States on a star algebra and semi-simplicity.

From combinatorics of universal problems to usual applications.

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Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
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Goal of this talk

From **nLab**. –

nLab says (see ncatlab "States on a star algebra"
in the Links section 9)

The concept of **state** on a star-algebra is the formalization of the general idea of states from the point of view of quantum probability theory and algebraic quantum theory.
In order to motivate the definition from more traditional formulations in physics, recall that there a state $\langle - \rangle$ is the information that allows to assign to each observable A the expectation value $\langle A \rangle$ that this observable has when the physical system is assumed to be in that state.
Often this is formalized in the Schrödinger picture where a Hilbert space of states $\mathcal{H}$ is taken as primary, and the observables are represented as suitable linear operators A on $\mathcal{H}$. Then for $\psi \in \mathcal{H}$ a state (pure state) the expectation value of A in this state is the inner product

$$\langle \psi | A | \psi \rangle = \langle \psi | A.\psi \rangle$$

Goal of this talk/2

nLab cont'd

This defines a linear function $\langle\psi| - |\psi\rangle$

$$\langle\psi|(-)|\psi\rangle \;:\; \mathcal{A} \longrightarrow \mathbb{C}$$

on the algebra of observables $\mathcal{A}$, satisfying some extra properties.

Evolution

1. The states can evolve (a function of $t \in I$ (real interval) or $z \in \Omega$ (a complex domain) and be (in many cases) of the form

$$\langle\psi|\mathcal{O}|\psi\rangle$$

   where $\mathcal{O}$ is a solution of an **evolution equation**.
2. In Hilbert-based models $\mathcal{O}$ is unitary and ψ is of norm one. Let us examine the linear form $\varphi : \mathcal{O} \mapsto \langle\psi|\mathcal{O}|\psi\rangle$ obtained in this case.

Towards states

Analysis

1. We start with a star-algebra (we will see examples later on) i.e. a $\mathbb{C}$-AAU equipped with a semi-linear, involutive anti-automorphism $x \mapsto x^*$.
2. In other words, identically
   - $(x+y)^* = x^* + y^*$
   - $(\alpha.x)^* = \overline{\alpha}.x^*$
   - $(x.y)^* = y^*.x^*$
   - $(x^*)^* = x$
3. Examples
   1. $\mathcal{M}(n,\mathbb{C}) = \mathbb{C}^{n\times n}$ with $M^* :=^t (\overline{M})$
   2. $\mathcal{C}(\Omega,\mathbb{C})$ with $f^* := \bar{f}$
   3. $\mathbb{C}\langle\langle X\rangle\rangle$ with $(\sum_{w\in X^*} \alpha_w\, w)^* := (\sum_{w\in X^*} \overline{\alpha_w}\, \tilde{w})^*$
   4. $\mathbb{C}[G]$ with $(\sum_{g\in G} \alpha_g\, g)^* := \sum_{g\in G} \overline{\alpha_g}\, g^{-1}$

Towards states/2

Analysis/2

4. With $||\psi|| = 1$, we have $\varphi : \ \mathcal{O} \mapsto \langle\psi|\mathcal{O}|\psi\rangle$ with the following properties
$$\varphi(1) = 1 \text{ and, (for all } x) \ \varphi(x^*.x) \geq 0$$
5. If, moreover, $\varphi(x^*.x) = 0 \Longrightarrow x = 0$, the state is said faithful. We the preceding examples (F=faithful, NF=non faithful)
   1. $\mathcal{M}(n, \mathbb{C})$ with $\varphi(M) = (1/n).tr(M)$ (F)
   2. With $a < b$, $\mathcal{A} = \mathcal{C}([a, b], \mathbb{C})$ with $\frac{1}{b-a}\int_a^b f(s)ds$ (F)
   3. $\mathbb{C}\langle\langle X\rangle\rangle$ avec $\varphi(S) = \langle S|1_{X^*}\rangle$ (NF)
   4. $\mathbb{C}[G]$ avec $\varphi(S) = \langle S|1_G\rangle$ (F)

Semisimple categories of modules/1

- Semi-simple categories of modules, see in general
  https://ncatlab.org/nlab/show/semisimple+category

Definition

Let R be a ring. We note R-**Mod** the category of R-modules (whatever the size) the arrows being that of R-linear mappings between objects.

