Statistics on permutation tableaux

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parts based on joint work with Sylvie Corteel (Paris-Sud)
and parts with Svante Janson (Uppsala)

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Permutation tableaux

Permutation tableau T : a Ferrers diagram of a partition λ filled with 0's and 1's such that :

1. Each column contains at least one 1.
2. There is no 0 which has a 1 above it in the same column *and* a 1 to its left in the same row.

0	0	1	0	0	1	1
0	0	1	0	1		
0	1	1	1	1		
0	0	0				
1						

Previous work

- introduced by Postnikov (2001)
- subsequently studied by Williams (2004), Steingrímsson and Williams (2005) (bijections with permutations)
- connections to PASEP (a particle model in statistical physics) Corteel and Williams (2006) and (2007).
- additional combinatorial work Corteel and Nadeau (2007) (more bijections), Burstein (2006) (some properties of permutation tableaux)

Statistics on T

- Length $\ell(T)$: no. rows plus no. columns

0	0	1	0	0	1	1
0	0	1	0	1		
0	1	1	1	1		
0	0	0				
1						

$\ell(T) = 12$

Number of permutation tableaux of length $n = n!$. $\mathcal{T}_n$ is the set of all permutation tableaux with $\ell(T) = n$.

Statistics on T

- Length $\ell(T)$: no. rows plus no. columns
- $U(T)$: number of unrestricted rows (a row is restricted if it has a 0 that has 1 above it)

$\rightarrow$	0	0	1	0	0	1	1
$\rightarrow$	0	0	1	0	1		
$\rightarrow$	0	1	1	1	1		
	0	0	0				
$\rightarrow$	1						

$U(T) = 4$

Statistics on T

- Length $\ell(T)$: no. rows plus no. columns
- $U(T)$: number of unrestricted rows (a row is restricted if it has a 0 that has 1 above it)
- $F(T)$: number of 1's in the first row

0	0	1	0	0	1	1
0	0	1	0	1		
0	1	1	1	1		
0	0	0				
1						

$F(T) = 3$

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- $F(T)$: number of 1's in the first row
- $R(T)$: number of rows

0	0	1	0	0	1	1
0	0	1	0	1		
0	1	1	1	1		
0	0	0				
1						

$R(T) = 5$

Statistics on T

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- $U(T)$: number of unrestricted rows (a row is restricted if it has a 0 that has 1 above it)
- $F(T)$: number of 1's in the first row
- $R(T)$: number of rows
- $S(T)$: number of superfluous 1's (1's below the top one in the column)

0	0	1	0	0	1	1
0	0	1	0	1		
0	1	1	1	1		
0	0	0				
1						

$S(T) = 3$

From tableaux of length $n-1$ to tableaux of length n

Let $T \in \mathcal{T}_{n-1}$ and suppose that it has U_{n-1} unrestricted rows. From the SW corner of the tableau we can extend its length by one by either:

0	0	1	0	0	1	1
0	0	1	0	1		
0	1	1	1	1		
0	0	0				
1						

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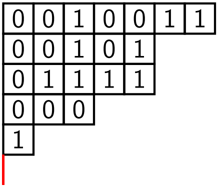

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	0	1	1	1	1		
	0	0	0				
	1						

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- moving S; this adds a row (unrestricted)
- moving W; this adds a column that has to be filled
  - Put zero in resticted rows
  - Put zero or one in unrestricted rows

0	0	0	1	0	0	1	1
0	0	0	1	0	1		
1	0	1	1	1	1		
0	0	0	0				
0	1						

Distribution of the number of unrestricted rows

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Let U_n be the number of unrestricted rows in the extension of T. Elementary calculations based on these earlier observations yield that for $1 \leq k \leq U_{n-1} + 1$

$$P(U_n = k) = \frac{1}{2^{U_{n-1}}} \binom{U_{n-1}}{k-1} = P(\mathrm{Bin}(U_{n-1}) = k - 1).$$

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This means that,

$$\mathcal{L}(U_n | U_{n-1}) = 1 + \mathrm{Bin}(U_{n-1}).$$

Change of measure

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We know $\frac{|\mathcal{T}_{n-1}|}{|\mathcal{T}_n|} = \frac{1}{n}$ but we don't want to use it yet.

Illustration

Theorem: *For every* $n \geq 0$ $|\mathcal{T}_{n+1}| = (n+1)!$.

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Hence, by the change of measure

$$\begin{aligned} E_n 2^{U_n} &= 2E_n \left(\frac{3}{2}\right)^{U_{n-1}} = 2\frac{|\mathcal{T}_{n-1}|}{|\mathcal{T}_n|} E_{n-1} 2^{U_{n-1}} \left(\frac{3}{2}\right)^{U_{n-1}} \\ &= 2\frac{|\mathcal{T}_{n-1}|}{|\mathcal{T}_n|} E_{n-1} 3^{U_{n-1}}. \end{aligned}$$

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This can be iterated and gives

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  - iterate.

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- For superfluous one's: $ES_n = (n-1)(n-2)/12$, $\mathrm{var}(S_n) = (n-2)(2n^2 + 11n - 1)/360$, and

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There is covergence to $N(0,1)$ in the first three cases, too.

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- But, there is no proof *independent* of the bijections between permutation tableaux and permutations. One writes

$$S = \sum_{2 \leq i < j \leq n} I_{\sigma_{i-1} > \sigma_j > \sigma_i}$$

  and proves the Central Limit Theorem for dependent random variables (Janson).

Sample easy proof (unrestricted rows)

For the characteristic function of U_n we have:

$$\begin{aligned} E_n e^{itU_n} &= E_n E\left(e^{itU_n} | U_{n-1}\right) = E_n E\left(e^{it(1+\mathrm{Bin}(U_{n-1}))} | U_{n-1}\right) \\ &= e^{it} E_n \left(\frac{e^{it}+1}{2}\right)^{U_{n-1}} = \frac{e^{it}}{n} E_{n-1} 2^{U_{n-1}} \left(\frac{e^{it}+1}{2}\right)^{U_{n-1}} \\ &= \frac{e^{it}}{n} E_{n-1} \left(e^{it}+1\right)^{U_{n-1}}, \end{aligned}$$

where we have used (in that order) conditioning, distributional properties of U_n, an obvious fact that for a complex number z, $Ez^{\mathrm{Bin}(m)} = \left(\frac{z+1}{2}\right)^m$, and the change of measure.

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where we have used (in that order) conditioning, distributional properties of U_n, an obvious fact that for a complex number z, $Ez^{\text{Bin}(m)} = \left(\frac{z+1}{2}\right)^m$, and the change of measure.
Applying the same procedure to the last expectation, this time with $z = 1 + e^{it}$ we see that

$$E_n e^{itU_n} = \frac{e^{it}(e^{it}+1)}{n(n-1)} E_{n-2} \left(e^{it}+2\right)^{U_{n-2}}.$$

Further iterations yield

$$\begin{aligned} E_n e^{itU_n} &= \left(\prod_{k=0}^{n-2} \frac{e^{it}+k}{n-k}\right) E_1(e^{it}+n-1)^{U_1} = \prod_{k=0}^{n-1} \frac{e^{it}+k}{k+1} \\ &= \prod_{k=1}^{n} (\frac{e^{it}}{k} + 1 - \frac{1}{k}). \end{aligned}$$

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Since the product corresponds to summing independent random variables, we get that the characteristic function of U_n is equal to that of $\sum_{k=1}^{n} J_k$.