Monotonic Subsequences

Three (Nice) Open Problems

Nabil H. Mustafa

Large monotone subsequence

Erdős–Szekeres Theorem

Given a sequence S of n reals,
there exists a monotonic subsequence of S of size at least $\sqrt{n}$.

10 1 9 8 7 5 3 11 6 12 4 2

10 **1** 9 8 7 **5** 3 **11** 6 **12** 4 2

10 1 **9** **8** **7** 5 3 11 **6** 12 **4** **2**

3 2 1 | 6 5 4 | 9 8 7 $\implies$ leads to the proof

Large monotone subsequence

Erdős–Szekeres Theorem

Given a sequence S of n reals,
there exists a monotonic subsequence of S of size at least $\sqrt{n}$.

10	1	9	8	7	5	3	11	6	12	4	2
1	1	2	2	2	2	2	3	3	4	3	2
10	1	9	8	7	5	3	11	6	12	4	2

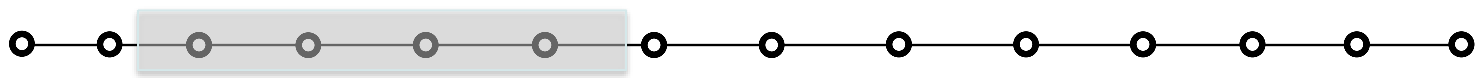
t_i : length of the longest increasing sequence ending at the i-th element

either $\exists\ i$ with $t_i \geq \sqrt{n}$ or the same integer appears $\frac{n}{\sqrt{n}}$ times

CONFLICT-FREE COLORINGS

Conflict-free colorings

[Even, Lotker, Ron, Smorodinsky]

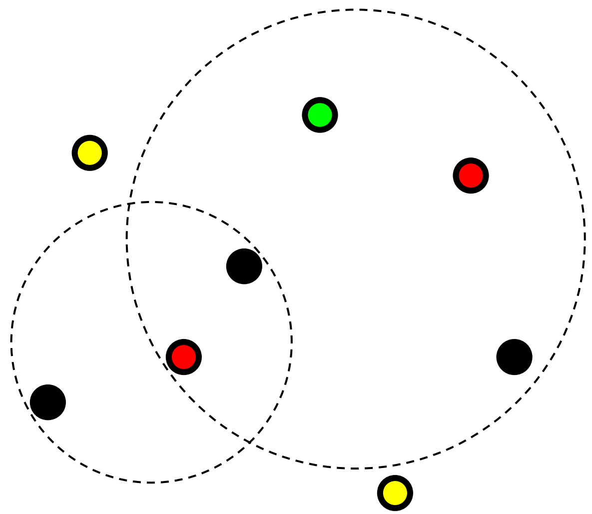


Goal : coloring such that each interval contains a unique color

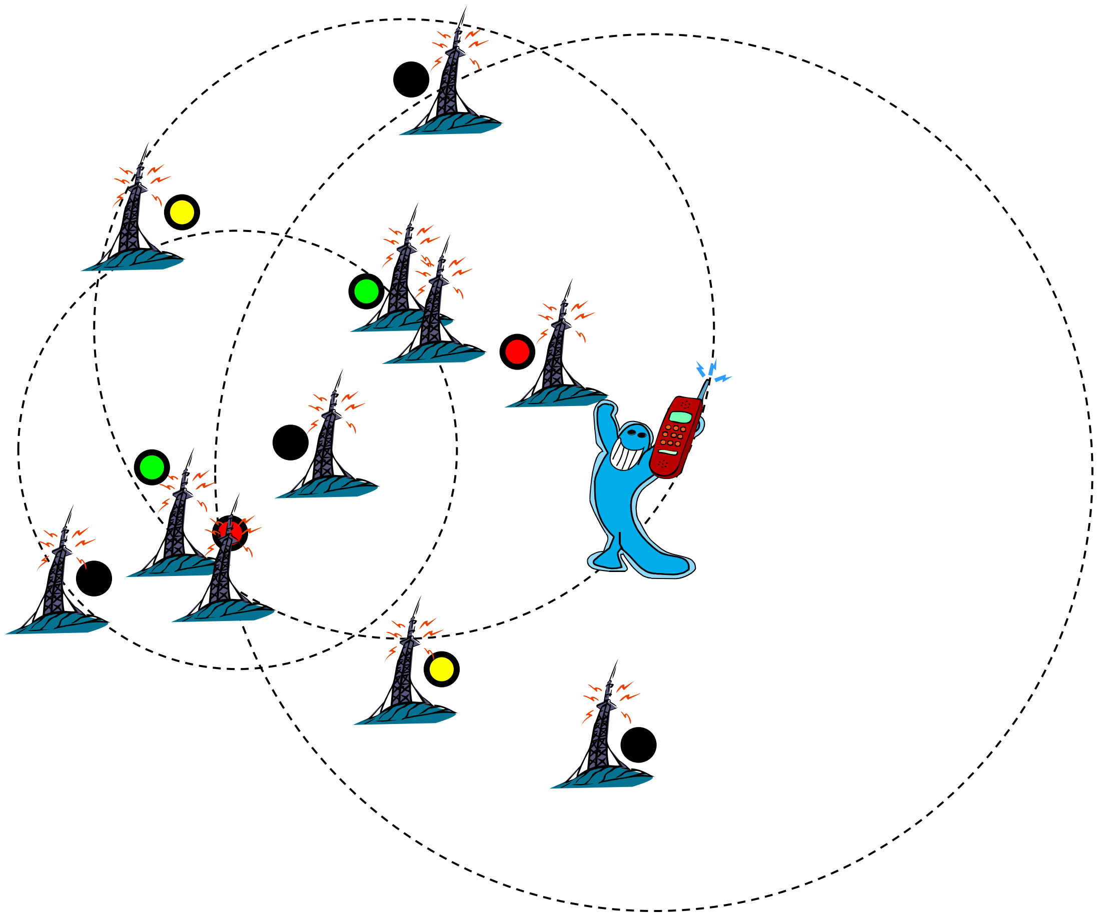
→ possible with $O(\log n)$ colors

→ need $\Omega(\log n)$ colors

→ **Disks** in $\mathbb{R}^2$



Conflict-free colorings



Conflict-free colorings

[Even, Lotker, Ron, Smorodinsky]

