Negative moments of orthogonal polynomials

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Outline

What is the combinatorial reciprocity theorem?

For a sequence $(f_n)_{n\in\mathbb{Z}}$, if both $|f_n|$ and $|f_{-n}|$ count some combinatorial objects of size $n \geq 1$, such a result is called a **combinatorial reciprocity theorem**.

Examples

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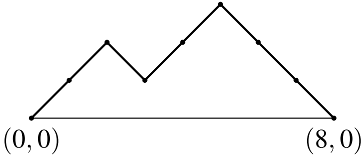
Examples

1. binomial coefficients $\binom{n}{k}$
2. chromatic polynomials $\chi_G(n)$
3. Ehrhart polynomials $\mathrm{Ehr}_P(n)$

Dyck paths and Motzkin paths

Dyck paths



$\in \mathrm{Dyck}_8$

weight = $\lambda_2\lambda_3\lambda_2\lambda_1$

Dyck paths and Motzkin paths

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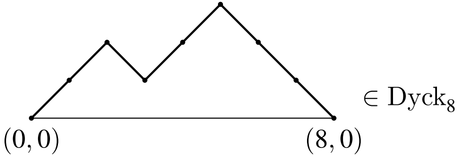

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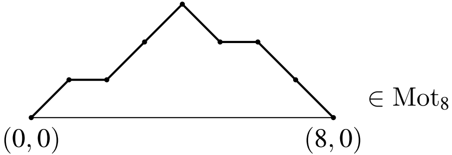
Motzkin paths



weight = $b_1\lambda_3 b_2\lambda_2\lambda_1$

Dyck paths and Motzkin paths

Question

- Is there a combinatorial object counted by $|\mathrm{Dyck}_{-n}|$ or $|\mathrm{Mot}_{-n}|$?

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- How to define $|\,\mathrm{Dyck}_{-n}\,|$ and $|\,\mathrm{Mot}_{-n}\,|$?

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- How to define $|\operatorname{Dyck}_{-n}|$ and $|\operatorname{Mot}_{-n}|$?

We have to introduce **bounded Dyck path** and **bounded Motzkin path**.

Previous results

Theorem (Cigler and Krattenthaler, 2020)

$$
\begin{aligned}
|\,\mathrm{Dyck}_{2n}^{\leq 2k-1}\,| &= |\,\mathrm{Alt}_{2n-1}^{\leq k}\,| \\
&:= |\{(a_1, \cdots, a_{2n-1}) : a_1 \leq a_2 \geq a_3 \leq \cdots \geq a_{2n-1}, 1 \leq a_i \leq k\}|.
\end{aligned}
$$

They also showed many other interesting results including a reciprocity between determinants of these numbers.

Orthogonal polynomials

- Polynomials $\{P_n(x)\}_{n\geq 0}$ are called **orthogonal polynomials** with respect to a linear functional $\mathcal{L}$ if $\deg P_n(x) = n$ and

$$\mathcal{L}(P_m(x)P_n(x)) = \delta_{m,n}c_n, \quad c_n \neq 0.$$

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- Let $\{P_n(x)\}_{n\geq 0}$ be monic polynomials that satisfy a three-term recurrence relation: $P_{-1}(x) = 0$, $P_0(x) = 1$, and for $n \geq 0$,

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x),$$

for some sequences $\boldsymbol{b} = (b_n)_{n\geq 0}$ and $\boldsymbol{\lambda} = (\lambda_n)_{n\geq 1}$.

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- It is well known that these are orthogonal polynomials with respect to a unique linear functional $\mathcal{L}$ with $\mathcal{L}(1) = 1$.
- The **moment** $\mu_n(\boldsymbol{b}, \boldsymbol{\lambda})$ of $P_n(x)$ is defined by $\mu_n(\boldsymbol{b}, \boldsymbol{\lambda}) = \mathcal{L}(x^n)$.

Combinatorics and Moments

Viennot found the following combinatorial interpretation for the moment:

$$\mathcal{L}(x^n) = \mu_n(\boldsymbol{b}, \boldsymbol{\lambda}) = \sum_{p \in \mathrm{Mot}_n} \mathrm{wt}(p).$$

Note that

$$\mu_n(\mathbf{0}, \boldsymbol{\lambda}) = \sum_{p \in \mathrm{Dyck}_n} \mathrm{wt}(p).$$

Bounded moments

The **bounded moments** $\mu_n^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})$ are defined by

$$\mu_n^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda}) = \sum_{p \in \mathrm{Mot}_n^{\leq k}} \mathrm{wt}(p).$$

The sequence $(\mu_n^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda}))_{n \geq 0}$ satisfies a homogeneous linear recurrence relation so that its negative version $(\mu_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda}))_{n \geq 1}$ is defined.