Remarks

1. This is a category with direct sums (coproducts) and products.
2. Subcategory of finite length modules (ex. finite dim when R is a **k** algebra) admit (finite) decompositions (Krull) in indecomposables. Another example will be subcategory of semi-simple modules (see below).
3. In the preceding case (finite dim when R is a **k** algebra) it is a subcategory
4. Link with non-degenerate bilinear forms + examples

Semisimple categories of modules/2

Definition: Simple and semi-simple modules

1. A module $M \in R$-**Mod** is said simple if it is not (0) and if its set of submodules is $\{(0), M\}$
2. A module $M \in R$-**Mod** is said semi-simple iff $M = \oplus_{i\in I} M_i$ where M_i are simple submodules of M.

Proposition [A]

Let $M \in R$-**Mod**

1. If M is such that $M = \sum_{i\in I} M_i$ where M_i are simple submodules of M and $N \subset_{submod} M$, then it exists $J \subset I$ such that $M = (\oplus_{i\in J} M_i) \oplus N$.
2. In particular a submodule or a quotient of a semi-simple module is semi-simple.

Proof

- A.1) Let $\mathfrak{S} \subset 2^I$ defined on 2^I

$$\mathfrak{S} = \{J \subset I | (\oplus_{i \in J} M_i) \oplus N \text{ is well defined}\} \tag{1}$$

  The set of $\mathfrak{S}$ is non-empty and of finite character. Then, by Tukey-Teichmüller theorem it admits at least a maximal element foor inclusion. Let J_0 be such an element. If $J_0 = I$ we are done, otherwise let $i \in I \setminus J_0$ and set $T = \big((\oplus_{i \in J_0} M_i) \oplus N\big)$. We cannot have $M_i \cap T = (0)$ otherwise we would get $J_0 \cup \{i\} \in \mathfrak{S}$ and $i \in J_0$, a contradiction. Remains $M_i \subset T$ because M_i is simple. Hence $(\forall i \in I \setminus J_0)(M_i \subset T)$ and this entails $M = T$.
  Remark that, setting N to (0), one obtains that if a module is a sum (direct or not) of simple submodules, then it is semi-simple.

- A.2) We suppose $M = \oplus_{i \in I} M_i$ to be semi-simple. Let $f : \ M \twoheadrightarrow Q$, . Setting $N = ker(f)$ in the preceding situation, we get a subfamily $(M_i)_{i \in J}$ such that $M = (\oplus_{i \in J} M_i) \oplus N$. Then, by f, $(\oplus_{i \in J} M_i) \simeq Q$ and we are done. Now, if N is any submodule of M, by (A.1), it is direct summand and we can write $M = N \oplus N_1$ with projectors p_N, p_{N_1}. From $p_N : \ M \twoheadrightarrow N$ we are done.

Case when R_s itself is semi-simple

Any ring R can be considered as a $R - R$ bimodule by the left and right actions (for $a, b \in R$), $\lambda_a(m) = a.m,\ \rho_b(m) = m.b.$ these two actions commute. By definition R_s is the left-module defined by the action $\lambda_a(m)$. We have the following

Proposition [B]

If R_s is semi-simple, all R-module is so.

Proof.

We suppose that R_s is semi-simple. Let M be a R-module, then for all $x \in M$ the (principal) R-submodule $R.x$ is a semi-simple image (that of $t \to t.x$), hence semisimple. The result is then a consequence of Proposition [A].1 in view of the fact that $M = \sum_{x \in M} R.x$. □

A sufficient condition for R_s to be semi-simple

Proposition [C]

Under the preceding conditions

1. If R_s is semi-simple then every left ideal is direct summand of R_s within the lattice of left ideals.
2. The converse is true in the case when this lattice[a] satisfies ACC+DCC chain conditions.

   `https://en.wikipedia.org/wiki/Ascending_chain_condition`

[a] The lattice of left ideals.

For hilbertian traces, see Dieudonné XV.6 [4].
In the category of modules, ACC is Noetherian, DCC is Artinian.
Next steps: Frobenius characteristics, characters, case of finite groups, the symmetric group, Kronecker, Littlewood-Richardson and Clebsch–Gordan coefficients.

