On the solutions of Knizhnik-Zamolodchikov differential equations by noncommutative Picard-Vessiot theory

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Outline

INTRODUCTION [1]

1. **Abstract :** In this work, basing on the algebraic combinatorics on non commutative formal series with holomorphic coefficients and, on the other hand, a Picard-Vessiot theory of noncommutative differential equations, we give a recursive construction of solutions of Knizhnik-Zamolodchikov equations satisfying asymptotic conditions.

Knizhnik-Zamolodchikov differential equations

Let $(\mathcal{H}(\mathcal{V}), 1_{\mathcal{H}(\mathcal{V})})$ be the ring of holomorphic functions over the manifold $\mathcal{V} = \widetilde{\mathbb{C}_*^n}$, the universal covering of the configuration space of n points, *i.e.*

$$\mathbb{C}_*^n := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ for } i \neq j\}.$$

Let $\mathcal{H}(\mathcal{V})\langle\langle\mathcal{T}_n\rangle\rangle$ be the ring of noncommutative series over the alphabet $\mathcal{T}_n := \{t_{i,j}\}_{1 \leq i < j \leq n}$ and with coefficients in $\mathcal{H}(\mathcal{V})$.

The following noncommutative differential equation is so called KZ_n

$$\mathbf{d}F(z) = \Omega_n(z)F(z), \quad \text{where} \quad \Omega_n(z) := \sum_{1 \leq i < j \leq n} \frac{t_{i,j}}{2\mathrm{i}\pi} d\log(z_i - z_j)$$

for which solutions can be computed by convergent iterations, for the discrete topology[2] of pointwise convergence over $\mathcal{H}(\mathcal{V})\langle\langle\mathcal{T}_n\rangle\rangle$, for instance

$$F_0(z) = 1_{\mathcal{H}(\mathcal{V})} \quad \text{and} \quad F_l(z) = \int_{z_0}^{z} \Omega_n(s) F_{l-1}(s).$$

Remark (dévissage)

$$\Omega_n(z) = \underbrace{\sum_{1 \leq i < j \leq n-1} \frac{t_{i,j}}{2\mathrm{i}\pi} \frac{d(z_j - z_i)}{z_j - z_i}}_{\Omega_{n-1}(z) \longleftrightarrow \mathcal{T}_{n-1}} + \underbrace{\sum_{j=1}^{n-2} \frac{t_{i,n}}{2\mathrm{i}\pi} \frac{d(z_n - z_j)}{z_n - z_j} + \frac{t_{n-1,n}}{2\mathrm{i}\pi} \frac{d(z_n - z_{n-1})}{z_n - z_{n-1}}}_{\textit{for } z_n \to z_{n-1}, \textit{ c.f. hyperlogarithms}}.$$

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2. $\forall S, T \in \mathcal{H}(\mathcal{V})\langle\langle\mathcal{T}_n\rangle\rangle, d(S, T) = 2^{\varpi(S-T)}$, where ϖ denotes the valuation, *i.e.* **If** $S \neq 0$ **then** $\varpi(S) = \inf\{|w|, w \in \mathrm{supp}(S)\}$ **else** $+\infty$.

Quadratic relations among $\{t_{i,j}\}_{1\leq i<j\leq n}$

According to Drinfel'd, KZ_n is completely integrable if $\Omega_n(z)$ is flat, *i.e.*

$$d\Omega_n(z) - \Omega_n(z) \wedge \Omega_n(z) = 0.$$

It turns out that this condition induces the following quadratic relations in $\{t_{i,j}\}_{1\leq i<j\leq n}$:

$$\mathcal{R}_n = \begin{cases} [t_{i,k} + t_{j,k}, t_{i,j}] = 0 & \text{for distinct } i,j,k & \text{and } 1 \leq i < j < k \leq n, \\ [t_{i,j} + t_{i,k}, t_{j,k}] = 0 & \text{for distinct } i,j,k & \text{and } 1 \leq i < j < k \leq n, \\ [t_{i,j}, t_{k,l}] = 0 & \text{for distinct } i,j,k,l & \text{and } \begin{cases} 1 \leq i < j \leq n, \\ 1 \leq k < l \leq n, \end{cases} \end{cases}$$

generating the Lie ideal $\mathcal{J}_{\mathcal{R}_n}$.

Solutions of KZ_n belong now to $\mathcal{H}(\mathcal{V})\langle\langle\mathcal{T}_n\rangle\rangle/\mathcal{J}_{\mathcal{R}_n}$.

Examples of KZ_n

Example (KZ_2 : trivial case)

One has $\mathcal{T}_2 = \{t_{1,2}\}$ and $\mathbf{d}F(z) = \Omega_2(z)F(z)$, where

$$\Omega_2(z) = (t_{1,2}/2\mathrm{i}\pi)d\log(z_1 - z_2),$$

is $F(z_1, z_2) = e^{(t_{1,2}/2\mathrm{i}\pi)\log(z_1-z_2)} = (z_1 - z_2)^{t_{1,2}/2\mathrm{i}\pi} \in \mathcal{H}(\widetilde{\mathbb{C}_*^2})\langle\langle\mathcal{T}_2\rangle\rangle$.

Example (KZ_3 : simplest non-trivial case)

One has $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$ and $\mathbf{d}F(z) = \Omega_3(z)F(z)$, where

$$\Omega_3(z) = \frac{1}{2\mathrm{i}\pi}\left(t_{1,2}\frac{d(z_1 - z_2)}{z_1 - z_2} + t_{1,3}\frac{d(z_1 - z_3)}{z_1 - z_3} + t_{2,3}\frac{d(z_2 - z_3)}{z_2 - z_3}\right).$$

Drinfel'd proposed a following solution on $]0, 1[$

$$F(z) = (z_1 - z_2)^{(t_{1,2}+t_{1,3}+t_{2,3})/2\mathrm{i}\pi} G\left(\frac{z_3 - z_2}{z_1 - z_2}\right),$$

where G satisfies the following noncommuative differential equation

$$(DE1) \quad dG(s) = \left(A\frac{ds}{s} - B\frac{ds}{1-s}\right)G(s), \quad \begin{cases} A := t_{1,2}/2\mathrm{i}\pi, \\ B := t_{2,3}/2\mathrm{i}\pi. \end{cases}$$

He stated that there is a unique solution G_0 (resp. G_1) satisfying

$$G_0(s) \sim_0 e^{A\log(s)} = s^A \quad (\text{resp. } G_1(s) \sim_1 e^{-B\log(1-s)} = (1-s)^{-B}),$$

and a unique series Φ_{KZ}, so-called Drinfel'd series [3], s.t. $G_0 = G_1\Phi_{KZ}$.

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3. Cartier, Gonzalez-Lorca, Racinet defined associators as group like series satisfying the relations duality, pentagonal and hexagonal : Φ_{KZ} is an associator.