We call $\mu_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})$ the **negative (bounded) moments** of the orthogonal polynomials $P_n(x; \boldsymbol{b}, \boldsymbol{\lambda})$.

Generalized bounded moments

Viennot showed that the generalized moment $\mu_{n,r,s}(\boldsymbol{b}, \boldsymbol{\lambda}) := \mathcal{L}(x^n P_r(x) P_s(x))$ has a similar combinatorial expression

$$\mu_{n,r,s}(\boldsymbol{b}, \boldsymbol{\lambda}) = \sum_{p \in \mathrm{Mot}_{n,r,s}} \mathrm{wt}(p).$$

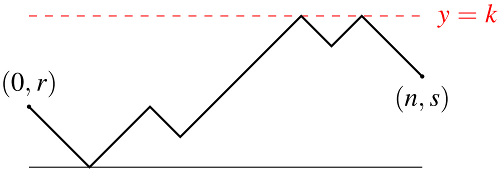
Definition

A **generalized bounded moment** $\mu^{\leq k}_{n,r,s}(\boldsymbol{b}, \boldsymbol{\lambda})$ is defined by

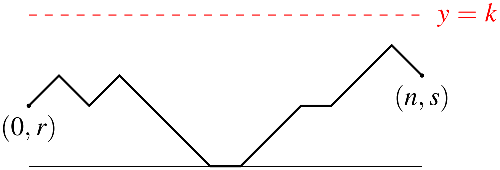
$$\mu^{\leq k}_{n,r,s}(\boldsymbol{b}, \boldsymbol{\lambda}) = \sum_{p \in \mathrm{Mot}^{\leq k}_{n,r,s}} \mathrm{wt}(p).$$

Bounded Dyck/Motzkin paths

$\mathrm{Dyck}_{n,r,s}^{\leq k}$



$\mathrm{Mot}_{n,r,s}^{\leq k}$



Homogeneous linear recurrence relation

Theorem (EC1, Theorem 4.1.1 and Proposition 4.2.3)

A sequence $(f_n)_{n\geq 0}$ satisfies a homogeneous linear recurrence relation if and only if

$$\sum_{n\geq 0} f_n x^n = \frac{P(x)}{Q(x)},$$

for some polynomials $P(x)$ and $Q(x)$ with $\deg P(x) < \deg Q(x)$ and $Q(0) \neq 0$.

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for some polynomials $P(x)$ and $Q(x)$ with $\deg P(x) < \deg Q(x)$ and $Q(0) \neq 0$. Moreover, in this case, we have

$$\sum_{n\geq 1} f_{-n} x^n = -\frac{P(1/x)}{Q(1/x)},$$

as rational functions.

The Proposition 4.2.3 is also known as 'Popoviciu's theorem'.

Generating function for the moments

Let $P_n^*(x) = x^n P_n(1/x)$, and let $\delta P(x; \boldsymbol{b}, \boldsymbol{\lambda})$ be a polynomial obtained from $P(x; \boldsymbol{b}, \boldsymbol{\lambda})$ by moving b_i to b_{i+1} and λ_i to λ_{i+1}.

Theorem (Viennot, 83')

Let r, s, k be integers with $0 \le r, s \le k$.

$$\sum_{n\geq 0} \mu_{n,r,s}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda}) x^n = \begin{cases} \frac{x^{s-r} P_r^*(x) \delta^{s+1} P_{k-s}^*(x)}{P_{k+1}^*(x)} & \text{if } r \le s, \\ \frac{P_s^*(x) \delta^{r+1} P_{k-r}^*(x)}{P_{k+1}^*(x)} \prod_{i=s+1}^{r} \lambda_i. & \text{if } r > s. \end{cases}$$

Generating function for the negative moments

Theorem (JKKSS, 2023)

Let r, s, k be integers with $0 \le r, s \le k$. Suppose that $\mu^{\le k}_{-n,r,s}(\boldsymbol{b}, \boldsymbol{\lambda})$ is well defined for $n \ge 1$. Then we have

$$\sum_{n \ge 1} \mu^{\le k}_{-n,r,s}(\boldsymbol{b}, \boldsymbol{\lambda}) x^n = \begin{cases} -\frac{x P_r(x) \delta^{s+1} P_{k-s}(x)}{P_{k+1}(x)} & \text{if } r \le s, \\ -\frac{x^{r-s+1} P_s(x) \delta^{r+1} P_{k-r}(x)}{P_{k+1}(x)} \prod_{i=s+1}^{r} \lambda_i. & \text{if } r > s. \end{cases}$$

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Proposition (JKKSS, 2023)

Let $\boldsymbol{b}^2 = (b_{n-1}b_n)_{n\geq 1} = (b_0 b_1, b_1 b_2, \ldots)$. The sequence $(\mu^{\leq k}_{-n,r,s}(\boldsymbol{b}, \boldsymbol{b}^2))_{n\geq 1}$ is well-defined if and only if $k \not\equiv 1 \pmod 3$.