Proof of Proposition [C]

1) In fact this is true of every semi-simple module by Proposition [A].1.
2) As in (1), this converse is true for every module satisfying the same conditions (i.e. every submodule is direct summand + ACC + DCC). Let M be such a module, we build the following double sequence

1. (Init.) $C_0 = ((0), M)$
2. (Running) $C_n = (\oplus_{i=1}^{n} N_i, Q_n)$ with N_i simple submodules of M and $\oplus_{i=1}^{n} N_i \oplus Q_n = M$
3. (Halt) $Q_n = (0)$ (then we are done)
4. (Next Step) Suppose $C_n = (\oplus_{i=1}^{n} N_i, Q_n)$ with $Q_n \neq (0)$ (non-halting step) then we choose a minimal submodule Q_{min} of M among those such that $(0) \subsetneq Q \subset Q_n$ (it is possible because M satisfies DCC). We set $N_{n+1} = Q_{min}$ and remark that the family $(N_i)_{1 \leq i \leq n+1}$ is in direct sum and, by hypothesis, it exists Q_{n+1} such that $\oplus_{i=1}^{n+1} N_i \oplus Q_{n+1} = M$ then set $C_{n+1} = (\oplus_{i=1}^{n+1} N_i, Q_{n+1})$

Proof of Proposition [C]/2 and first applications

- **Proof that this algorithm halts**) unless $M = (0)$ there is at least one step. Let $n+1$ be any valid rank of a step. By construction $\oplus_{i=1}^{n} N_i \subsetneq \oplus_{i=1}^{n+1} N_i$, a strictly increasing sequence of submodules. By ACC this sequence must be finite.
- **Semi-simplicity**) Let m is the last index of the sequence C_n. We have $Q_m = (0)$ and then $M = \oplus_{i=1}^{m} N_i$. CQFD

Finite groups, states and semi-simplicity

1. **Applies to** Every finite dimensional $*$-algebra which admits a faithful state (FS) then is semi-simple. See below.
2. **and in particular to** $\mathcal{A} = \mathbb{C}[G]$ where G is a finite group. With

$$\big(\sum_{g\in G} \alpha(g)\, g\big)^* := \big(\sum_{g\in G} \overline{\alpha(g)}\, g^{-1}\big)$$

and $\varphi(Q) = \langle 1_G | Q\rangle$

States, isometries, orbits and orthogonality in star-algebras.

- First of all, note that spectral theory fails dramatically in the absence of a complete norm [a].
- Let $\mathcal{A}$ be an $*$-algebra ($x \to x^*$ is semi-linear, involutive and an anti-automorphism)
- $\mathcal{C}_+(\mathcal{A})$, generated by elements of the form $\sum_{i\in F} x_i x_i^*$ (with F finite) is an hermitian (self-dual) convex cone
- $State(\mathcal{A})$ is the set of linear forms $f \in \mathcal{A}^*$ such that $z \in \mathcal{C}_+(\mathcal{A}) \Longrightarrow f(z) \geq 0$ and $f(1) = 1$
- A faithful state (FS) is such that

$$z \in \mathcal{C}_+(\mathcal{A}) \text{ and } f(z) = 0 \Longrightarrow z = 0$$

[a] See discussion in https://math.stackexchange.com/questions/1520974

cont'd/1

Proposition [D]

A finite dimensional star-algebra with a non-degenerate state is semi-simple with all its commutants $\simeq \mathbb{C}$.

Unfolding [D]

1. Let φ be one of these faithful states and set

$$\langle x|y\rangle = \varphi(x^* y) \tag{2}$$

it is a non-degenerate hermitian form such that, identically $\langle x|a.y\rangle = \langle a^*.x|y\rangle$. Then, if $\mathcal{J}$ is a left-ideal of the algebra $\mathcal{A}$, then it is easy to prove that $\mathcal{J}^\perp$ is a left ideal

2. In particular with the preceding setting ($\mathcal{A} = \mathbb{C}[G]$ where G is a finite group, star-structure and state) we have the result.

3. We decompose $\mathcal{A}$ into minimal left ideals with algorithm of slide 11 and get $\mathcal{A} = \oplus_{j\in F}\mathcal{J}_j$. We can then write $1_\mathcal{A} = \sum_{i\in F} p_i$.

Construction of the matrix units.