$\log \Phi_{KZ}$ determined by Drinfel'd

1. Assuming that $[A, B] = 0$, he proposed an approximation solution for $(DE1)$ over $]0, 1[$, $z^A(1-z)^B$ (a group like series) satisfying standard asymptotic conditions. Hence, the logarithm of such approximation solution of KZ_3 belongs to
$$\mathcal{L}ie_{\mathcal{H}(\widetilde{\mathbb{C}^3_*})}\langle\langle t_{1,2}, t_{1,3}, t_{2,3}\rangle\rangle/[\mathcal{L}ie_{\mathcal{H}(\widetilde{\mathbb{C}^3_*})}\langle\langle t_{1,2}, t_{2,3}\rangle\rangle, \mathcal{L}ie_{\mathcal{H}(\widetilde{\mathbb{C}^3_*})}\langle\langle t_{1,2}, t_{2,3}\rangle\rangle].$$

2. He also proposed, over $]0, 1[$,
$$G_0(z) = z^A(1-z)^B V_0(z) \quad \text{and} \quad G_1(z) = z^A(1-z)^B V_1(z).$$
V_0 and V_1 have continuous extensions to $]0, 1[$ and are group like solutions of the following noncommutative differential equation
$$(DE2) \quad \mathbf{d}S(z) = Q(z)S(z), \quad Q(z) := e^{\mathrm{ad}_{-\log(1-z)B}} e^{\mathrm{ad}_{-\log(z)A}} \frac{B}{z-1} \in \mathfrak{p},$$
with the initial conditions $V_0(0) = 1, V_1(1) = 1$ and $\mathfrak{p}$ is the topological free Lie algebra generated by $\{\mathrm{ad}_A^k \mathrm{ad}_B^l [A, B]\}_{k,l\geq 0}$.

3. Since $G_9 = G_1\Phi_{KZ}$ then the group like series Φ_{KZ} equals to $V(0)V(1)^{-1}$, where V is a solution of $(DE2)$ and then the coefficients $\{c_{k,l}\}_{k,l\geq 0}$ of $\log \Phi_{KZ}$ are obtained, in $\mathfrak{p}/[\mathfrak{p}, \mathfrak{p}]$, by
$$\log \Phi_{KZ} = \sum_{k,l\geq 0} c_{k,l} B^{k+1}A^{l+1} = \int_0^1 Q(z)dz \mod [\mathfrak{p}, \mathfrak{p}].$$

Polylogarithms

Denoting $(X^*, 1_{X^*})$ the monoid generated by $X = \{x_0, x_1\}$, recall that

$$\mathrm{L}(s) := \sum_{w \in X^*} \mathrm{Li}_w(s) w \in \mathcal{H}(\tilde{B})\langle\langle X \rangle\rangle, \quad \text{where} \quad B := \mathbb{C} \setminus \{0, 1\}$$

where $\mathrm{Li}_\bullet$ is the character of $(\mathcal{H}(\tilde{B})\langle X \rangle, ш, 1_{X^*})$ defined by

$$\mathrm{Li}_{1_{X^*}} = 1_{\mathcal{H}(\tilde{B})}, \quad \mathrm{Li}_{x_0}(s) = \log(s), \quad \mathrm{Li}_{x_1}(s) = \log(1-s)$$

and, for any $x_i w \in \mathcal{L}ynX \setminus X$,

$$\mathrm{Li}_{x_i w}(s) = \int_0^s \omega_i(\sigma) \mathrm{Li}_w(\sigma), \quad \text{where} \quad \begin{cases} \omega_0(s) = ds/s, \\ \omega_1(s) = ds/(1-s). \end{cases}$$

$\{\mathrm{Li}_l\}_{l \in \mathcal{L}ynX}$ (resp. $\{\mathrm{Li}_w\}_{w \in X^*}$) are $\mathbb{C}$-algebraically (resp. linearly) free.

By the Friedrichs crirerion, L is group like. Thus [4],

$$\mathrm{L}(s) = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\mathrm{Li}_{S_l}(s) P_l} \quad \text{and then} \quad \begin{cases} \lim\limits_{z \to 0} \mathrm{L}(s) e^{-x_0 \log z} & = & 1, \\ \lim\limits_{z \to 1} e^{x_1 \log(1-z)} \mathrm{L}(s) & = & \Phi_{KZ}, \end{cases}$$

and Φ_{KZ} admits $\{\mathrm{Li}_l(1)\}_{l \in \mathcal{L}ynX \setminus X}$ as convergent locale coordinates

$$\Phi_{KZ} := \prod_{l \in \mathcal{L}ynX \setminus X}^{\searrow} e^{\mathrm{Li}_{S_l}(1) P_l} \in \mathbb{R}\langle\langle X \rangle\rangle, \quad \text{for} \quad \begin{cases} x_0 = & t_{1,2}/2\mathrm{i}\pi, \\ x_1 = & -t_{2,3}/2\mathrm{i}\pi. \end{cases}$$

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4. $\{P_l\}_{l \in \mathcal{L}yn\mathcal{T}_n}$ is the basis of $\mathcal{L}ie_{\mathcal{H}(\tilde{B})}\langle X \rangle$ over which are constructed the PBW basis $\{P_w\}_{w \in \mathcal{T}_n^*}$ of $\mathcal{U}(\mathcal{L}ie_{\mathcal{H}(\tilde{B})}\langle X \rangle)$ and its dual, $\{S_w\}_{w \in X^*}$, containing the pure transcendence basis $\{S_l\}_{l \in \mathcal{L}ynX}$

BACKGROUND ON
PV THEORY OF NONCOMMUTATIVE
DIFFERENTIAL EQUATIONS

Differential ring of holomorphic functions

- $\mathcal{V}$: simply connected manifold of $\mathbb{C}^n$ $(n > 0)$.
- $\mathcal{A} = (\mathcal{H}(\mathcal{V}), \partial_1, \ldots, \partial_n)$: the differential ring of holomorphic functions on $\mathcal{V}$ and equipped $1_{\mathcal{H}(\mathcal{V})}$ as the neutral element.

  For any $f \in \mathcal{H}(\mathcal{V})$, one has $df = (\partial_1 f)dz_1 + \ldots + (\partial_n f)dz_n$.
- Let $\mathcal{C}$ be a sub differential ring of $\mathcal{A}$ (*i.e.* $\partial_i \mathcal{C} \subset \mathcal{C}$, for $1 \leq i \leq n$) and let $\varsigma \rightsquigarrow z$ denotes a path (with fixed endpoints, (ς, z)) over $\mathcal{V}$, *i.e.* the parametrized curve $\gamma : [0,1] \longrightarrow \mathcal{V}$ such that

  $$\gamma(0) = \varsigma = (\varsigma_1, \ldots, \varsigma_n) \quad \text{and} \quad \gamma(1) = z = (z_1, \ldots, z_n).$$
- For any integers i, j such that $1 \leq i < j \leq n$, let $\omega_{i,j}$ denote the 1-differential forms [5], in $\Omega^1(\mathcal{V})$, $\omega_{i,j} = d\xi_{i,j}$, with $\xi_{i,j} \in \mathcal{C}$.

Example $(\xi_{i,j}(z) = \log(z_i - z_j), 1 \leq i < j \leq n)$

Let $\mathcal{C}_0 := \mathbb{C}[\{(\partial_1 \xi_{i,j})^{\pm 1}, \ldots, (\partial_n \xi_{i,j})^{\pm 1}\}_{1 \leq i < j \leq n}]$.

