A Classification of Restricted Lattice Walks with Small Steps

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Lattice Walks in the Quarter Plane

Given: A set of directions
Count: Number of integer lattice walks in the first quadrant using these steps.

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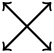
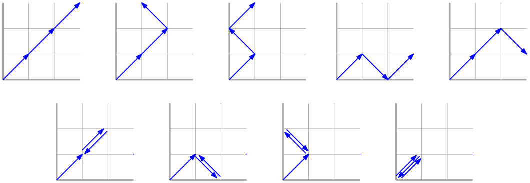
For instance, given the step set $S = \{NE, SE, NW, SW\}$



there are 9 walks of length 3:



Realistic Goals

Finding an exact expression for the number of walks of length n is too hard in general - we seek **asymptotic estimates**.

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If we can classify the generating function as algebraic or D-Finite (satisfies a linear ODE with polynomial coefficients) then we know the form of its growth.

Nature of $F(t)$	Growth of $[t^n]F(t)$
Algebraic	$\frac{c\beta^n n^s}{\Gamma(s+1)}$
D-Finite	$A\beta^n n^s \log(n)^r$
Neither	?

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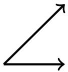
calculate the number of such walks efficiently;

answer questions about related physical systems (limiting free energy).

Is it D-Finite?



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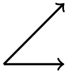

YES - it is rational!

Asymptotics:

$$[t^n]F(t) \sim 2^n.$$

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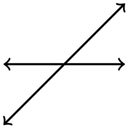

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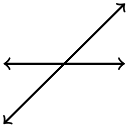

Yes - it is algebraic!

Asymptotics:

$$[t^n]F(t) \sim \frac{4\sqrt{3}}{3\Gamma\left(\frac{1}{3}\right)} \cdot \frac{4^n}{n^{2/3}}.$$

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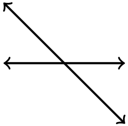

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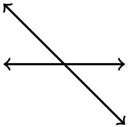

Yes! (But not algebraic)

Asymptotics:

$$[t^n]F(t) \sim \frac{8}{\pi} \cdot \frac{4^n}{n^2}.$$

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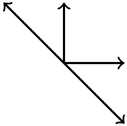

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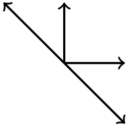

NO!

Asymptotics:

$$[t^n]F(t) \sim \tau \cdot 4^n, \quad \tau \in \mathbb{R}.$$

D-Finiteness is a desirable property...
But how do we find it?

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Closure properties of D-Finite functions

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Asymptotic Form (results of probability and G-Functions)

Searching for D-Finiteness

History

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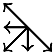
Also, subsets of



will never leave the origin, so these are also not considered.

Tools

The generating function

$$Q(x,y;t) = \sum_{n,i,j \geq 0} q(i,j;n) x^i y^j t^n$$

which counts the number of walks of length n ending at (i,j) satisfies an obvious functional equation.

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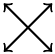
which counts the number of walks of length n ending at (i, j) satisfies an obvious functional equation. For example, with the step set

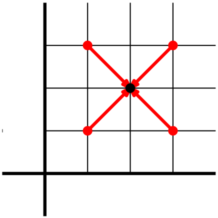


we define the *characteristic* function

$$S(x, y) := xy + \frac{y}{x} + \frac{1}{xy} + \frac{x}{y}$$

A functional equation

$$Q(x, y; t) = 1 + tS(x, y)Q(x, y; t)$$



A functional equation

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&\quad - t\left(\frac{y}{x} + \frac{1}{xy}\right)Q(0,y;t)
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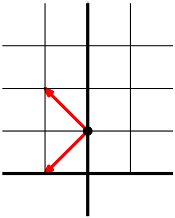

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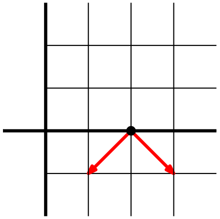

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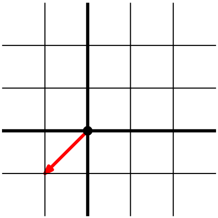

The Kernel Equation

Begin with the functional equation, re-group the terms and multiply by xy to give:

$$\boxed{K(x,y)\cdot xyQ(x,y) = xy - t(y^2+1)Q(0,y) - t(x^2+1)Q(x,0) + tQ(0,0)} \qquad (\mathcal{K})$$

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We now define a group G of bi-rational transformations of the xy-plane which preserves $S(x,y)$ - and thus $K(x,y)$.