Goal : Given a set P of n points, find $Q \subseteq P$ such that

if disk D contains points of Q

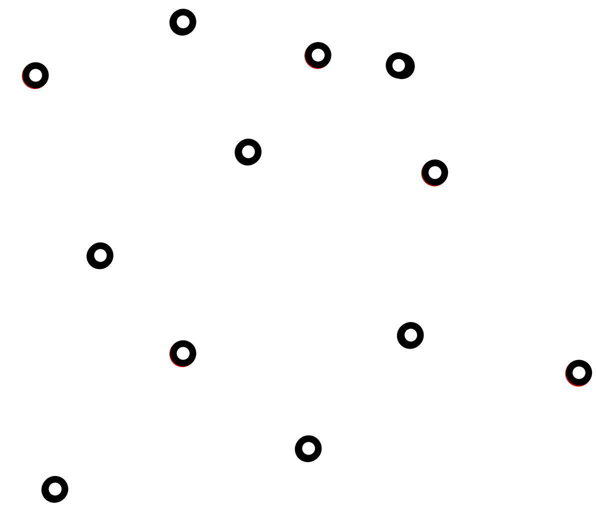
then D **must** also contain from $P \setminus Q$

Coloring procedure:

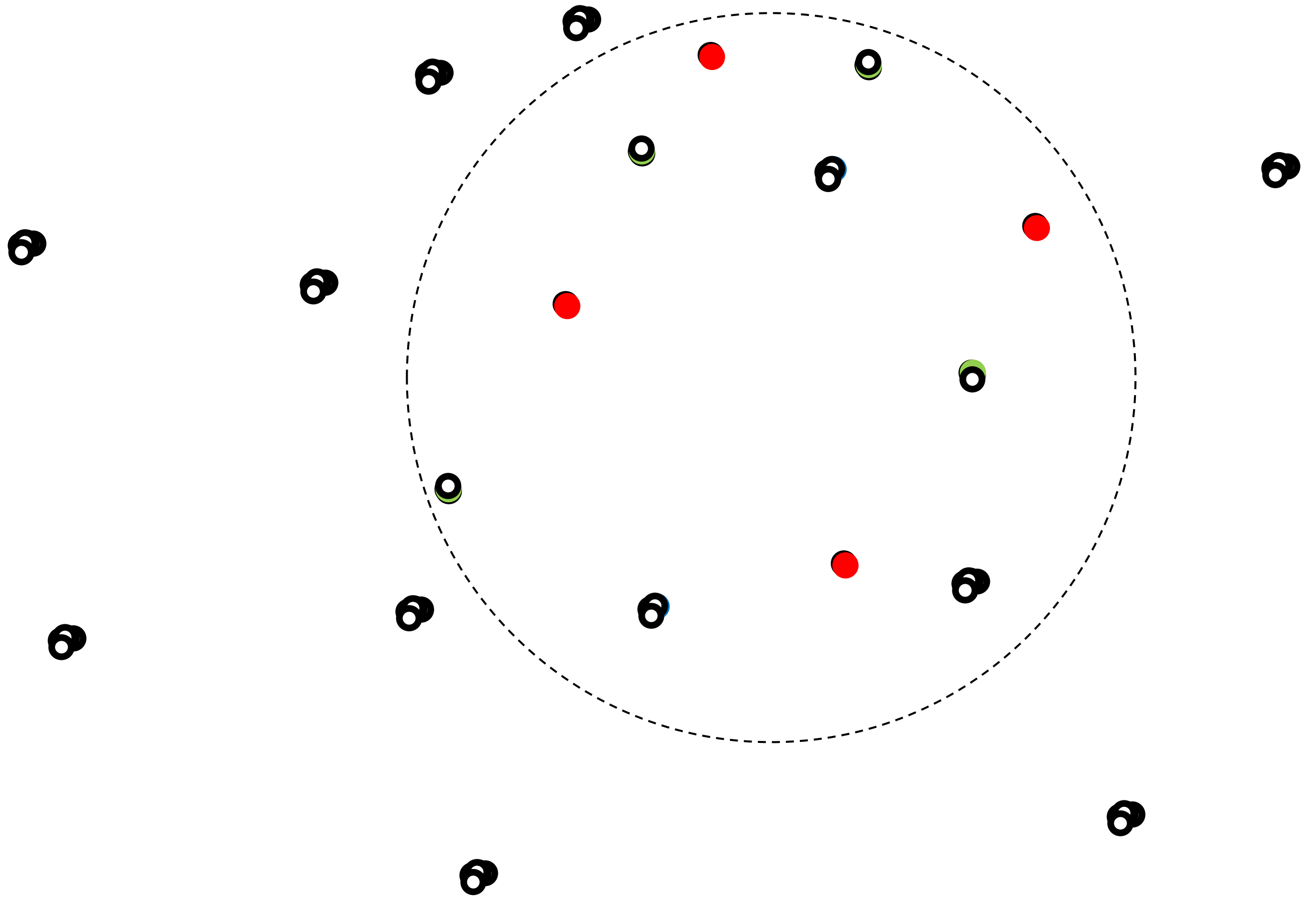
while (P not empty)

find Q and color with the same new color

$P = P - Q$



Conflict-free colorings



Conflict-free colorings

Goal : Given a set P of n points, find $Q \subseteq P$ such that

if disk D contains points of Q

then D **must** also contain from $P \setminus Q$

Intervals : Q exists of size $\frac{n}{2}$ $\rightarrow$ coloring with $\Theta(\log n)$ colors

Disks : Q exists of size $\frac{n}{4}$ $\rightarrow$ coloring with $\Theta(\log n)$ colors

(4-color theorem)

Rectangles : ??