Then $\mathcal{C}_0$ is a sub differential ring of $\mathcal{A}$.

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5. Over $\mathcal{V}$, the holomorphic function $\xi_{i,j}$ is called a primitive for $\omega_{i,j}$ which is said to be a exact form and then is a closed form (*i.e.* $d\omega_{i,j} = 0$).

Notations

- $(\mathcal{T}_n{}^*, 1_{\mathcal{T}_n{}^*})$ is the free monoid generated by $\mathcal{T}_n$.
- $\mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle$ (resp. $\mathcal{A}\langle\mathcal{T}_n\rangle$) is the set of series (resp. polynomials) over $\mathcal{T}_n$ with coefficients in $\mathcal{A}$. $\mathcal{L}yn\mathcal{T}_n$ (resp. $\mathcal{L}yn\mathcal{T}$) is the set of Lyndon words over $\mathcal{T}_n$ (resp. $\mathcal{T}$).
- $T_k := \{t_{j,k}\}_{1\le j\le k-1}, \mathcal{T} := \{T_2, \ldots, T_n\}$ s.t. $\mathcal{T}_k = T_k \sqcup \mathcal{T}_{k-1}, k \le n$. $|\mathcal{T}_n| = n(n-1)/2$ and $|T_n| = n-1$. If $n \ge 4$ then $|\mathcal{T}_{n-1}| \ge |T_n|$.

Example

- $\mathcal{T}_5 = \{t_{1,2}, t_{1,3}, t_{1,4}, t_{1,5}, t_{2,3}, t_{2,4}, t_{2,5}, t_{3,4}, t_{3,5}, t_{4,4}\}$, one has $T_5 = \{t_{1,5}, t_{2,5}, t_{3,5}, t_{4,5}\}$ and $\mathcal{T}_4$.
- $\mathcal{T}_4 = \{t_{1,2}, t_{1,3}, t_{1,4}, t_{2,3}, t_{2,4}, t_{3,4}\}$, one has $T_4 = \{t_{1,4}, t_{2,4}, t_{3,4}\}$ and $\mathcal{T}_3$.
- $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$, one has $T_3 = \{t_{1,3}, t_{2,3}\}$ and $\mathcal{T}_2 = \{t_{1,2}\}$.

- In $(\mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle, \partial_1, \ldots, \partial_n)$, for any $S \in \mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle$, one defines
$$\partial_i S = \sum_{w\in\mathcal{T}_n^*} (\partial_i\langle S|w\rangle)w \quad \text{and} \quad \mathbf{d}S = \sum_{i=1}^{n} (\partial_i S) dz_i.$$
$\mathrm{Const}(\mathcal{A}) = \mathbb{C}.1_{\mathcal{H}(\Omega)}$ and $\mathrm{Const}(\mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle) = \mathbb{C}\langle\langle\mathcal{T}_n\rangle\rangle$.

Lazard elimination : $\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle = \mathcal{I}_n \oplus \mathcal{L}ie_{\mathcal{A}}\langle T_n\rangle$

Let ρ the right normed bracketing which is the unique linear endomorphism of $\mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle$ defined, by $\rho(1_{\mathcal{T}_n^*}) = 0$ and, for $w = t_1 \ldots t_k \in \mathcal{T}_n^*$, by

$$\rho(w) = [t_1, [\ldots, [t_{k-1}, t_k] \ldots] = \mathrm{ad}_{t_1} \ldots \mathrm{ad}_{t_{k-1}} t_k.$$

$\mathcal{I}_n$: Lie subalg. generated by $\{\mathrm{ad}^k_{-T_n} t_{i,j}\}^{k\geq 0}_{t_{i,j}\in\mathcal{T}_{n-1}} = \{(-1)^{|v|}\rho(vt)/\,|v|!\}_{\substack{v\in T_n^*\\ t\in\mathcal{T}_{n-1}}}$.

By PBW, $\mathcal{U}(\mathcal{I}_n)$ is freely generated by

$$\{\mathrm{ad}^{k_1}_{-T_n} t_1 \ldots \mathrm{ad}^{k_p}_{-T_n} t_p\}^{k_1,\ldots,k_p\geq 0, p\geq 0}_{t_1,\ldots,t_p\in\mathcal{T}_{n-1}}$$
$$= \{\rho((-T_n)^* t_1)\cdots\rho((-T_n)^* t_k)\}^{k\geq 0}_{t_1,\ldots,t_k\in\mathcal{T}_{n-1}}$$
$$= \{(-1)^{|v_1\ldots v_k|}\,|v_1|!^{-1}\ldots|v_k|!^{-1}\rho(v_1 t_1)\cdots\rho(v_k t_k)\}^{k\geq 0}_{v_1,\ldots,v_k\in T_n^*, t_1,\ldots,t_k\in\mathcal{T}_{n-1}}$$

which are associated to the following family of polynomials of $\mathcal{U}(\mathcal{I}_n)^\vee$

$$\{t_1(\bar{T}_n^{k_1} ⧢ (\cdots ⧢ (t_p \bar{T}_n^{k_p})\ldots))\}^{k_1,\ldots,k_p\geq 0, p\geq 0}_{t_1,\ldots,t_p\in\mathcal{T}_{n-1}},$$
$$= \{t_1(\bar{v}_1 ⧢ (\cdots ⧢ (t_p \bar{v}_p)\ldots))\}^{k_1,\ldots,k_p\geq 0, p\geq 0}_{v_1\in T_n^{k_1},\ldots,v_p\in T_n^{k_p}, t_1,\ldots,t_k\in\mathcal{T}_{n-1}}$$
$$= \{(t_1\bar{v}_1)\circ\cdots\circ(t_p\bar{v}_p)\}^{k_1,\ldots,k_p\geq 0, p\geq 0}_{v_1\in T_n^{k_1},\ldots,v_p\in T_n^{k_p}, t_1,\ldots,t_k\in\mathcal{T}_{n-1}},$$
$$= \{(t_1\bar{T}_n^{k_1})\circ\cdots\circ(t_p\bar{T}_n^{k_p})\}^{k_1,\ldots,k_p\geq 0, p\geq 0}_{t_1,\ldots,t_p\in\mathcal{T}_{n-1}},$$

where [6] $\bar{T}_n^k = \{\bar{v} \in T_n^k, |v| = k\}$ and the composite operator $\circ$ is defined, for any H and $R \in \mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle$ and $t \in \mathcal{T}_{n-1}$, by

If $R \neq 1_{\mathcal{T}_n^*}$ **then** $(tH)\circ R = t(H ⧢ R)$ **else** $(tH)\circ R = tH$.

6. $\bar{v}$ is the polynomial $t_1 ⧢ \ldots ⧢ t_k$ associated to $v = t_1 \ldots t_k$.

Lexicographic ordering

$\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle$ is the set of Lie polynomials over $\mathcal{T}_n$ with coefficients in $\mathcal{A}$ and is equipped with the basis $\{P_l\}_{l\in\mathcal{L}yn\mathcal{T}_n}$ over which are constructed the PBW basis $\{P_w\}_{w\in\mathcal{T}_n^*}$ of $\mathcal{U}(\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle)$ and its dual, $\{S_w\}_{w\in\mathcal{T}_n^*}$, containing the pure transcendence basis $\{S_l\}_{l\in\mathcal{L}yn\mathcal{T}_n}$ of[7] $(\mathcal{A}\langle\mathcal{T}_n\rangle, ш, 1_{\mathcal{T}_n^*})$.