The Group of a Walk

To begin, write

$$S(x,y) = \frac{1}{y}A_{-1}(x) + A_0(x) + yA_1(x) = \frac{1}{x}B_{-1}(y) + B_0(y) + xB_1(y).$$

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We let G be the group generated by the involutions

$$\tau : (x,y) \mapsto \left(x, \frac{A_{-1}(x)}{yA_1(x)}\right) \qquad \psi : (x,y) \mapsto \left(\frac{B_{-1}(y)}{xB_1(y)}, y\right).$$

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In our case, we have

$$\psi : (x,y) \mapsto \left(\frac{1}{x}, y\right) \qquad \tau : (x,y) \mapsto \left(x, \frac{1}{y}\right),$$

so G contains 4 elements.

A Partial Result

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23 of the 79 walks correspond to a finite group, with 22 of them admitting a D-Finite generating function. The remaining 56 walks correspond to an infinite group.

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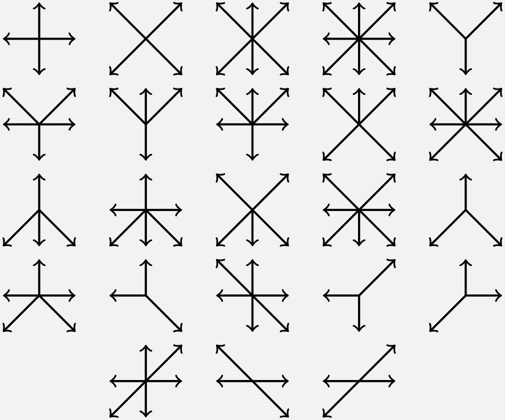
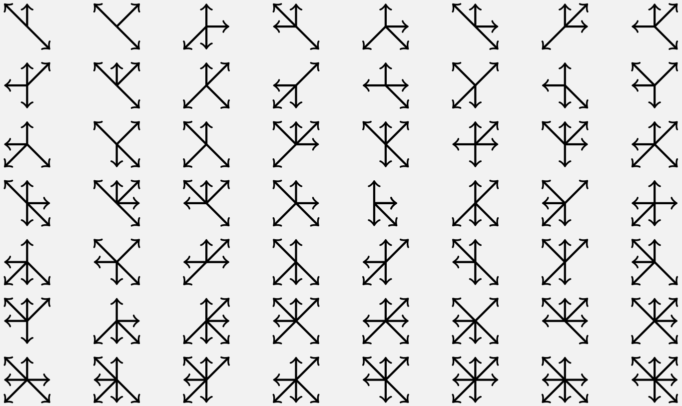
Conjecture (Mishna 2007)

The generating functions $Q(1,1;t)$ of the 56 walks with infinite group are not D-Finite.

Finite Group Walks (D-Finite):



Infinite Group Walks (Non D-Finite?):



Walks with Infinite Groups

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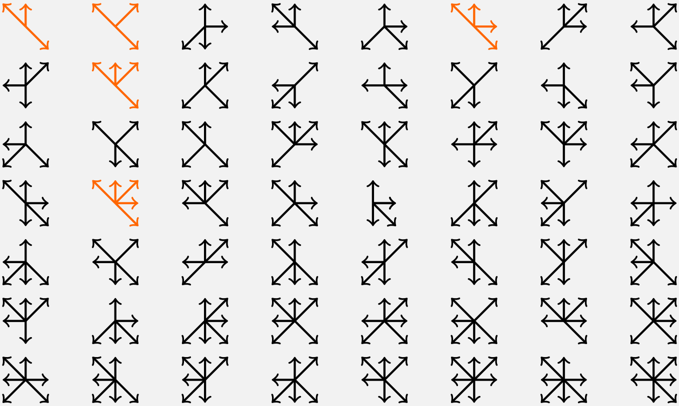
[M. & Mishna 2012]
The final 3 walks have $Q(1,1;t)$ non D-Finite

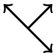
Infinite Group Walks (Non D-Finite*):



3 Techniques to Prove Non D-finiteness

The Iterated Kernel Method [M., Mishna, Rechnitzer]

For the step set



we have the functional equation

$$xyK(x,y) \cdot Q(x,y) = xy - ty^2 Q(y,0) - tx^2 Q(x,0) \qquad (\mathcal{K})$$

where

$$xyK(x,y) = 1 - tS(x,y) = 1 - t(x^2 + y^2 + x^2y^2).$$

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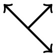

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Note that

$$Q(1,1) = \frac{1 - 2tQ(1,0)}{1 - 3t}.$$

The Iterated Kernel Method [M., Mishna, Rechnitzer]

We write

$$Q(1,0;t) = \sum_{n\geq 0} (-1)^n Y_n(t) Y_{n+1}(t),$$

where the Y_n are explicit algebraic functions (degree 2) determined from the kernel.