Conflict-free colorings

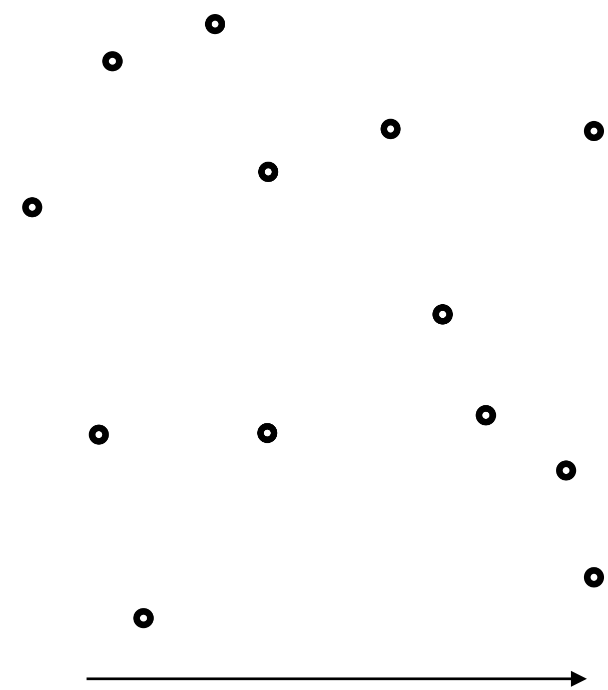
Goal : Given a set P of n points, find $Q \subseteq P$ such that