Example (in KZ_3, $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$ and $t_{1,2} \prec t_{1,3} \prec t_{2,3}$)

$\forall k \geq 0, i = 1$ or 2, $\quad t_{1,2}^k t_{i,3} \in \mathcal{L}yn\mathcal{T}_3, \quad P_{t_{1,2}^k t_{i,3}} = \mathrm{ad}_{t_{1,2}}^k t_{i,3}, S_{t_{1,2}^k t_{i,3}} = t_{1,2}^k t_{i,3}.$

In the sequel, let $\mathcal{L}yn\mathcal{T}_n$ (resp. T_k) be the set of Lyndon words over $\mathcal{T}_n$ (resp. T_k) equipped the following total order over T_k $(n \geq k \geq 2)$:

$$t_{1,k} \succ \ldots \succ t_{k-1,k}, \quad T_2 \succ \ldots \succ T_n, \quad \mathcal{L}yn T_2 \succ \ldots \succ \mathcal{L}yn T_n.$$

By the standard factorization[8] of Lyndon words, one has

$$\mathcal{L}yn\mathcal{T}_{n-1} \succ \mathcal{L}yn T_n.\mathcal{L}yn\mathcal{T}_{n-1} \succ \mathcal{L}yn T_n,$$

More generally, for any $(t_1, t_2) \in T_{k_1} \times T_{k_2}, 2 \leq k_1 < k_2 \leq n$, one also has

$$t_2 t_1 \in \mathcal{L}yn\mathcal{T}_{k_2} \subset \mathcal{L}yn\mathcal{T}_n \quad \text{and} \quad t_2 \prec t_2 t_1 \prec t_1.$$

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7. in which one defines $\Delta_{ш} x = x \otimes 1_{\mathcal{T}_n^*} + 1_{\mathcal{T}_n^*} \otimes x$, or equivalently, $u \,ш\, 1_{\mathcal{T}_n^*} = 1_{\mathcal{T}_n^*} \,ш\, u = u$ and $xu \,ш\, yv = x(u \,ш\, yv) + y(xu \,ш\, v)$.

8. *i.e.* $st(l) = (l_1, l_2)$, where l_2 is the longest nontrivial proper right factor of a Lyndon word l, or equivalently, its smallest such for the lexicographic ordering.

Diagonal series (for KZ_n, $n \geq 4$)

1. If $l \in \mathcal{L}ynT_{k-1}$ and $t \in T_k, 2 \leq k \leq n$ then $tl \in \mathcal{L}yn\mathcal{T}_n$ and $t \prec tl \prec l$.
2. If $l_1 \in \mathcal{L}ynT_{k_1}$ and $l_2 \in \mathcal{L}ynT_{k_2}$ (for $2 \leq k_1 < k_2 \leq n$) then $l_2 l_1 \in \mathcal{L}yn\mathcal{T}_{k_2} \subset \mathcal{L}yn\mathcal{T}_n$ and $l_2 \prec l_2 l_1 \prec l_1$.
3. If $l_1 \in \mathcal{L}ynT_k$ and $l_2 \in \mathcal{L}yn\mathcal{T}_{k-1}$ (for $2 \leq k_1 < k_2 \leq n$) then $l_1 l_2 \in \mathcal{L}yn\mathcal{T}_n$ and $l_1 \prec l_1 l_2 \prec l_2$.

In $\mathcal{A}\langle\mathcal{T}_n\rangle\hat{\otimes}\mathcal{A}\langle\mathcal{T}_n\rangle$, let $\nabla S = S - 1_{\mathcal{T}_n^*} \otimes 1_{\mathcal{T}_n^*}$. The diagonal series is defined by

$$\mathcal{D}_{\mathcal{T}_n} := \mathcal{M}^*, \quad \text{with} \quad \mathcal{M} := \sum_{t \in \mathcal{T}_n} t \otimes t,$$

and is the unique solution of $\nabla S = \mathcal{M}S$ and $\nabla S = S\mathcal{M}$. Then

$$\mathcal{D}_{\mathcal{T}_n} = \mathcal{D}_{\mathcal{T}_{n-1}} \Big(\prod^{\searrow}_{\substack{l = l_1 l_2 \\ l_2 \in \mathcal{L}yn\mathcal{T}_{n-1}, l_1 \in \mathcal{L}ynT_n}} e^{S_l \otimes P_l} \Big) \mathcal{D}_{T_n}, \quad \text{for} \quad n > 2.$$

where $\mathcal{D}_{\mathcal{T}_{n-1}}$ (resp. $\mathcal{D}_{T_n}$) denote the diagonal series, over $\mathcal{T}_{n-1}$ (resp. T_n), and

$$\mathcal{D}_{\mathcal{T}_{n-1}} = \prod^{\searrow}_{l \in \mathcal{L}yn\mathcal{T}_{n-1}} e^{S_l \otimes P_l}, \quad \text{and} \quad \mathcal{D}_{T_n} = \prod^{\searrow}_{l \in \mathcal{L}ynT_n} e^{S_l \otimes P_l}.$$

More about notations

Let us back to the relations

$$\mathcal{R}_n = \begin{cases} [t_{i,k}+t_{j,k}, t_{i,j}] = 0 & \text{for distinct } i,j,k \quad \text{and } 1 \le i < j < k \le n, \\ [t_{i,j}+t_{i,k}, t_{j,k}] = 0 & \text{for distinct } i,j,k \quad \text{and } 1 \le i < j < k \le n, \\ [t_{i,j}, t_{k,l}] = 0 & \text{for distinct } i,j,k,l \quad \text{and } \begin{cases} 1 \le i < j \le n, \\ 1 \le k < l \le n, \end{cases} \end{cases}$$

generating the Lie ideal $\mathcal{J}_{\mathcal{R}_n}$.

- The monoid (resp. the set of Lyndon words) generated by $\mathcal{T}_n$ satisfying the relations $\mathcal{R}_n$ is denoted by $\langle \mathcal{T}_n^*; \mathcal{J}_{\mathcal{R}_n} \rangle$ (resp. $\langle \mathcal{L}yn T_n; \mathcal{J}_{\mathcal{R}_n} \rangle$).
- The set of noncommutative polynomials (resp. series) with coefficients in $\mathcal{A}$, over $\mathcal{T}_n$, satisfying $\mathcal{R}_n$, is denoted by $\mathcal{A}\langle \mathcal{T}_n \rangle / \mathcal{J}_{\mathcal{R}_n}$ (resp. $\mathcal{A}\langle\langle \mathcal{T}_n \rangle\rangle / \mathcal{J}_{\mathcal{R}_n}$).
- The set of Lie polynomials (resp. Lie series) with coefficients in $\mathcal{A}$, over $\mathcal{T}_n$, satisfying $\mathcal{R}_n$, is denoted by $\mathcal{L}ie_{\mathcal{A}}\langle \mathcal{T}_n \rangle / \mathcal{J}_{\mathcal{R}_n}$ (resp. $\mathcal{L}ie_{\mathcal{A}}\langle\langle \mathcal{T}_n \rangle\rangle / \mathcal{J}_{\mathcal{R}_n}$).
- $H_{\sqcup\!\sqcup}(\mathcal{T}_n)/\mathcal{J}_{\mathcal{R}_n}$ denotes $(\mathcal{A}\langle \mathcal{T}_n \rangle / \mathcal{J}_{\mathcal{R}_n}, \mathtt{conc}, \Delta_{\sqcup\!\sqcup}, 1_{\mathcal{T}_n^*})$.