Question

What is a combinatorial meaning for $\mu^{\leq k}_{-n,r,s}(\boldsymbol{b}, \boldsymbol{b}^2)$?

peak-valley sequences

Definition

An **(ℓ, r, s)-peak-valley sequence** of length n is a sequence $(a_1, \ldots, a_n)$ of nonnegative integers such that for $i = 0, \ldots, n+1$,

- if $a_i \equiv 0 \pmod{\ell}$, then a_i is a valley, that is, $a_{i-1} > a_i < a_{i+1}$,
- if $a_i \equiv -1 \pmod{\ell}$, then a_i is a peak, that is, $a_{i-1} < a_i > a_{i+1}$,

where we set $a_0 = r$ and $a_{n+1} = s$.

Denote by $\mathrm{PV}_{n,r,s}^{\ell,k}$ the set of (ℓ, r, s)-peak-valley sequences $(a_1, \ldots, a_n)$ of length n with $0 \le a_i \le k$ for all $i = 1, \ldots, n$.

$\mathrm{PV}_n^{\ell,k} = \mathrm{PV}_{n,0,0}^{\ell,k}$: ℓ-peak-valley sequence.

The *weight* of a sequence $\pi = (a_1, \ldots, a_n)$ is defined by

$$\mathrm{wt}(\pi) = V_{a_1} \cdots V_{a_n}.$$

Examples

Let $r = 2$ and $s = 3$.

Example ($\ell = 2$)

- $\pi = 5\ 2\ 3\ 0\ 7\ 4\ 9\ 2\ 7\ 4\ 5$

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Example ($\ell = 3$)

- $\pi = 5\ 4\ 4\ 0\ 8\ 6\ 7\ 8\ 3\ 4\ 7$

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- $\pi = \mathbf{5}\,4\,4\,\mathbf{0\,8\,6}\,7\,\mathbf{8\,3}\,4\,7$
- 2,5,8 : peaks, and 0,3,6 : valleys
- $\pi \in PV^{3,8}_{11,2,3}$

Continued fraction

By Flajolet's combinatorial theory of continued fractions, Viennot showed that

$$\sum_{n\geq 0}\mu_n^{\leq k}(\boldsymbol{b},\boldsymbol{\lambda})x^n=\cfrac{1}{1-b_0x-\cfrac{\lambda_1x^2}{1-b_1x-\ddots-\cfrac{\lambda_kx^2}{1-b_kx}}}.$$

Continued fraction for the negative moments

Let $\boldsymbol{b}^2 = (b_{n-1}b_n)_{n\geq 1} = (b_0b_1, b_1b_2, \dots)$ and $b_i = -V_i^{-1}$.

$$\sum_{n\geq 1} \mu_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})x^n = \cfrac{-1}{1 - b_0x^{-1} - \cfrac{\lambda_1 x^{-2}}{1 - b_1x^{-1} - \cfrac{\lambda_2 x^{-2}}{1 - b_2x^{-1} - \ddots - \cfrac{\lambda_k x^{-2}}{1 - b_kx^{-1}}}}}.$$

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$$\sum_{n\geq 1} \mu_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})x^n = \cfrac{-x}{x - b_0 - \cfrac{\lambda_1}{x - b_1 - \cfrac{\lambda_2}{x - b_2 - \ddots - \cfrac{\lambda_k}{x - b_k}}}}.$$

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$$\sum_{n\geq 1} \mu_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda}) x^n = \cfrac{b_0^{-1}x}{1 - b_0^{-1}x - \cfrac{b_0^{-1}b_1^{-1}\lambda_1}{1 - b_1^{-1}x - \cfrac{b_1^{-1}b_2^{-1}\lambda_2}{1 - b_2^{-1}x - \ddots - \cfrac{b_{k-1}^{-1}b_k^{-1}\lambda_k}{1 - b_k^{-1}x}}}}.$$

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$$\sum_{n\geq 1} \mu_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{b}^2)x^n = \cfrac{V_0x}{-V_0x - 1 - \cfrac{1}{-V_1x - 1 - \ddots - \cfrac{1}{-V_kx - 1}}}.$$

Combinatorial interpretation

Theorem (JKKSS, 2023)

Let $b_i = -V_i^{-1}$ for all i. We have

$$\mu_{-n}^{\leq 3k-1}(\boldsymbol{b}, \boldsymbol{b}^2) = V_0 \sum_{\pi \in \mathrm{PV}_{n-1}^{3,3k-1}} \mathrm{wt}(\pi).$$

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Let $b_i = -V_i^{-1}$ for all i. We have

$$\mu_{-n}^{\leq 3k}(\boldsymbol{b}, \boldsymbol{b}^2) = V_0 \sum_{\pi \in \widetilde{\mathrm{PV}}_{n-1}^{3,3k}} \mathrm{wt}(\pi).$$

