Phase Transition for the mixing time of Glauber Dynamics on Regular Trees at Reconstruction: Colorings and Independent Sets.

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JOINT WORK WITH

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Coloring graphs

Given

- A graph $G = (V, E)$ on n vertices and maximum degree Δ
- A set of k colors

A k-coloring of G is an assignment $f : V \to \{1, ..., k\}$ such that

$$\text{for all } (u, v) \in E,\ f(u) \neq f(v)$$

Given a graph with maximum degree Δ

- How to construct k-colorings
  - Trivial for $k > \Delta$
- How to sample (uniformly at) random k-colorings
  - Non-trivial even for $k > \Delta$

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Random Coloring graphs

Why are we interested?

- How to sample (uniformly at) random k-colorings?

- How do random k-colorings look?
- Random colorings is the Gibbs distribution for (zero-temperature) anti-ferromagnetic Potts model.
- Efficient sampler yields approximation algorithm for counting colorings, which is #P-complete.

Glauber dynamics

Let Ω denote the set of all proper k-colorings of G.

Glauber dynamics (heat bath version)

Given $X_t \in \Omega$,

- Take $v \in V$ uniformly at random (u.a.r.)
- Take c u.a.r. from available colors for v in X_t:

$$\mathcal{A}_{X_t} = \{c : c \notin X_t(N(v))\}.$$

- Obtain $X_{t+1} \in \Omega$ by recoloring v to color c.

Glauber dynamics

A natural threshold: $k \geq \Delta + 2$

For $k \geq \Delta + 2$:

- Glauber dynamics is always ergodic.
- The (unique) stationary distribution is uniform over Ω, independently of initial coloring.

$t \to \infty$: distribution of $X_t \to$ uniform dist. on Ω.

- Run chain long enough to get close to uniform state.

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- Some graphs are not even colorable for $k \leq \Delta$.

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Theorem (Hayes and Sinclair 05)

$T_{\text{mix}} = \Omega(n \ln n)$ *for general graphs.*

- Intuitively, time necessary to see all vertices.

Conjecture (folklore)

In general graphs, for $k \geq \Delta + 2$ the mixing time is optimal, i.e.,

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Towards the conjecture (selection of results):

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$$T_{\mathrm{mix}} = O(n \ln n)$$

- $k > 2\Delta$: [Jerrum '95]
- $k > 11\Delta/6$: ($T_{\mathrm{mix}} = O(n^2)$) [Vigoda '99]
- Girth and/or max degree assumptions: [Dyer-Frieze'01], [Molloy'02], [Hayes'03],[Hayes-Vigoda'03],[Frieze-Vera'04],[Dyer-Frieze-Hayes-Vigoda'04]
- Δ-regular trees, any fix boundary: $k \geq \Delta + 3$, [Martinelli-Sinclair-Weitz '04]
- Planar graphs, $k \geq 100\Delta / \ln \Delta$: $T_{\mathrm{mix}} = O^*(n^3)$ [Hayes-Vera-Vigoda '07]
- For Δ-regular trees, $k = C\Delta / \ln \Delta$: $T_{\mathrm{mix}} = n^{\Theta(\min(1,1/C))}$ [Lucier-Molloy '08],[Goldberg-Jerrum-Karpinski '08]

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- What happens below $\Delta/\ln\Delta$?

Goal: Get detailed picture on trees.

- Better understanding for planar and sparse random graphs.

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Outline

Significance of $\Delta / \ln \Delta$:

Consider the complete tree with branching factor Δ and height h.

Recall

For $k = \Delta + 1$ there are colorings that “freeze” the root

- Colors of leaves determine color of root

- But this is not true for “typical” colorings

Question

For which values of k does a random coloring of the leaves freeze the root?

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Reconstruction

Generating a random coloring of the tree: Broadcasting model

1. Choose a random color for the root, call it $\sigma(r)$.
2. For each vertex v, given the color of its parent $\sigma(p(v))$, choose a random different color.

Reconstruction

Reconstruction holds if the leaves have a non-vanishing (as $h \to \infty$) influence on the root in expectation.

$$\lim_{h \to \infty} \mathrm{E}_{\sigma_L}\left[\left|\mu(\tau(r)|\tau(L) = \sigma_L) - \frac{1}{k}\right|\right] > 0.$$

- Given (random) coloring of leaves can guess color of root

Reconstruction threshold

Threshold is at $\approx \Delta / \ln \Delta$ [Sly'08, Bhatnagar-Vera-Vigoda'08]

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Connections of reconstruction to the efficiency of local algorithms on trees and tree-like graphs

- $T_{\text{mix}} = O(n \ln n)$ on the complete tree implies non-reconstruction [Berger-Kenyon-Mossel-Peres '05]
- "Clustering of solution space" in reconstruction region for several constraint satisfaction problems, including colorings, on sparse random graphs [Achlioptas,Coja-Oghlan '08]

We prove:

Mixing time of the Glauber dynamics for random colorings of the complete tree undergoes a phase transition. Critical point appears to coincide with the reconstruction threshold.