Iterated integrals and Chen series

The iterated integral associated, of the 1-differential forms $\{\omega_{i,j}\}_{1\leq i<j\leq n}$ and along the path $\varsigma \rightsquigarrow z$, is given by $\alpha_{\varsigma}^{z}(1_{\mathcal{T}_n^*}) = 1_{\mathcal{H}(\mathcal{V})}$ and, for any $w = t_{i_1,j_1}t_{i_2,j_2}\ldots t_{i_k,j_k} \in \mathcal{T}_n^*$,

$$\alpha_{\varsigma}^{z}(w) := \int_{\varsigma}^{z} \omega_{i_1,j_1}(s_1) \int_{\varsigma}^{s_1} \omega_{i_2,j_2}(s_2) \ldots \int_{\varsigma}^{s_{k-1}} \omega_{i_k,j_k}(s_k) \in \mathcal{H}(\mathcal{V}),$$

where $(\varsigma, s_1 \ldots, s_{k-1}, z)$ is a subdivision of $\varsigma \rightsquigarrow z$.

The Chen series, of the differential forms $\{\omega_{i,j}\}_{1\leq i<j\leq n}$ and along a path $\varsigma \rightsquigarrow z$, is the following noncommutative generating series

$$C_{\varsigma\rightsquigarrow z} := \sum_{w\in\mathcal{T}_n^*} \alpha_{\varsigma}^{z}(w)w \in \mathcal{H}(\mathcal{V})\langle\langle\mathcal{T}_n^*\rangle\rangle.$$

Proposition

1. *$\forall u, v$ in $\mathcal{T}_n^*, \alpha_{\varsigma}^{z}(u \sqcup\!\sqcup v) = \alpha_{\varsigma}^{z}(u)\alpha_{\varsigma}^{z}(v)$ (Chen's lemma).*
2. *$\forall t \in \mathcal{T}_n, k \geq 0, \alpha_{\varsigma}^{z}(t^k) = (\alpha_{\varsigma}^{z}(t))^k/k!$ and then $\alpha_{\varsigma}^{z}(t^*) = e^{\alpha_{\varsigma}^{z}(t)}$.*
3. *For any compact $K \subset \mathcal{V}$, there is $c > 0$ and a morphism of monoids $\mu : \mathcal{T}_n^* \longrightarrow \mathbb{R}_{\geq 0}$ s.t. $\|\langle C_{\varsigma\rightsquigarrow z}|w\rangle\|_K \leq c\mu(w)\,|w|!^{-1}$, for $w \in \mathcal{T}_n^*$, and then $C_{\varsigma\rightsquigarrow z}$ is said to be exponentially bounded from above.*

Basic triangular theorem over a differential ring

Let $\mathcal{C}$ be a sub differential ring of $\mathcal{A}$.
For any $S \in \mathcal{C}\langle\langle \mathcal{T}_n \rangle\rangle$, let $\mathcal{F}(S) := \mathrm{span}_{\mathcal{C}}\{\langle S|w\rangle\}_{w\in\mathcal{T}_n^*}$

Lemma

The following assertions are equivalent[9]

1. *The following map is injective*

$$(\mathcal{C}\langle \mathcal{T}_n \rangle, ш, 1_{\mathcal{T}_n^*}) \longrightarrow (\mathcal{H}(\mathcal{V}), *, 1_{\mathcal{H}(\mathcal{V})}), \qquad w \longmapsto \alpha_\varsigma^z(w).$$

2. $\{\alpha_\varsigma^z(w)\}_{w\in\mathcal{T}_n^*}$ *is linearly free over* $\mathcal{C}$.
3. $\{\alpha_\varsigma^z(l)\}_{l\in\mathcal{L}yn\mathcal{T}_n}$ *is algebraically free over* $\mathcal{C}$.
4. $\{\alpha_\varsigma^z(t)\}_{t\in\mathcal{T}_n}$ *is algebraically free over* $\mathcal{C}$.
5. $\{\alpha_\varsigma^z(t)\}_{t\in\mathcal{T}_n\cup\{1_{\mathcal{T}_n^*}\}}$ *is linearly free over* $\mathcal{C}$.
6. *For any* $C \in \mathcal{L}ie_{\mathcal{C}}\langle\langle \mathcal{T}_n \rangle\rangle$, *there is an automorphism* ψ *of* $\mathcal{F}(C_{\varsigma\rightsquigarrow z})$ *such that* $\psi(C_{\varsigma\rightsquigarrow z}) = C_{\varsigma\rightsquigarrow z}e^{C}$.

9. This is the abstract form, over ring, of (Deneufchâtel, Duchamp, HNM & Solomon, 2011).

Noncommutative differential equations

$$(NCDE) \quad \mathbf{d}S = M_n S, \quad \text{where}^{10} \quad M_n = \sum_{1\le i<j\le n} \omega_{i,j} t_{i,j}.$$

Proposition

1. *$C_{\varsigma\rightsquigarrow z}$, satisfying (NCDE), is group-like and $\log C_{\varsigma\rightsquigarrow z}$ is primitive :*

$$C_{\varsigma\rightsquigarrow z} = \prod_{l\in\mathcal{L}yn\mathcal{T}_n}^{\searrow} e^{\alpha_\varsigma^z(S_l)P_l} \quad \text{and} \quad \log C_{\varsigma\rightsquigarrow z} = \sum_{w\in\mathcal{T}_n^*} \alpha_\varsigma^z(w)\pi_1(w),$$

where $\pi_1(w) = \sum_{k\ge1} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k\in\mathcal{T}_n\mathcal{T}_n^*} \langle w | u_1 ⧢ \ldots ⧢ u_k\rangle u_1\ldots u_k.$

2. *Let $C \in \mathbb{C}\langle\langle\mathcal{T}_n\rangle\rangle, \langle C|1_{\mathcal{T}_n^*}\rangle = 1$. Then $C_{\varsigma\rightsquigarrow z}C$ satisfies (NCDE). Moreover, $C_{\varsigma\rightsquigarrow z}C$ is group-like if and only if C is group-like.*

From this, it follows that the differential Galois group of (NCDE) + group-like solutions is[11] the group $\{e^C\}_{C\in\mathcal{L}ie_{\mathbb{C}.1_{\mathcal{H}(\mathcal{V})}}\langle\langle\mathcal{X}\rangle\rangle}$. Which leads to the definition of the PV extension related to (NCDE) as $\widehat{\mathcal{C}_0.\mathcal{X}}\{C_{z_0\rightsquigarrow z}\}$.