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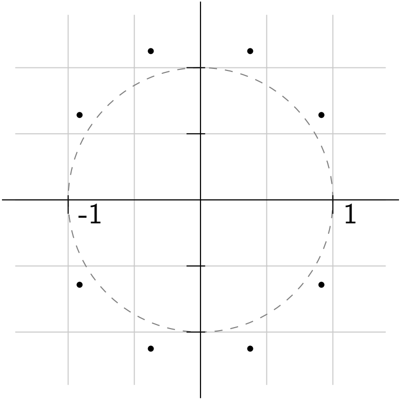
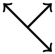
We prove that each Y_n has a unique singularity, so $Q(1,0)$ has an infinite number of singularities and is not D-Finite.

We can also extract asymptotics and quickly count the number of such walks.

Plots of Singularities



Singularities of Y_n for 

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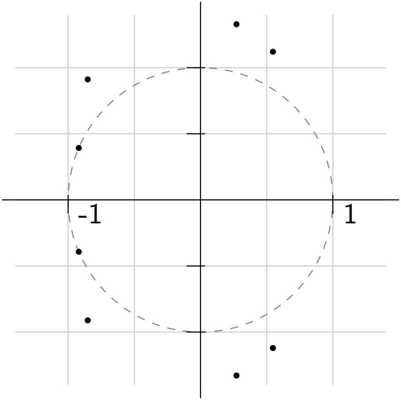
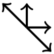


Singularities of Y_n for 

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Note that this does not imply that $Q(1,1;t)$ is non D-Finite.

Excursion Method [Bostan, Raschel, Salvy]

A recent probabilistic result [Denisov & Wachtel 2011] implies that for walks in the quarter plane

$$[t^n]Q(0,0;t) \sim K \cdot \rho^n \cdot n^\alpha,$$

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c and ρ found by solving a polynomial system.

As $Q(0,0;t)$ is a G-function, the growth exponent α must be rational if $Q(0,0;t)$ is D-Finite.

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This proves that $Q(0,0;t)$ is non D-Finite (and thus so is $Q(x,y;t)$) for 51 walks.

Longer Steps
[Bostan, Bousquet-Mélou, M. - in preparation]

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We group the walks by counting sequence, then apply the above method to filter walks which appear to be D-Finite.

Results with One and Two Long Steps

Walks with one large step (4 degenerative):

For 643 sequences, $Q(0,0;t)$ proven non D-Finite

For 37 sequences, α shown to be rational

32 of 37 sequences have differential equations guessed

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For 37 sequences, α shown to be rational

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Walks with two large steps (11 degenerative):

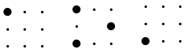
For 5754 sequences, $Q(0,0;t)$ proven non D-Finite

For 156 sequences, α shown to be rational (69 have guessed equations)

Extensions to New Walks
(3D Walks)

Basic Idea

Now, we look at walks in the xyz-plane restricted to the positive octant.



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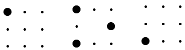

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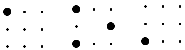

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We focus on the $83,682$ with 5 steps or less. Bostan and Kauers conjectured (up to equivalence) 35 D-Finite steps.

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There are 8 sets with an *infinite group*.

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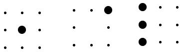

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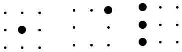

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\begin{aligned}
&(x) \qquad & b &\geq (c + d + e) \\
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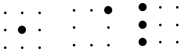

and 'mark' the steps by a, b, c, d, e. Then

$$\begin{array}{ll} (x) & b \geq (c + d + e) \\ (y) & b + c \geq e \\ (z) & (c + d + e) \geq a \end{array}$$

If (x) is satisfied then (y) must be satisfied!

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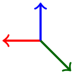

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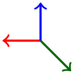

We can write $Q(x, y, z; t) = Q'(X, Y; T)$, where

$$T = t(1 + y + y^2)^{1/3} \qquad X = \frac{zT}{t} \qquad Y = \frac{xyt}{T}.$$

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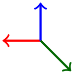

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As $Q'(x, 1/x; t)$ is algebraic, so is $Q\left(x, y, \frac{1}{xy}; t\right)$.

Picking D-Finite Cases

Thus, we can filter the D-Finite walks as follows:

(1) Filter all walks that reduce to 2D

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31 with a finite group that can't be proven using existing methods.

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We would like to:

have a better characterization of D-Finiteness

understand the role of the group better

develop more robust methods - start looking at steps with multiple colours

References

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THANK YOU