if rectangle R contains points of Q

then R **must** also contain from $P \setminus Q$

Claim : Such a Q of size $\Omega(\sqrt{n})$ exists

$\rightarrow$ sort by x-coordinate



Conflict-free colorings

Goal : Given a set P of n points, find $Q \subseteq P$ such that

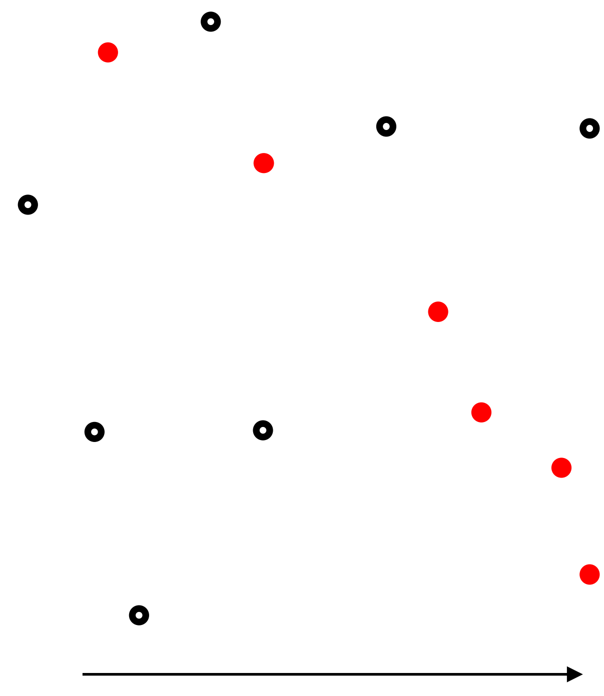
if rectangle R contains points of Q

then R **must** also contain from $P \setminus Q$

Claim : Such a Q of size $\Omega(\sqrt{n})$ exists

→ sort by x-coordinate

→ monotone subsequence of size $\Omega(\sqrt{n})$



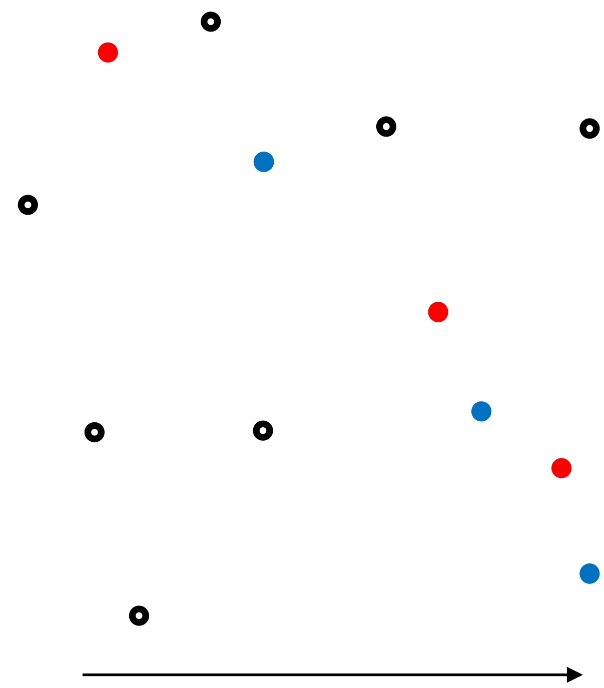
Conflict-free colorings

Goal : Given a set P of n points, find $Q \subseteq P$ such that

if rectangle R contains points of Q

then R **must** also contain from $P \setminus Q$

Claim : Such a Q of size $\Omega(\sqrt{n})$ exists



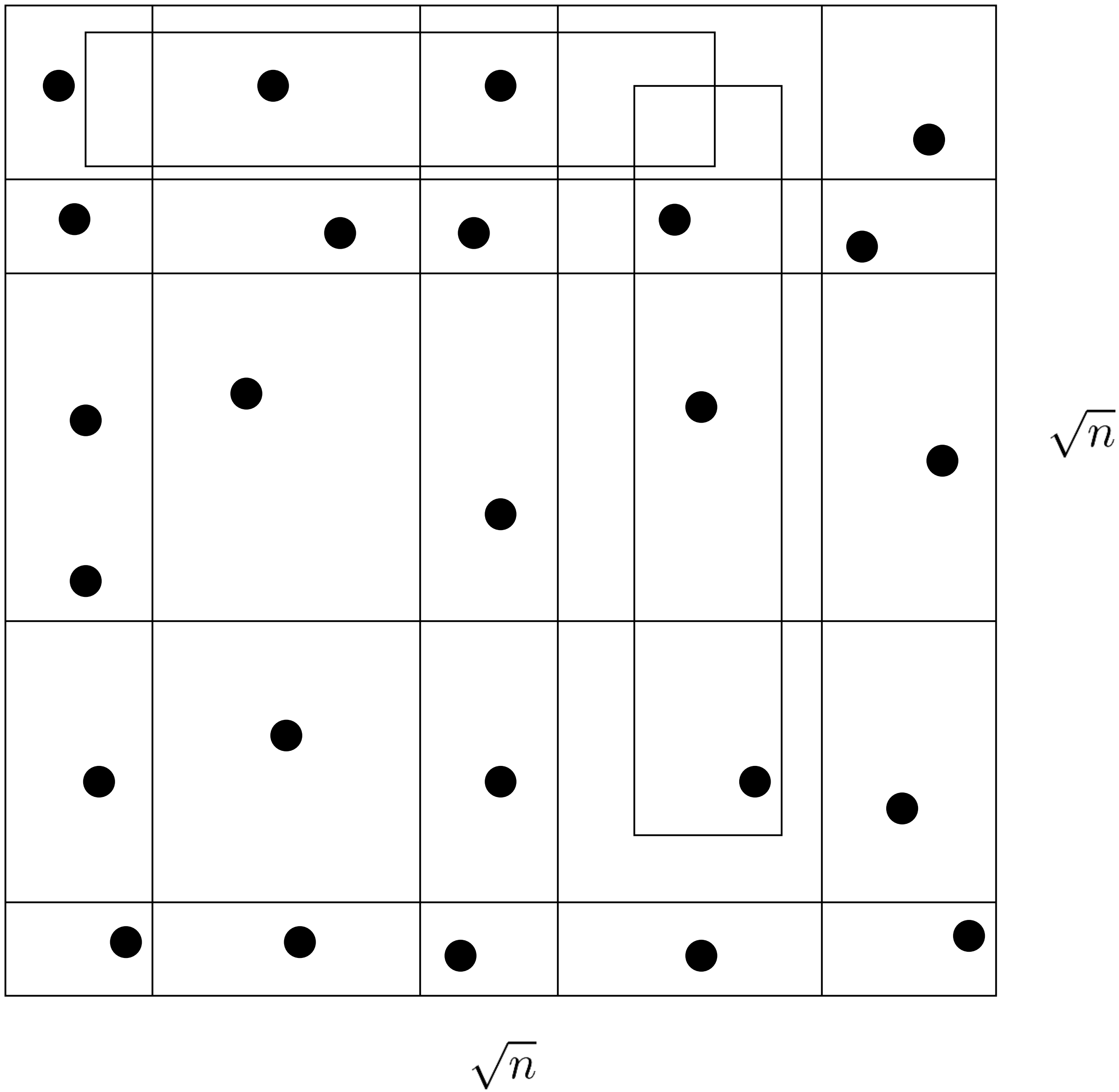
$\rightarrow$ sort by x-coordinate

$\rightarrow$ monotone subsequence of size $\Omega(\sqrt{n})$

$\rightarrow$ Q: alternate points in this subsequence

$\rightarrow$ coloring with $O(\sqrt{n})$ colors

Conflict-free colorings



Conflict-free colorings

each column has $n^{\frac{1}{2}}$ points

$\rightarrow$ pick a monotone subsequence of size $\Omega\left(n^{\frac{1}{4}}\right)$

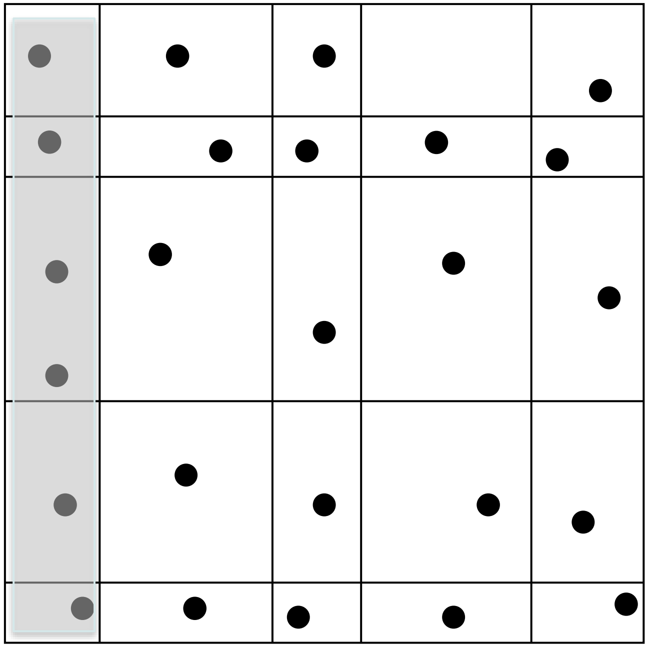
for each row :

$\rightarrow$ monotonic subsequence of points in it

Worst case :

first $n^{\frac{1}{4}}$ rows full of chosen points for all columns

Q has size: $\Omega\left(\sqrt{n^{\frac{1}{2}}} \cdot n^{\frac{1}{4}}\right) = \Omega\left(n^{\frac{1}{2}}\right)$



Insight : *many* monotone subsequences

Conflict-free colorings

each column has $n^{\frac{1}{2}}$ points

$\rightarrow$ partition into $O\left(n^{\frac{1}{4}}\right)$ monotonic subsequences

$\rightarrow$ pick one uniformly at random

expected points in each row : $O\left(\frac{1}{n^{\frac{1}{4}}} \cdot n^{\frac{1}{2}}\right) = O\left(n^{\frac{1}{4}}\right)$

