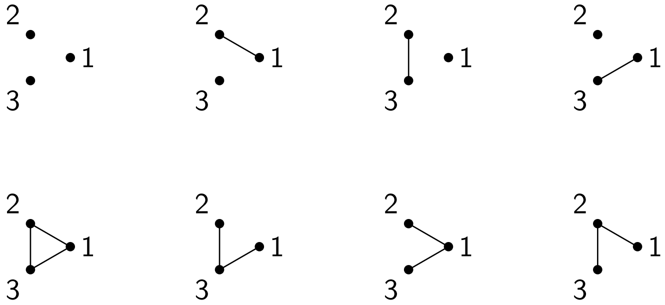
Graphs

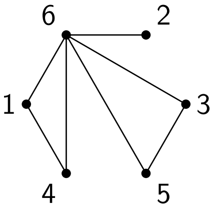
Let f_n be the number of labelled graphs with n vertices.



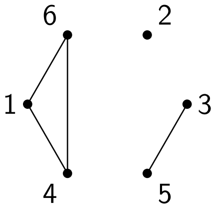
$$f_n = 2^{\binom{n}{2}}$$

Connected graphs

Let g_n be the number of connected labelled graphs with n vertices.



connected graph



disconnected graph

$$(g_n) = 1, 1, 4, 38, 728, 26704, 1866256, \ldots$$

Every graph is a disjoint union (SET) of connected graphs.

Probability of graph to be connected

Question. What is the probability $p_n = \dfrac{g_n}{f_n}$ of a random graph with n vertices to be connected as $n \to \infty$?

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- Wright, 1970:

$$p_n = 1 - \binom{n}{1}\frac{2}{2^n} - \binom{n}{3}\frac{2^7}{2^{3n}} - 3\binom{n}{4}\frac{2^{13}}{2^{4n}} + O\left(\frac{n^5}{2^{5n}}\right)$$

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- Can we have all terms at once? What is the interpretation?

Asymptotics for p_n

- Monteil, N., 2019:

  as $n \to \infty$, for every $r \geqslant 1$

$$p_n = 1 - \sum_{k=1}^{r-1} h_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

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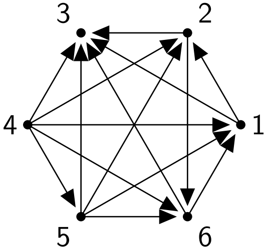
  where h_k counts irreducible labelled tournaments of size k.

$$(h_k) = 1, 0, 2, 24, 544, 22320, 1677488, \ldots$$

Tournaments

A tournament is a complete directed graph.



The number of labelled tournaments with n vertices is equal to

$$f_n = 2^{\binom{n}{2}}$$

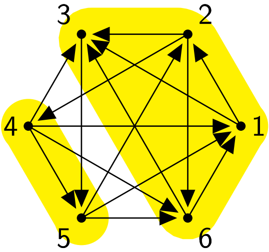
Irreducible tournaments

A tournament is called irreducible
(or strongly connected tournament),

if for every partition of vertices $V = A \sqcup B$

1. there exist an edge from A to B and
2. there exist an edge from B to A.

$V = \{1, 2, 3, 4, 5, 6\}$



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Irreducible tournaments

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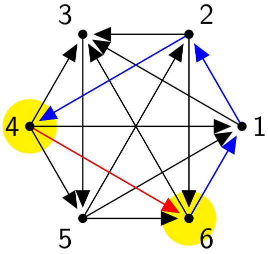
1. there exist an edge from A to B and
2. there exist an edge from B to A.

Equivalently, for each two vertices u and v

1. there is a path from u to v and
2. there is a path from v to u.

$V = \{1, 2, 3, 4, 5, 6\}$

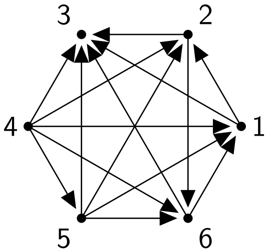


$u = 4$

$v = 6$

Tournaments as a sequence

Lemma. Every labelled tournament can be uniquely decomposed into a sequence (SEQ) of irreducible labelled tournaments.



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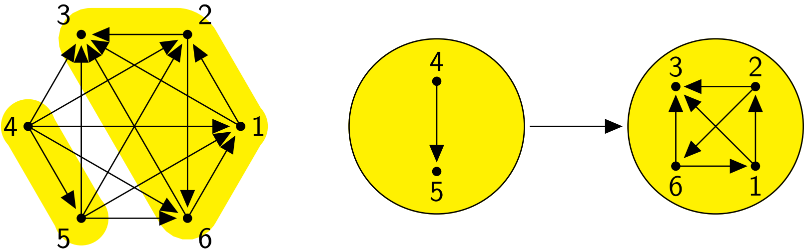