10. $M_n \in \Omega^1(\mathcal{V})\langle\mathcal{T}_n\rangle$ and $\Delta_{⧢} M_n = 1_{\mathcal{T}_n^*} \otimes M_n + M_n \otimes 1_{\mathcal{T}_n^*}$.
11. In fact, the Hausdorff group (group of characters) of $(\mathcal{A}\langle\mathcal{T}_n\rangle, ⧢, 1_{\mathcal{T}_n^*})$.

ALGORITHMIC AND COMPUTATIONAL ASPECTS OF SOLUTIONS OF KZ_n BY DEVISSAGE

Solutions of $(NCDE)$ by $\{V_m(\varsigma, z)\}_{m\geq 0}$ (1/2)

$$V_m(\varsigma,z) = V_0(\varsigma,z) \sum_{t_{i,j}\in\mathcal{T}_{n-1}} \int_\varsigma^z e^{\sum_{t\in\mathcal{T}_n} \mathrm{ad}_{-\alpha_\varsigma^s(t)t}} \omega_{i,j}(s) t_{i,j} V_{m-1}(\varsigma,s),$$

$$V_0(\varsigma,z) = \prod_{l\in\mathcal{L}yn T_n}^{\searrow} e^{\alpha_\varsigma^z(S_l)P_l} \mod [\mathcal{L}ie_{\mathcal{A}}\langle\langle T_n\rangle\rangle, \mathcal{L}ie_{\mathcal{A}}\langle\langle T_n\rangle\rangle]$$
$$= e^{\sum_{t\in T_n} \alpha_\varsigma^z(t)t}.$$

1. $(\alpha_\varsigma^z \otimes \mathrm{Id})\mathcal{D}_{T_n}$ satisfies the differential equation $\mathbf{d}F = N_{n-1}F$, where.
$$N_{n-1} := \sum_{k=1}^{n-1} \omega_{k,n} t_{k,n} \quad \in \quad \mathcal{L}ie_{\Omega^1(\mathcal{V})}\langle T_n\rangle.$$
2. V_0 satisfies the partial differential equation $\partial_n f = N_{n-1}f$.
3. For any $m \geq 1$, on obtains explicitly
$$V_m(\varsigma,z) = \sum_{w=t_{i_1,j_1}\ldots t_{i_m,j_m}\in\mathcal{T}_{n-1}^*} \int_\varsigma^z \omega_{i_1,j_1}(s_1)\cdots\int_\varsigma^{s_{m-1}} \omega_{i_m,j_m}(s_m)\kappa_w(z,s_1,\cdots,s_m),$$
where (using the identity $e^{-a}be^a = e^{\mathrm{ad}_{-a}}b$)
$$V_0(\varsigma,z)^{-1}\kappa_w(z,s_1,\cdots,s_m)$$
$$= \prod_{p=1}^m e^{\mathrm{ad}_{-\sum_{t\in T_n}\alpha_\varsigma^{s_p}(t)t}} t_{i_p,j_p} = \sum_{q_1,\cdots,q_k\geq 0} \prod_{p=1}^m \frac{1}{q_p!} \mathrm{ad}^{q_p}_{-\sum_{t\in T_n}\alpha_\varsigma^{s_p}(t)t}\, t_{i_p,j_p}.$$

Solutions of $(NCDE)$ by $\{V_m(\varsigma, z)\}_{m\geq 0}$ (2/2)

Proposition

1. *$(NCDE)$ admits $V_0(\varsigma,z)G(\varsigma,z)$ as solution, with*

$$G(\varsigma,z) = (\alpha_\varsigma^z \otimes \mathrm{Id})\sum_{k\geq 0}\sum_{\substack{v_{i_1,j_1},\ldots,v_{i_k,j_k}\in T_n^* \\ t_{i_1,j_1},\ldots,t_{i_k,j_k}\in\mathcal{T}_{n-1}}} \frac{(-1)^{|v_{i_1,j_1}\ldots v_{i_k,j_k}|}}{|\,v_{i_1,j_1}\,|!\ldots|\,v_{i_k,j_k}\,|!} (t_{i_1,j_1}\bar{v}_{i_1,j_1})\circ\cdots\circ(t_{i_k,j_k}\bar{v}_{i_k,j_k})\otimes\rho(v_{i_1,j_1}t_{i_1,j_1})\ldots\rho(v_{i_k,j_k}t_{i_k,j_k})$$

2. *There is a diffeomorphism g of $\mathcal{V}$ s.t. $G(\varsigma,z)$ is group like series and is the Chen series, along the path $g(\varsigma \rightsquigarrow z)$ and of the differential forms $\{\omega_{i,j}\}_{1\leq i<j\leq n-1}$, and then satisfies*

$$\mathbf{d}S = \mathcal{M}^*{}_{n-1}S, \quad \text{where} \quad \mathcal{M}^*{}_{n-1} = \sum_{1\leq i<j\leq n-1} g^*\omega_{i,j}t_{i,j} \in \mathcal{L}ie_{\Omega^1(\mathcal{V})}\langle\mathcal{T}_{n-1}\rangle.$$

3. *If the restricted $\sqcup\!\sqcup$-morphism α_ς^z, on $\mathbb{C}\langle\mathcal{T}_n\rangle$, is injective then there is a primitive series $C\in\mathcal{L}ie_{\mathbb{C}}\langle\langle\mathcal{T}_{n-1}\rangle\rangle$ such that*

$$G(\varsigma,z) = \Big(\sum_{w\in T_{n-1}^*}\alpha_\varsigma^z(w)w\Big)e^{C}.$$

Solutions of KZ_n ($n \geq 4$)

For any $1 \leq i < j \leq n-1$, let $(P_{i,j}) : z_i - z_j = 1$.

Theorem ($\omega_{i,j}(z) = d\log(z_i - z_j), t_{i,j} \leftarrow t_{i,j}/2\mathrm{i}\pi$)

For $z_n \to z_{n-1}$, solution of $\mathbf{d}F = M_n F$ can be put in the form $f(z)G(z_1, \ldots, z_{n-1})$ such that

1. $f(z) \sim (z_{n-1} - z_n)^{t_{n-1,n}}$ *satisfying* $\partial_n f = N_{n-1} f$, *where*
$$N_{n-1}(z) = \sum_{k=1}^{n-1} t_{k,n} \frac{dz_n}{z_n - z_k} = \sum_{k=1}^{n-1} t_{k,n} \frac{ds}{s - s_k}, \quad \text{with} \begin{cases} s = z_n, \\ s_k = z_n - z_k. \end{cases}$$
2. $G(z_1, \ldots, z_{n-1})$ *is solution of* $\mathbf{d}S = M_{n-1}^{t_{\bullet,n}} S$, *where*
$$M_{n-1}^{t_{\bullet,n}}(z) \sim \sum_{1 \leq i < j \leq n-1} \varphi_{t_{\bullet,n}}^{(\varsigma,z)}(t_{i,j}) d\log(z_i - z_j),$$
$$\varphi_{t_{\bullet,n}}^{(\varsigma,z)}(t_{i,j}) = e^{\mathrm{ad}_{-\sum_{1\leq k<n} \log(z_k - z_{n-1}) t_{k,n}}} t_{i,j} \mod \mathcal{J}_{\mathcal{R}_n}.$$
Moreover, $M_{n-1}^{t_{\bullet,n}}$ *exactly coincides with* M_{n-1} *in the intersection of affine planes* $\bigcap_{1 \leq i < n-1} (P_{i,n-1})$.