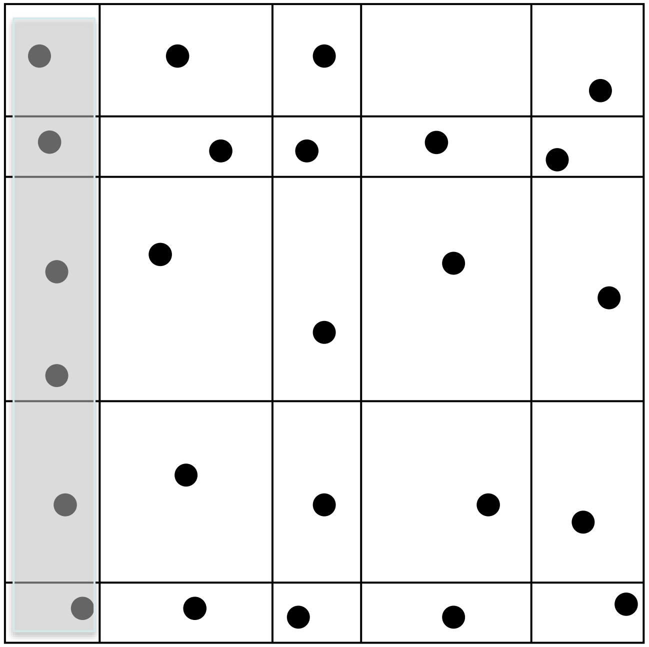
$\rightarrow$ monotonic subsequence has size $O\left(n^{\frac{1}{8}}\right)$

$\rightarrow$ strongly concentrated (Chernoff's bound)

Q has size: $\tilde{O}\left(n^{\frac{1}{2}} \cdot n^{\frac{1}{8}}\right) = \tilde{O}\left(n^{\frac{5}{8}}\right)$

$\rightarrow$ coloring with $\tilde{O}\left(n^{\frac{3}{8}}\right)$ colors

[Elbassioni, M.]



Conflict-free colorings

Grid case :

→ coloring with $O\left(n^{\frac{3}{8}}\right) = O\left(n^{0.375}\right)$ colors

General case with $O(n^{1-\epsilon})$ Steiner points :

→ coloring with $O\left(n^{\frac{3(1+\epsilon)}{8}}\right)$ colors

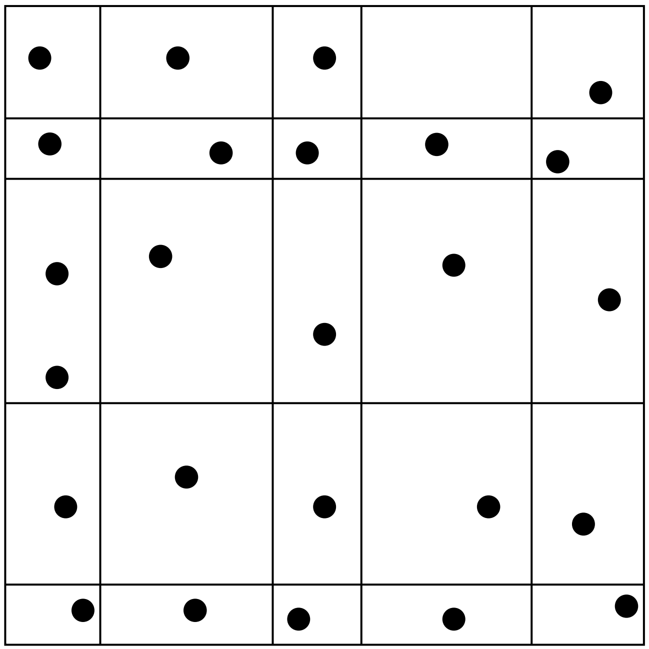
General case :

→ coloring with $O\left(n^{0.382}\right)$ colors

[Ajwani, Elbassioni, Govindarajan, Ray]

→ coloring with $O\left(n^{0.368}\right)$ colors

[Chan]