Combinatorial interpretation

Corollary (JKKSS, 2023)

We have

$$\left|\mathrm{Mot}_{=n}^{\leq 3k-1}\right| = \left|\mathrm{PV}_{n-1}^{3,3k-1}\right|.$$

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matrix representation

We define the tridiagonal matrix $A^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})$ by

$$A^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda}) = \begin{pmatrix} b_0 & 1 & & & & \\ \lambda_1 & b_1 & 1 & & & \\ & & \ddots & & & \\ & & \lambda_{k-1} & b_{k-1} & 1 \\ & & & \lambda_k & b_k \end{pmatrix}.$$

By the definition of $\mu^{\leq k}_{n,r,s}(\boldsymbol{b}, \boldsymbol{\lambda})$,

$$\mu^{\leq k}_{n,r,s}(\boldsymbol{b}, \boldsymbol{\lambda}) = \epsilon_r^T \left(A^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})\right)^n \epsilon_s.$$

Combinatorial interpretation

Proposition (Hopkins and Zaimi, 2023)

For $r, s, k, n \in \mathbb{Z}_{\geq 0}$ with $r, s \leq k$ and $n \geq 1$, if $A^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})$ is invertible, then

$$\mu_{-n,r,s}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda}) = \epsilon_r^T \left(A^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})\right)^{-n} \epsilon_s.$$

Theorem (JSSKK, 2023)

Let $b_i = -V_i^{-1}$ for all i. We have

$$\mu_{-n,r,s}^{\leq 3k-1}(\boldsymbol{b}, \boldsymbol{b}^2) = (-1)^{\lfloor r/3 \rfloor + \lfloor s/3 \rfloor} \frac{V_0 \cdots V_s}{V_0 \cdots V_{r-1}} \sum_{\pi \in \mathrm{PV}_{n-1,r,s}^{3,3k-1}} \mathrm{wt}(\pi).$$

Combinatorial interpretation

Corollary (JKKSS, 2023)

We have

$$\left|\mathrm{Mot}_{-n,r,s}^{\leq 3k-1}\right| = \left|\mathrm{PV}_{n-1,r,s}^{3,3k-1}\right|.$$

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We have

$$\left|\mathrm{Mot}_{-n,r,s}^{\leq 3k}\right| = \left|\widetilde{\mathrm{PV}}_{n-1,r,s}^{3,3k}\right|.$$

Reciprocity between determinants

Let $R^{(n)}$ be the operator defined on polynomials in b_i's and λ_i's that replaces each b_i by b_{n-i} and each λ_i by λ_{n+1-i}. We have the general reciprocity theorem as follows.

Theorem (JSSKK, 2023)

For positive integers k and m, we have

$$\det\left(\mu^{\le k+m-1}_{n+i+j+2m-2}(\boldsymbol{b},\boldsymbol{\lambda})\right)_{i,j=0}^{k-1} = C \cdot R^{(k+m-1)}\left(\det\left(\mu^{\le k+m-1}_{-n-i-j}(\boldsymbol{b},\boldsymbol{\lambda})\right)_{i,j=0}^{m-1}\right),$$

where $C = \left(\prod_{i=1}^{k+m-1}\lambda_i^{k-i}\right)\det\left(A^{\le k+m-1}(\boldsymbol{b},\boldsymbol{\lambda})\right)^{n+2m-2}$.

This implies the result of Cigler and Krattenthaler, which is the general reciprocity theorem for Dyck paths version (that is, for $\boldsymbol{b} = \boldsymbol{0}$).

Consequences

We prove Conjectures 50 and 53 of Cigler and Krattenthaler (2020).

Theorem (JKKSS, 2023)

For all nonnegative integers n, k, m, we have

$$\det\left(\sum_{s=0}^{2k+2m-1} \mu^{\leq 2k+2m-1}_{n+i+j+2m-1,0,s}(\mathbf{0},\mathbf{1})\right)_{i,j=0}^{k-1} = (-1)^{\left(\binom{k}{2}+\binom{m}{2}\right)(n+1)} \det\left(\left|\mathrm{Alt}^{k+m}_{n+i+j}\right|\right)_{i,j=0}^{m-1}.$$

Theorem (JKKSS, 2023)

For all positive integers n, k, m with $k+m \not\equiv 2 \pmod 3$, we have

$$\det\left(\mu^{\leq k+m-1}_{n+i+j+2m-2}(\mathbf{1},\mathbf{1})\right)_{i,j=0}^{k-1} = (-1)^{n\lfloor (k+m)/3 \rfloor} \det\left(\mu^{\leq k+m-1}_{-n-i-j}(\mathbf{1},\mathbf{1})\right)_{i,j=0}^{m-1}.$$

Merci !