Conversely, if f satisfies $\partial_n f = N_{n-1} f$ and $G(z_1, \ldots, z_{n-1})$ satisfies $\mathbf{d}S = M_{n-1}^{t_{\bullet,n}} S$ then $f(z)G(z_1, \ldots, z_{n-1})$ satisfies $\mathbf{d}F = M_n F$.

Solutions of KZ_n ($n \geq 4$) with asymptotic conditions

Let $F_\bullet : (\mathbb{C}\langle\mathcal{T}_n\rangle, ш, 1_{\mathcal{T}_n^*}) \to (\mathcal{H}(\mathcal{V}), *, 1_{\mathcal{H}(\mathcal{V})})$ be the character defined by $F_{1_{\mathcal{T}_n^*}} = 1_{\mathcal{H}(\mathcal{V})}$, $\forall t_{i,j} \in \mathcal{T}_n$, $F_{t_{i,j}}(z) = \log(z_i - z_j)$, $\forall t_{i,j} w \in \mathcal{L}yn\mathcal{T}_n \setminus \mathcal{T}_n$,

$$F_{t_{i,j}w}(z) = \int_0^z \omega_{i,j}(s) F_w(s), \quad \text{where} \quad \omega_{i,j}(z) = d\log(z_i - z_j).$$

Corollary ($\omega_{i,j}(z) = d\log(z_i - z_j)$, $t_{i,j} \leftarrow t_{i,j}/2\mathrm{i}\pi$)

1. $\{F_t\}_{t \in \mathcal{T}_n \cup \{1_{\mathcal{T}_n^*}\}}$ *are* $\mathcal{C}_0$*-linearly free.*
2. *The graph of* $F_\bullet$, F, *is unique solution of* $\mathbf{d}F = M_n F$ *and*

$$\mathrm{F}(z) = \prod_{l \in \mathcal{L}yn\mathcal{T}_n}^{\searrow} e^{F_{S_l}(z) P_l} \sim_{\substack{z_i \rightsquigarrow z_{i-1} \\ 1 < i \leq n}} (z_{i-1} - z_i)^{t_{i-1,i}} G_i(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$$

where $G_i(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$ *satisfies* $\mathbf{d}S = M_{n-1}^{t_{\bullet,n}} S$ *and, for* $y_1 = z_1, \ldots, y_{i-1} = z_{i-1}, y_i = z_{i+1}, \ldots, y_{n-1} = z_n$*, one has*

$$M_{n-1}^{t_{\bullet,n}}(y) = \sum_{1 \leq i < j \leq n-1} e^{\mathrm{ad}_{-\sum_{1 \leq k \leq n-1} \log(y_k - y_{n-1}) t_{k,n}}} t_{i,j} d\log(y_i - y_j) \mod \mathcal{J}_{\mathcal{R}_n}$$

and $M_{n-1}^{t_{\bullet,n}}$ *exactly coincides with* M_{n-1} *in* $\bigcap_{1 \leq k < n-1} (P_{i,n-1})$.

3. *In* $\mathcal{L}ie_{\mathcal{A}}\langle\langle\mathcal{T}_n\rangle\rangle / [\mathcal{L}ie_{\mathcal{A}}\langle\langle T_n\rangle\rangle, \mathcal{L}ie_{\mathcal{A}}\langle\langle T_n\rangle\rangle]$*, one has*

$$\mathrm{F}(z) = e^{\sum_{i=1}^{n-1} \log(z_n - z_i) t_{i,n}} \sum_{\substack{k \geq 0, l_1, \ldots, l_k \geq 0 \\ t_1, \ldots, t_k \in \mathcal{T}_{n-1}}} F_{(t_1 \bar{T}_n^{l_1}) \circ \ldots \circ (t_k \bar{T}_n^{l_k})}(z) \prod_{1 \leq j \leq k} \mathrm{ad}_{-T_n}^{l_j} t_j.$$

KZ_3 : Simplest non-trivial case (1/3)

One has $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$ and

$$\Omega_3(z) = \frac{1}{2\mathrm{i}\pi}\left(t_{1,2}\frac{d(z_1 - z_2)}{z_1 - z_2} + t_{1,3}\frac{d(z_1 - z_3)}{z_1 - z_3} + t_{2,3}\frac{d(z_2 - z_3)}{z_2 - z_3}\right).$$

Solution of $\mathbf{d}F(z) = \Omega_3(z)F(z)$ can be computed as limit of the sequence $\{F_l\}_{l\geq 0}$, in $\mathcal{H}(\widetilde{\mathbb{C}_*^3})\langle\langle\mathcal{T}_3\rangle\rangle$, by convergent Picard's iteration :

$$F_0(z) = 1_{\mathcal{H}(\mathcal{V})} \quad \text{and} \quad F_l(z) = \int_0^z \Omega_3(s)F_{l-1}(s).$$

Let us compute, by another way, a solution of $\mathbf{d}F(z) = \Omega_3(z)F(z)$ as the limit of the sequence $\{V_l\}_{l\geq 0}$, in $\mathcal{H}(\widetilde{\mathbb{C}_*^3})\langle\langle\mathcal{T}_3\rangle\rangle$, iteratively obtained by

$$\begin{aligned}
V_0(z) &= e^{(t_{1,2}/2\mathrm{i}\pi)\log(z_1 - z_2)},\\
V_l(z) &= \int_0^z e^{(t_{1,2}/2\mathrm{i}\pi)(\log(z_1 - z_2) - \log(s_1 - s_2))}\tilde{\Omega}_2(s)V_{l-1}(s)\\
&= V_0(z)\int_0^z e^{-(t_{1,2}/2\mathrm{i}\pi)\log(s_1 - s_2)}\tilde{\Omega}_2(s)V_{l-1}(s),\\
\text{with } \tilde{\Omega}_2(z) &= \frac{1}{2\mathrm{i}\pi}\left(t_{1,3}\frac{d(z_1 - z_3)}{z_1 - z_3} + t_{2,3}\frac{d(z_2 - z_3)}{z_2 - z_3}\right).
\end{aligned}$$

KZ_3 : Simplest non-trivial case (2/3)

Explicit solution is $F = V_0 G$, where $V_0(z) = (z_1 - z_2)^{t_{1,2}/2\mathrm{i}\pi}$ and

$$G(z) = \sum_{\substack{t_{i_1,j_1}\ldots t_{i_m,j_m} \in \{t_{1,3},t_{2,3}\}^* \\ m\geq 0}} \int_0^z \omega_{i_1,j_1}(s_1)\varphi^{s_1}(t_{i_1,j_1}) \ldots \int_0^{s_{m-1}} \omega_{i_m,j_m}(s_m)\varphi^{s_m}(t_{i_m,j_m}),$$

where $\omega_{1,3}(z) = d\log(z_1 - z_3)$ and $\omega_{2,3}(z) = d\log(z_2 - z_3)$ and φ is the following automorphism of Lie algebra, $\mathcal{L}ie_{\mathcal{H}(\widetilde{\mathbb{C}_*^n})}\langle \mathcal{T}_3 \rangle$,

$$\varphi^z = e^{\mathrm{ad}_{-(t_{1,2}/2\mathrm{i}\pi)\log(z_1-z_2)}} = \sum_{k\geq 0} \frac{\log^k(z_1 - z_2)}{(-2\mathrm{i}\pi)^k k!} \mathrm{ad}^k_{t_{1,2}}.$$