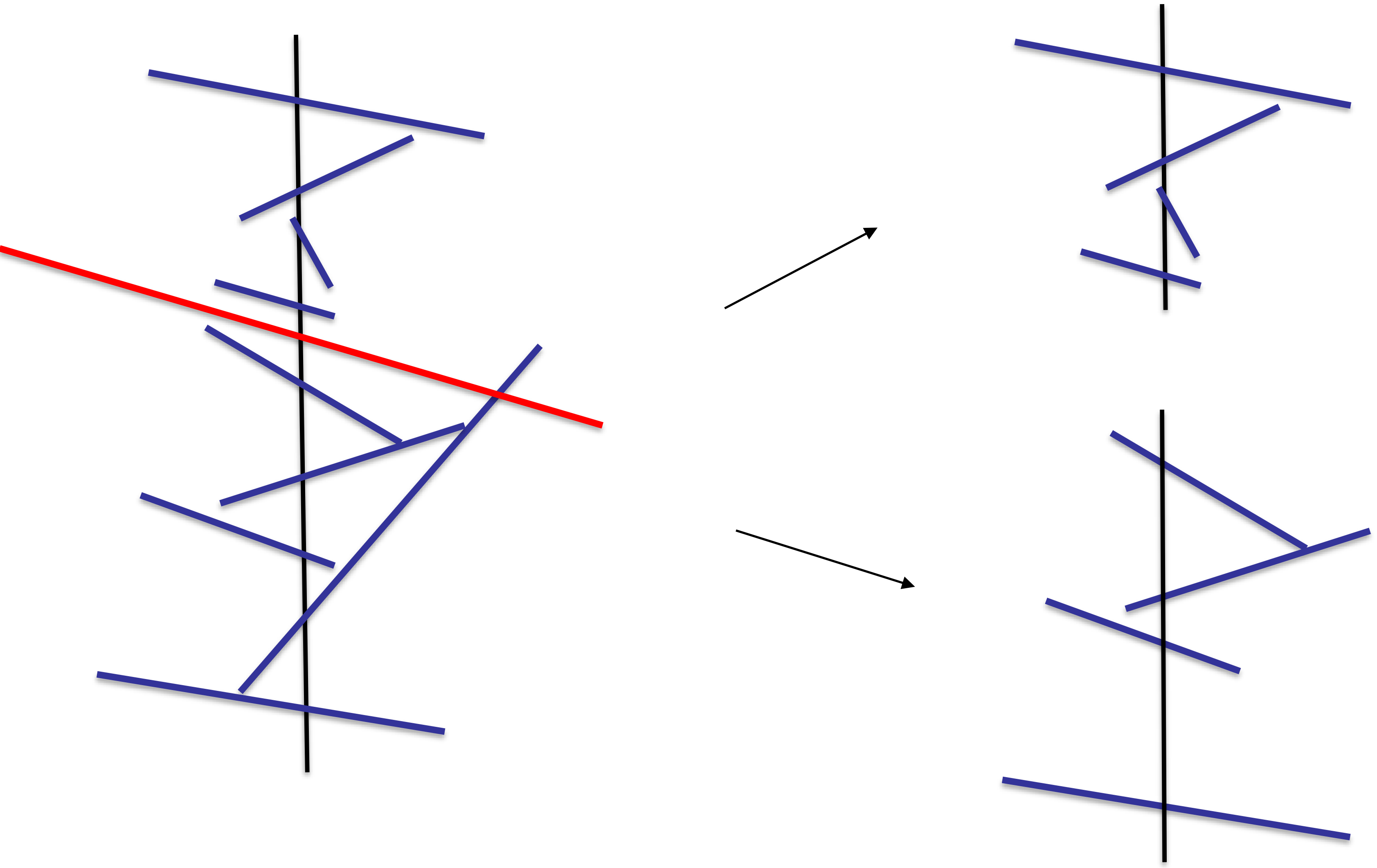


Open Problem 1

Independent Sets

Line segments

Goal: linear separation



Question: how many can be separated ?

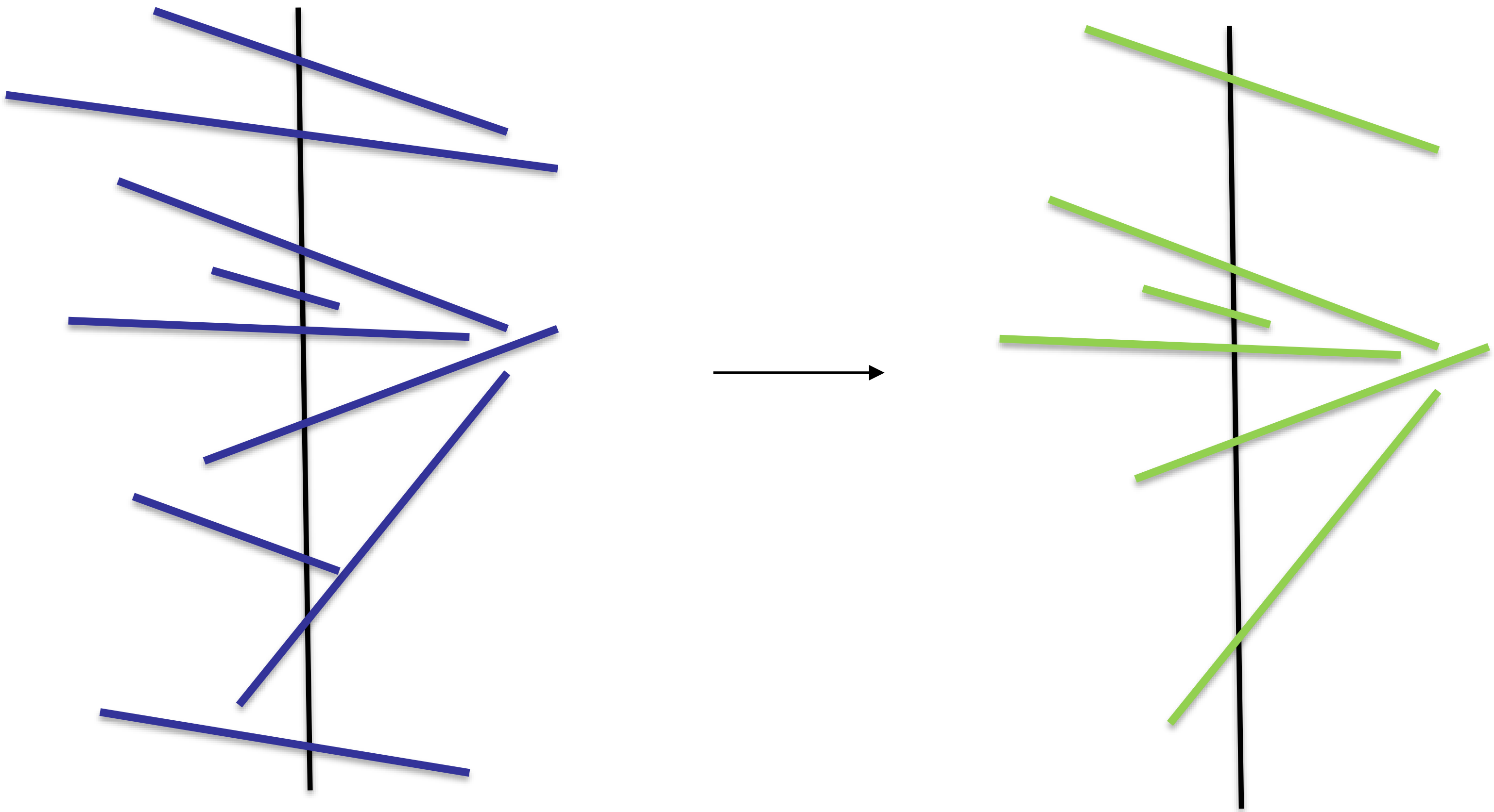
→ approximation for independent set

Line segments

Claim : possible to get $\Omega(\sqrt{n})$ segments separated [Pach, Tardos]