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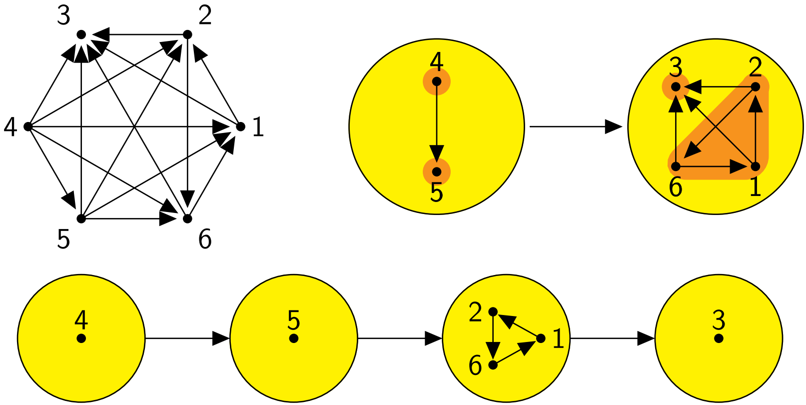
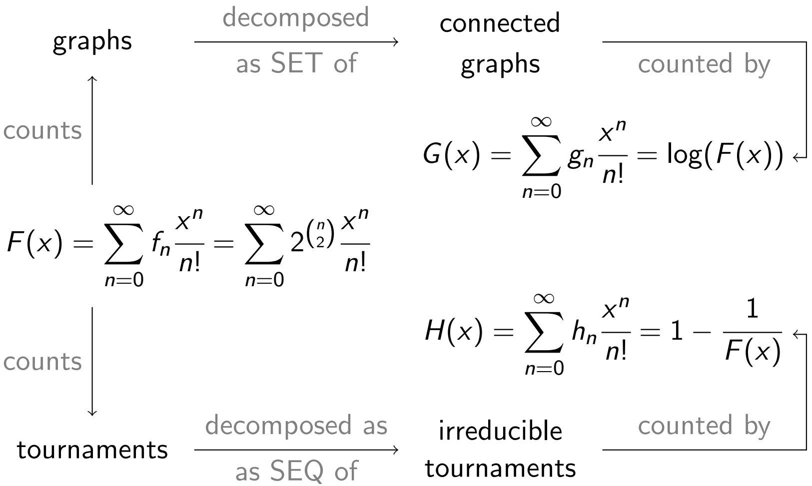


SET vs SEQ



graphs

decomposed

as SET of

connected graphs

counted by

$$G(x) = \sum_{n=0}^{\infty} g_n \frac{x^n}{n!} = \log(F(x))$$

counts

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{x^n}{n!}$$

$$H(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!} = 1 - \frac{1}{F(x)}$$

counts

tournaments

decomposed as

as SEQ of

irreducible tournaments

counted by

Notations

- $\mathcal{F} = \mathrm{SET}(\mathcal{G}), \qquad G(x) = \log(F(x));$
- $\mathcal{F} = \mathrm{SEQ}(\mathcal{H}), \qquad H(x) = 1 - \dfrac{1}{F(x)};$
- $\mathcal{T}^{(m)} = \mathrm{SEQ}_m(\mathcal{H}), \qquad T^{(m)}(x) = \big(H(x)\big)^m;$
- $H^{(m)}(x) = 1 - \dfrac{1}{(F(x))^m} = 1 - (1 - H(x))^m.$

General result

Let $G(x) = \log(F(x)), \ H(x) = 1 - \frac{1}{F(x)}, \ H^{(2)}(x) = 1 - \frac{1}{F^2(x)}.$

Theorem (Monteil, N., 2019+)

If $f_n \neq 0$ *for all* $n \in \mathbb{N}$ *and there exists* $r \geqslant 1$ *such that*

$$(i) \ n \cdot \frac{f_{n-1}}{f_n} \to 0 \qquad \textit{and} \qquad (ii) \ \sum_{k=r}^{n-r} \binom{n}{k} f_k f_{n-k} = O\big(n^r f_{n-r}\big),$$

Then

$$(a) \qquad p_n = \frac{g_n}{f_n} = 1 - \sum_{k=1}^{r-1} h_k \cdot \binom{n}{k} \cdot \frac{f_{n-k}}{f_n} + O\left(n^r \cdot \frac{f_{n-r}}{f_n}\right).$$

$$(b) \qquad p_n^{(1)} := \frac{h_n}{f_n} = 1 - \sum_{k=1}^{r-1} h_k^{(2)} \cdot \binom{n}{k} \cdot \frac{f_{n-k}}{f_n} + O\left(n^r \cdot \frac{f_{n-r}}{f_n}\right).$$

General result

Let $$G(x) = \log(F(x)),\ H(x) = 1 - \frac{1}{F(x)},\ H^{(m)}(x) = 1 - \frac{1}{F^m(x)}.$$

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Then for all $m \geqslant 1$

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$$(c) \qquad \frac{1}{m}\frac{h_n^{(m)}}{f_n} = 1 - \sum_{k=1}^{r-1} h_k^{(m+1)} \cdot \binom{n}{k} \cdot \frac{f_{n-k}}{f_n} + O\left(n^r \cdot \frac{f_{n-r}}{f_n}\right).$$