Since $t_{1,2} \prec t_{1,3} \prec t_{2,3}$ and, for $k \geq 0$ and $i = 1$ or 2, $t^k_{1,2}t_{i,3} \in \mathcal{L}yn\mathcal{T}_3$ then

$$P_{t^k_{1,2}t_{i,3}} = \mathrm{ad}^k_{t_{1,2}}\, t_{i,3} \quad \text{and} \quad S_{t^k_{1,2}t_{i,3}} = t^k_{1,2}t_{i,3}$$

and then

$$\varphi^z(t_{i,3}) = \sum_{k\geq 0} \frac{\log^k(z_1 - z_2)}{(-2\mathrm{i}\pi)^k k!} P_{t^k_{1,2}t_{i,3}}, \quad \check{\varphi}^z(t_{i,3}) = \sum_{k\geq 0} \frac{\log^k(z_1 - z_2)}{(-2\mathrm{i}\pi)^k k!} S_{t^k_{1,2}t_{i,3}},$$

where $\check{\varphi}$ (adjoint to φ) is the following automorphism of $(\mathcal{A}\langle \mathcal{T}_3 \rangle, ⧢, 1_{\mathcal{T}_3^*})$

$$\check{\varphi}^z = e^{-(t_{1,2}/2\mathrm{i}\pi)\log(z_1-z_2)} = \sum_{k\geq 0} \frac{\log^k(z_1 - z_2)}{(-2\mathrm{i}\pi)^k k!} t^k_{1,2}.$$

KZ_3 : Simplest non-trivial case (3/3)

Belonging to $\mathcal{H}(\widetilde{\mathbb{C}_*^3})\langle\langle\mathcal{T}_3\rangle\rangle$, G satisfies $\mathbf{d}G(z) = \bar{\Omega}_2(z)G(z)$, where

$$\bar{\Omega}_2(z) = \frac{1}{2\mathrm{i}\pi}\left(\varphi^z(t_{1,3})\frac{d(z_1 - z_3)}{z_1 - z_3} + \varphi^z(t_{2,3})\frac{d(z_2 - z_3)}{z_2 - z_3}\right).$$

In the affine plan $(P_{1,2}) : z_1 - z_2 = 1$, one has

$$\log(z_1 - z_2) = 0 \quad \text{and then} \quad \varphi \equiv \mathrm{Id}.$$

Setting $x_0 = t_{1,3}/2\mathrm{i}\pi, x_1 = -t_{2,3}/2\mathrm{i}\pi$ and $z_1 = 1, z_2 = 0, z_3 = s$, one has

$$\bar{\Omega}_2(z) = \frac{1}{2\mathrm{i}\pi}\left(t_{1,3}\frac{d(z_1 - z_3)}{z_1 - z_3} + t_{2,3}\frac{d(z_2 - z_3)}{z_2 - z_3}\right) = x_1\frac{ds}{1-s} + x_0\frac{ds}{s}.$$

KZ_3 admits then the noncommutative generating series of polylogarithms, L, as the actual solution satisfying the Drinfel'd asymptotic conditions.

Via L and the homographic substitution $g : z_3 \longmapsto (z_3 - z_2)/(z_1 - z_2)$, mapping $\{z_2, z_1\}$ to $\{0, 1\}$, $\mathrm{L}((z_3 - z_2)/(z_1 - z_2))$ is a particular solution of KZ_3, in $(P_{1,2})$. So is $\mathrm{L}((z_3 - z_2)/(z_1 - z_2))(z_1 - z_2)^{(t_{1,2}+t_{1,3}+t_{2,3})/2\mathrm{i}\pi}$.

To end with KZ_3, by braid relations, $[t_{1,2} + t_{2,3} + t_{1,3}, t] = 0$, for $t \in \mathcal{T}_3$, meaning that t commutes with $(z_1 - z_2)^{(t_{1,2}+t_{2,3}+t_{1,3})/2\mathrm{i}\pi}$ and then $\mathcal{A}\langle\langle\mathcal{T}_3\rangle\rangle$ commutes with $(z_1 - z_2)^{(t_{1,2}+t_{1,3}+t_{2,3})/2\mathrm{i}\pi}$.
Thus, KZ_3 also admits $(z_1 - z_2)^{(t_{1,2}+t_{1,3}+t_{2,3})/2\mathrm{i}\pi}\mathrm{L}((z_3 - z_2)/(z_1 - z_2))$ as a particular solution in $(P_{1,2})$.

Other example of non-trivial case : KZ_4 $(t_{i,j} \leftarrow t_{i,j}/2\mathrm{i}\pi)$

For $n = 4$, one has $\mathcal{T}_4 = \{t_{1,2}, t_{1,3}, t_{1,4}, t_{2,3}, t_{2,4}, t_{3,4}\}$ and then $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$ and $T_4 = \{t_{1,4}, t_{2,4}, t_{3,4}\}$. Then

$$\varphi_{T_4}^{(\varsigma,z)} = e^{\mathrm{ad}_{-\sum_{t\in T_4} \alpha_\varsigma^z(t)t}},$$

and for any $t_{i,j} \in \mathcal{T}_3$,

$$\varphi_{t_{\bullet,4}}^{(\varsigma,z)}(t_{i,j}) = \varphi_{T_4}^{(\varsigma,z)}(t_{i,j}) \mod \mathcal{J}_{\mathcal{R}_n}.$$

If $z_4 \to z_3$ then

$$F(z) = V_0(z)G(z_1, z_2, z_3), \quad \text{where} \quad V_0(z) = e^{\sum_{1\le i\le 4} t_{i,4} \log(z_i - z_4)}$$

and $G(z_1, z_2, z_3)$ satisfies $\mathbf{d}S = M_3^{t_{\bullet,4}} S$ with

$$\begin{aligned} M_3^{t_{\bullet,4}}(z) &= \varphi_{t_{\bullet,4}}^{(z^0,z)}(t_{1,2}) d\log(z_1 - z_2) \\ &+ \varphi_{t_{\bullet,4}}^{(z^0,z)}(t_{1,3}) d\log(z_1 - z_3) \\ &+ \varphi_{t_{\bullet,4}}^{(z^0,z)}(t_{2,3}) d\log(z_2 - z_3). \end{aligned}$$

Considering $(P_{1,4}) : z_1 - z_4 = 1$, $(P_{2,4}) : z_2 - z_4 = 1$, $(P_{3,4}) : z_3 - z_4 = 1$, in the intersection $(P_{1,3}) \cap (P_{2,3})$, one has $\log(z_1 - z_3) = \log(z_2 - z_3) = 0$ and $\varphi_{t_{\bullet,4}} \equiv \mathrm{Id}$ and then $M_3^{t_{\bullet,4}}$ exactly coincides with M_3.

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