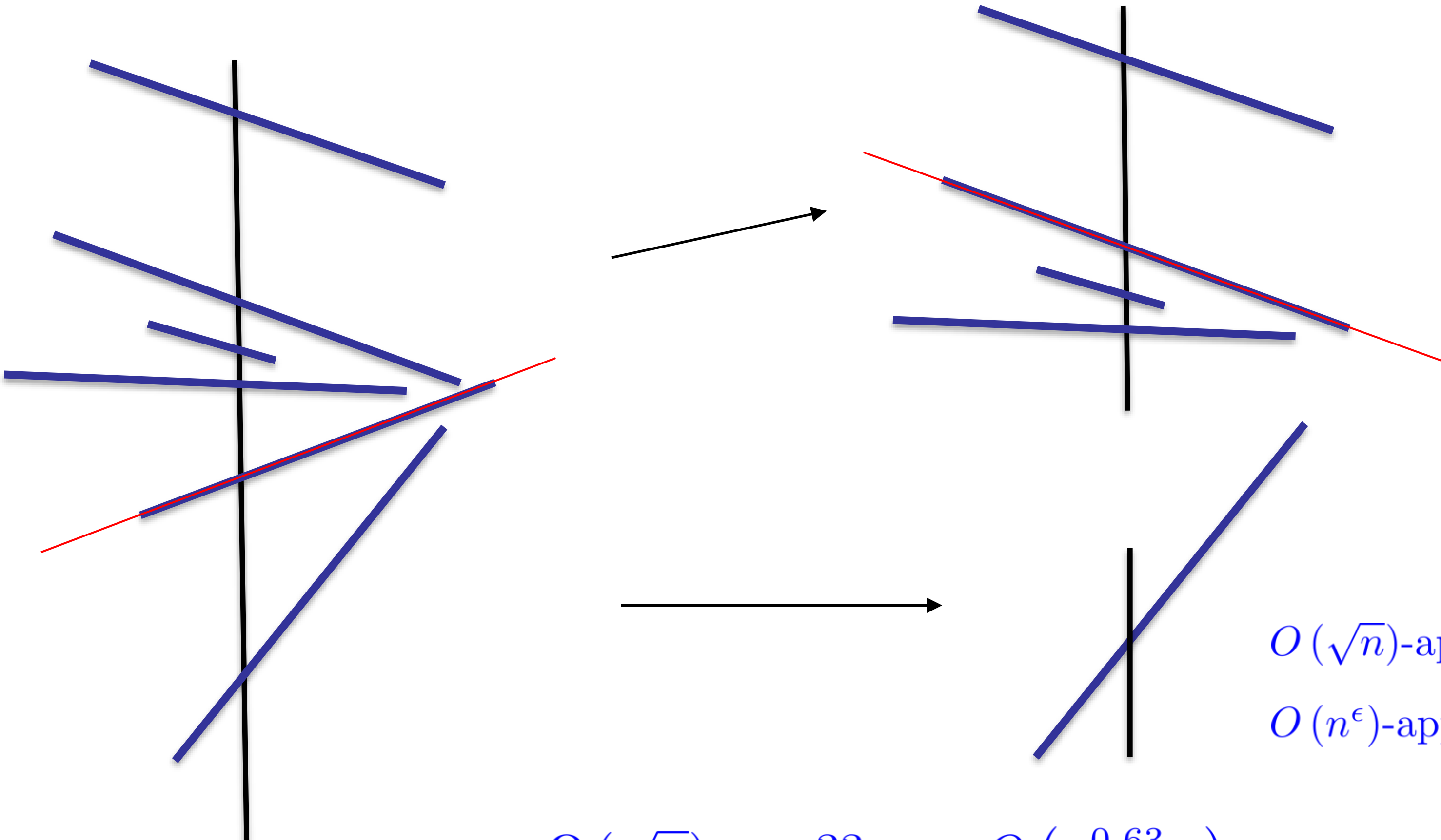
→ sort segments by intersection with line

→ monotonic subsequence by slopes



Line segments

Claim : can separate a monotonic subsequence



$O\left(\sqrt{n}\right)$-approximation

$O\left(n^{\epsilon}\right)$-approximation

[Fox, Pach]

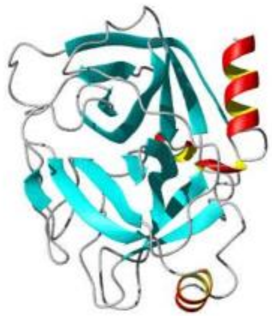
$$\Omega\left(\sqrt{n}\right) \;=\; ?? \;=\; O\left(n^{0.63\ldots}\right)$$