Main Result

Theorem

Let $k = C\Delta / \ln \Delta$. There exists Δ_0 such that, for all $\Delta > \Delta_0$, the Glauber dynamics on the complete Δ-tree on n vertices satisfies:

1. **For $C \geq 1$:**

$$\Omega(n \ln n) \leq T_{\text{mix}} \leq O\left(n^{1+o_\Delta(n)} \ln^2 n\right)$$

2. **For $C < 1$:**

$$\Omega\left(n^{1/C+o_\Delta(n)}\right) \leq T_{\text{mix}} \leq O\left(n^{1/C+o_\Delta(n)} \ln^2 n\right)$$

Next

Ideas on lower bound for reconstruction region ($C < 1$)

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Lowerbound for $C < 1$

Usually reconstruction is proven via a *reconstruction algorithm*

Reconstruction Algorithm

Function $A : \Omega_L \to \{0, 1\}$ (ideally efficiently computable)

- For any σ, $A(\sigma_L)$ and $\sigma(r)$ are positively correlated.
- Assume: when coloring of L freezes the root, A gives correct answer

Given reconstruction algorithm A

- Let
$$S_c = \{\sigma \in \Omega : A(\sigma_L) = c\}$$
- $S_c \supseteq \{\sigma \in \Omega : \sigma_L \text{ freezes } r \text{ to } c\}$.

Intuitive key Idea:

Under reconstruction: If initial coloring in S_R it is "difficult" to get to coloring in S_B.

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Under reconstruction: If initial coloring in S_R it is ”difficult” to get to coloring in S_B.

Conductance

Let $S \subseteq \Omega$ and $\bar{S} = \Omega \setminus S$. Define

$$\Phi_S = \frac{\sum_{\sigma \in S} \sum_{\eta \in \bar{S}} \pi(\sigma) P(\sigma, \eta)}{\pi(S)\pi(\bar{S})}$$

- Related to probability of escaping from S in one step

Theorem (Lawler-Sokal ’88. Sinclair-Jerrum ’89)

For all $S \subseteq \Omega$ $$T_{\text{mix}} \geq \Omega(1/\Phi_S)$$

Formalized Key idea

Show S_c has small conductance

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Relation between reconstruction and conductance

Goal

Show S_c has small conductance

Theorem

Under reconstruction, for any reconstruction function A,

$$\Phi_{S_c} = O\left(\mathrm{E}_\sigma\left[\Psi_A(\sigma) \middle| \sigma \in S_c\right]\right)$$

where

$$\Psi_A(\sigma) = \#\{v \in L : \exists d \in [k]\ A(\sigma^{v,d}) \neq A(\sigma)\}.$$

- Sensitivity of A at σ: For how many leaves, changing color of leaf will change outcome of A.

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Lowerbound for $C < 1$

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Actually, let $S = \cup_{c<k/2} S_c$.

- [Goldberg, Jerrum, and Karpinski - 08] For $0 < C < 1/2$

$$\Phi_S = O(n^{-\frac{1}{6C}})$$

- We prove for $C < 1$

$$\Phi_S = O^*(n^{-1/C})$$

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Does a similar phenomenon hold for independent sets?

No, more interesting phenomenon occurs.

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Hard-core model

- Graph $G = (V, E)$ with n vertices and maximum degree Δ.
- Independent set is a subset $S \subset V$ where for all $(v, w) \in E$, either $v \notin S$ and/or $w \notin S$.
- Activity (or fugacity) $\lambda > 0$.
- Hard-core distribution (i.e., Gibbs measure): $\mu(S) \sim \lambda^{|S|}$.

Reconstruction threshold for the hard-core model

Consider the complete tree with branching factor Δ and height h.
Let ω be the solution to $\lambda = \omega(1+\omega)^{\Delta}$.

Broadcasting model:

1. Occupy the root with probability $p = \omega/(1+\omega)$ and leave it unoccupied with $1-p$.
2. For each vertex v, if the parent is unoccupied, occupy v with probability p.

Reconstruction is said to hold if the leaves have a non-vanishing (as $h \to \infty$) influence on the root in expectation:

$$\lim_{h\to\infty} \mathrm{E}_{\sigma_L}\left[\left|\mu(r \in \tau | \tau(L) = \sigma_L) - \frac{\omega}{1+\omega}\right|\right] > 0.$$

Reconstruction threshold and mixing of the Glauber dynamics?

- Reconstruction threshold: $\omega_r \approx \frac{\ln \Delta + \ln \ln \Delta}{\Delta}$
  [Bhatnagar-Sly-Tetali '10],[Brightwell-Winkler '04]
- Rapid mixing for free boundary: For the complete tree on n vertices, $T_{\text{mix}} = O(n \log n)$ for all λ [Martinelli-Sinclair-Weitz '04]

So, no slow down at reconstruction?

- Free boundary does not correspond to the broadcast process for the hard-core model.
  - it does for colorings.
- There exist boundary conditions with a slow down at reconstruction.

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Results for hard-core model:

Theorem

For the Glauber dynamics on the hard-core model with activity $\lambda = \omega(1+\omega)^\Delta$ on the complete Δ-tree with n vertices:

1. **For all** $\omega \leq \ln \Delta/\Delta$**:** $\Omega(n) \leq T_{\mathrm{rel}} \leq O^*(n)$.
2. **For all** $\delta > 0$ **and** $\omega = (1+\delta)\ln \Delta/\Delta$**:**
   1. *For every boundary condition,*
   $$T_{\mathrm{rel}} \leq O^*(n^{1+\delta}).$$
   2. *Exists a sequence of boundary conditions with* $h \to \infty$ *such that,*
   $$T_{\mathrm{rel}} \geq \Omega^*(n^{1+\delta/2}).$$

Current and Future Work

- Similar analysis of other CSPs (spin systems).
  - e.g. k-SAT
- Analysis of more general graphs.
- Poisson tree closely related to sparse random graph $G(n, d/n)$.
  - For constant d, k, $k \geq \mathrm{poly}(d)$, $T_{\mathrm{mix}}(Col) = \mathrm{poly}(n)$ whp. [Mossel-Sly '08]
  - Open: Prove “rapid mixing” down to $d/\ln d$ colors.
- Explore more general relation between reconstruction and ”local algorithms”