General result

Let $$H(x) = 1 - \frac{1}{F(x)}, \quad T^{(m)} = (H(x))^m.$$

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Then for all $m \geqslant 1$

$$(d) \quad p_n^{(m+1)} = \frac{t_n^{(m+1)}}{f_n} = \sum_{k=1}^{r-1} c_k^{(m+1)} \cdot \binom{n}{k} \cdot \frac{f_{n-k}}{f_n} + O\left(n^r \cdot \frac{f_{n-r}}{f_n}\right),$$

where $$c_k^{(m+1)} = (m+1)\big(t_k^{(m)} - 2t_k^{(m+1)} + t_k^{(m+2)}\big).$$

Probability for graphs and tournaments

- f_n counts labelled graphs / tournaments,
- g_n counts connected labelled graphs,
- h_n counts irreducible labelled tournaments.
- $t_n^{(m)}$ counts irreducible labelled tournaments with exactly m irreducible components.

$\mathbb{P}\{\text{graph is connected}\} =$

$$= \frac{g_n}{f_n} = 1 - \sum_{k=1}^{r-1} h_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right).$$

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$\mathbb{P}\{\text{tournament is irreducible}\} =$

$$= \frac{h_n}{f_n} = 1 - \sum_{k=1}^{r-1} h_k^{(2)} \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where $\left(h_k^{(2)}\right) = 2, -2, 4, 32, 848, 38032 \ldots$

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$\mathbb{P}\{\text{tournament has exactly 2 irreducible components}\} =$

$$= \frac{t_n^{(2)}}{f_n} = \sum_{k=1}^{r-1} c_k^{(2)} \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where $\left(c_k^{(2)}\right) = 2, -8, 16, -16, 368, 22528\ldots$

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$\mathbb{P}\{$tournament has exactly 3 irreducible components$\}$ =

$$= \frac{t_n^{(3)}}{f_n} = \sum_{k=1}^{r-1} c_k^{(3)} \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where $\left(c_k^{(3)}\right) = 0, 6, -36, 120, 0, 9744 \ldots$

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$\mathbb{P}\{$tournament has exactly 4 irreducible components$\} =$

$$= \frac{t_n^{(4)}}{f_n} = \sum_{k=1}^{r-1} c_k^{(4)} \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where $\qquad (c_k^{(4)}) = 0, 0, 24, -192, 960, 960 \ldots$

Probability for graphs and tournaments

- f_n counts labelled graphs / tournaments,
- g_n counts connected labelled graphs,
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- $t_n^{(m)}$ counts irreducible labelled tournaments with exactly m irreducible components.

$\mathbb{P}\{$tournament has exactly $(m+1)$ irreducible components$\} =$

$$= \frac{t_n^{(m+1)}}{f_n} = (n)_m \cdot \frac{2^{m(m+1)/2}}{2^{nm}} + O\left(\frac{n^{m+1}}{2^{n(m+1)}}\right),$$

where $$(n)_m = n(n-1)(n-2)\ldots(n-m+1).$$

Surface applications

	square-tiled surfaces	polygons model
f_n	translation surfaces obtained by gluing squares $\{(\sigma,\tau) \mid \sigma,\tau \in S_n^2\}$	surfaces obtained by gluing polygons $\{(\sigma,\tau) \mid \tau$ is perfect matching$\}$
g_n	connected surfaces	connected surfaces
h_n	$\{(\sigma,\tau) \mid \tau$ is indecomposable permutation$\}$	$\{(\sigma,\tau) \mid \tau$ is indecomposable perfect matching$\}$
p_n	$\mathbb{P}\{$surface is connected$\}$	$\mathbb{P}\{$surface is connected$\}$
$p_n^{(1)}$	$\mathbb{P}\{$permutation is indecomposable$\}$	$\mathbb{P}\{$perfect matching is indecomposable$\}$
f_n	$n!$	$n!(n-1)!!,\quad n$ is even
g_n	$1, 3, 26, 426, 11064\dots$	$0, 2, 0, 60, 0, 8880\dots$
h_n	$h_n = n! \cdot m_n$ $(m_n) = 1, 1, 3, 13, 71, 461\dots$	$h_n = n! \cdot m_n$ $(m_n) = 0, 1, 0, 2, 0, 10, 0, 74\dots$

Asymptotics for $G(n, p)$

Consider $G(n, p)$ model, $q = 1 - p$.

Question. What is the probability p_n of a random graph with n vertices to be connected as $n \to \infty$?

- Gilbert, 1959: $p_n = 1 - nq^{n-1} + O\left(n^2 q^{3n/2}\right)$

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- Monteil, N., 2020:
$$p_n = 1 - \sum_{k=1}^{r-1} c_k(q) \cdot \binom{n}{k} \cdot q^{nk-k^2} + O\big(n^r q^{nr}\big),$$
where $c_k(q) \in \mathbb{Z}[q]$, $\deg c_k \leqslant \binom{k}{2}$. Particularly,
$c_1(q) = 1$, $c_2(q) = 1 - 2q$, $c_3(q) = 1 - 6q^2 + 6q^3$.
- What is the interpretation of $c_k(q)$?

The end

Thank you for your attention!