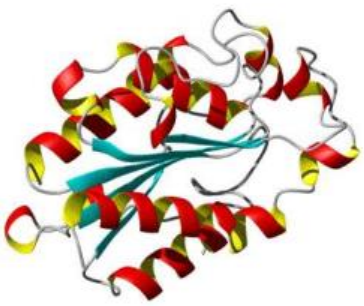
Open Problem 2

Contact-map matching

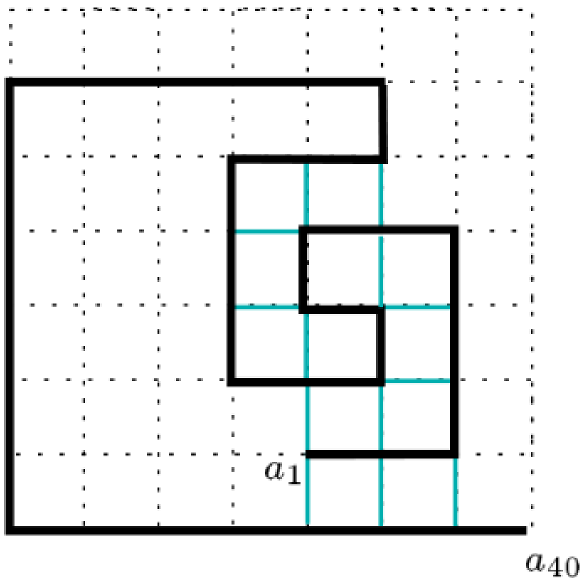
Contact-map similarity

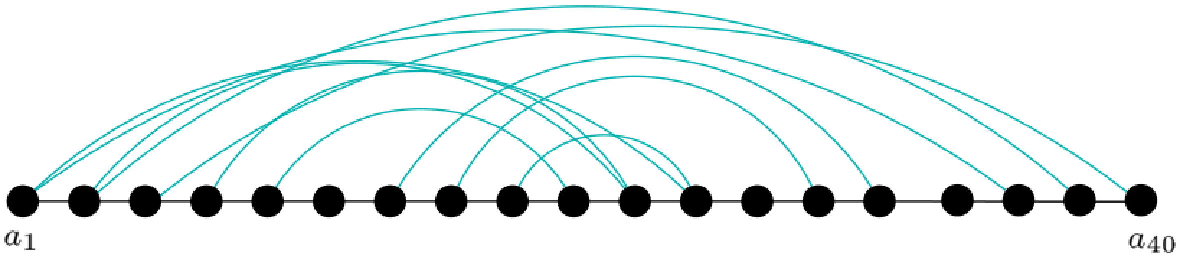
Measuring protein similarity





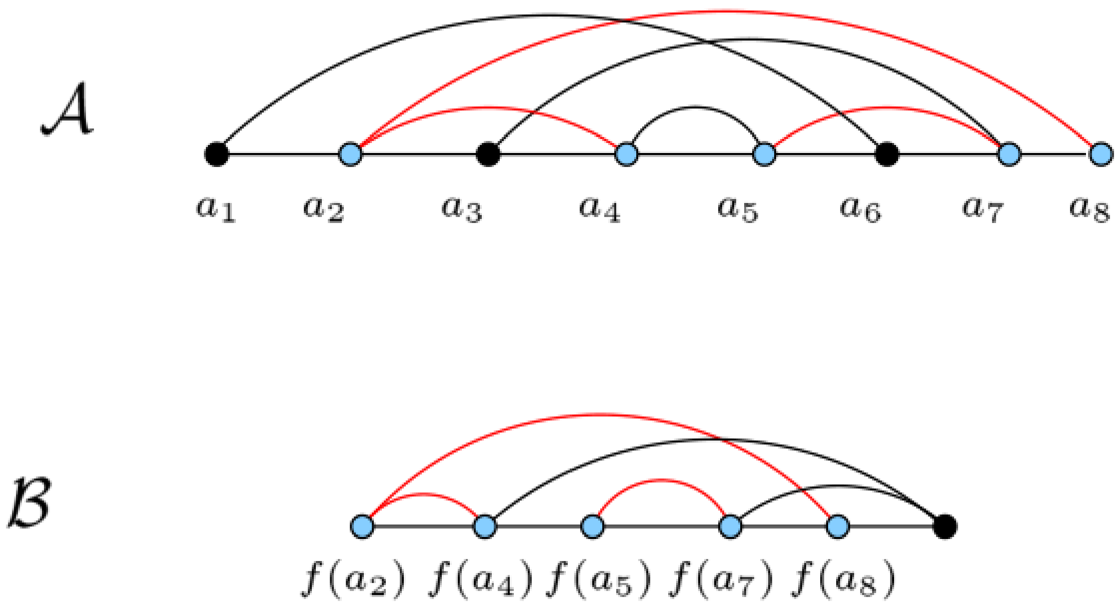
→ contact-maps





Contact-map similarity

→ order-preserving mapping $f(\cdot)$



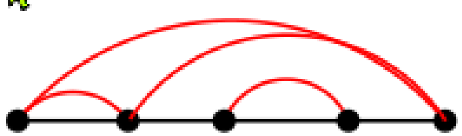
→ NP-hard

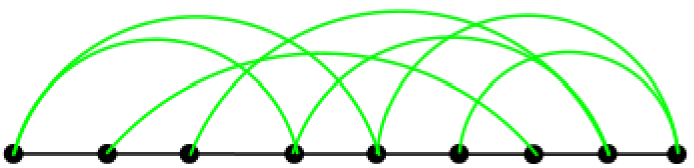
Contact-map similarity

→ In $\mathbb{R}^2$, a nice decomposition is possible

Claim: Contact-map in $\mathbb{R}^2$ decomposed into 2 stacks and 1 queue

[Goldman, Istrail, Papadimitriou]





Claim: Optimal matching of a stack and a contact-map

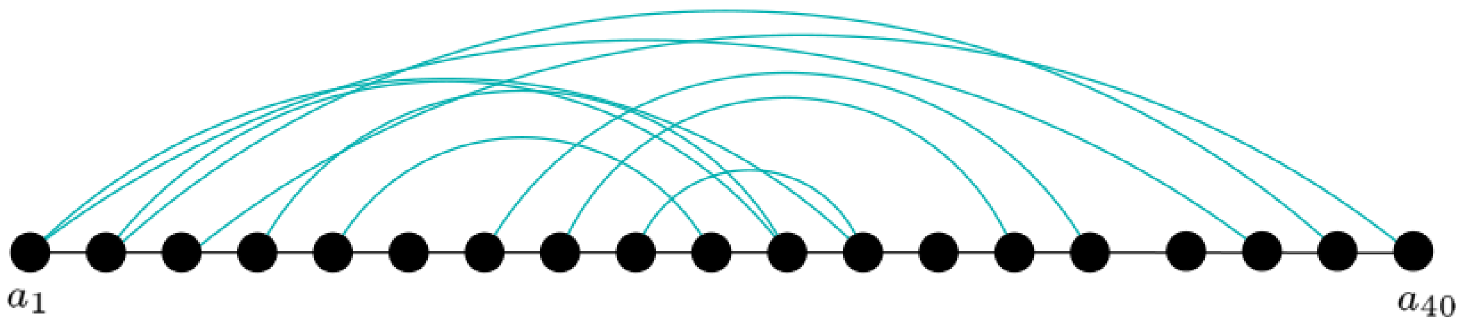
Approximate matching of a queue and a contact-map

→ 3-approximation in $\mathbb{R}^2$

Contact-map similarity

→ In $\mathbb{R}^3$? [Agarwal, M., Wang]

Claim: Contact-map in $\mathbb{R}^3$ decomposed into $O(\sqrt{n})$ stacks and queues



→ increasing subsequence is a **queue**

→ decreasing subsequence is a **stack**

in practice, small number of stacks and queues

Open Problem 3

References

Agarwal, M., Wang: Fast Molecular Shape Matching Using Contact Maps. J. Comput. Biol., 2007.

Guo, Sun: Combinatorics of Contacts in Protein Contact Maps. B. of Math. Bio., 2017.

Agarwal, M.: Independent set of intersection graphs of convex objects in 2D, CGTA 2006.

M., Raman, Ray: Settling the APX-Hardness Status for Geometric Set Cover, FOCS 2014.

Smorodinsky: Conflict-Free Coloring and its Applications, 2013.

Thank you