4. One can prove that $\mathcal{J}_i = \mathcal{A}\, p_j$ and $p_j p_i = p_i p_j = \delta_{ij} p_i$ (complete orthogonal family of minimal idempotents)
5. The lemma $Hom_{\mathcal{A}}(\mathcal{A}.e, \mathcal{A}.f) \simeq e.\mathcal{A}.f$ (sandwich) gives Wedderburn's decomposition.
6. For e, f idempotents then $eaf \to (x \to xeaf)$ is an iso of **k**-spaces between $e.\mathcal{A}.f$ and $Hom_{\mathcal{A}}(\mathcal{A}.e, \mathcal{A}.f)$ the inverse being $f \to f(e)$ (note that $f(e) \in e.\mathcal{A}.f$).
7. Return to $1_{\mathcal{A}} = \sum_{i \in F} p_i$ (each p_i is minimal) and set $i \sim j \Longleftrightarrow e_i.\mathcal{A}.e_j \neq (0)$ (block equivalence)
8. Take a block C, order $C = \{i_1 < i_2 < \cdots < i_m\}$ totally.
9. For $1 \leq j < m$ choose $a_{i_j} \in e_{i_j}.\mathcal{A}.e_{i_{j+1}} \setminus (0)$
10. **Remark** that 4 to 9 applies to every finite dimensional F-semi-simple algebras, in particular, with $F = \mathbb{R}$ you can get **Real**, **Complex** and **Quaternion** blocks.

Classification of finite-dim. semi-simple F- and $\mathbb{R}$-algebras

Wedderburn-Artin Theorem [1], p116 and [18].

Let F be a field and $\mathcal{A}$ a finite-dimensional F-algebra, then $\mathcal{A}$ is semi-simple iff it exists numbers $(n_i)_{1\leq i\leq r}$ and $(D_i)_{1\leq i\leq r}$ finite-dimensional F-division algebras such that

$$\mathcal{A} \simeq D_1^{n_1\times n_1} \times \cdots \times D_r^{n_r\times n_r} \tag{3}$$

Classification of finite-dim. semi-simple $\mathbb{R}$-algebras

As there are only three types of finite-dim. $\mathbb{R}$-division algebras i.e. $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, we have a complete description of the finite-dim. semi-simple $\mathbb{R}$-algebras: you have **Real**, **Complex** and **Quaternion** blocks.

Classification of finite-dim. semi-simple $\mathbb{C}$-algebras

In this case you have only **Complex** blocks.

Case of an *-algebra ($\mathbf{k} = \mathbb{C}$) with a (FS).

11. Let now $\mathcal{A}$ be a finite-dimensional *-algebra (over $\mathbb{C}$) with a (FS) φ.
12. From (2), we get that $\mathcal{A}$ is semi-simple.
13. On the other hand, one checks that the left-regular representation constructed by $\gamma_a(x) = a.x$ and $\rho(a) = \gamma_a$ is a *-faithful *-representation of $\mathcal{A}$
14. $\mathcal{A}$ can be given, through this faithful representation the structure of C*-algebra (not in general with the norm $||x||_1 = \sqrt{\langle x|x \rangle}$ though[a])
15. If $\mathcal{A}$ has the structure of C*-algebra for the norm $||x||_1$ then $\mathcal{A} \simeq \mathbb{C}$.
16. Proof of ball 14 (Sketch). – Using the decomposition (3) through orthogonal ideal process, we get that the p_i are self-adjoint, then if $|F| > 1$ we can set $F = F_1 \sqcup F_2$ with $|F_i| > 0$, then with $e_i = \sum_{j \in F_i} p_j$, we get two self-adjoint idempotents and $||e_i|| = ||e_1 + e_2|| = ||e_1 - e_2|| = 1$ makes the parallelogram rule[b] fail.

[a] If it were the case, one can prove that $dim_{\mathbb{C}}(\mathcal{A}) = 1$.
[b] $||x + y||^2 + ||x - y||^2 = 2.(|x||^2 + ||y||^2)$.

A remark

Remark[a]. – The fact that A be a star-algebra of finite dimension, sum of matrix algebras is by no means sufficient to imply that the projectors on the blocks are *-invariant nor $A \simeq \mathbb{C}$ as shows the following counterexample. Take $B = \mathbb{C}^{n \times n}$ (algebra of complex square matrices of dimension $n > 0$) and $A = B \oplus B$ with the anti-automorphism $(X, Y)^\star = (Y^*, X^*)$. Then $(A, \star)$ is easily checked to be a star algebra (i.e. involutive algebra). It is of finite dimension, sum of matrix algebras but $dim_{\mathbb{C}} = 2n^2 \neq 1$. Indeed, the existence of a faithful state is crucial as there is none over A.

[a] For a counterexample with words, see `https://math.stackexchange.com/questions/829182`.

Exercises/1

Ex1: States and pre-states

Let $\mathcal{A}$ be a complex finite-dimensional $*$-algebra. A FS (Faithful State) is a linear form $\varphi \in \mathcal{A}^*$ such that

$$(\forall x \in \mathcal{A} \setminus \{0\})(\varphi(x^* x) > 0) \tag{4}$$

1. Prove that the bilinear form $\langle x|y\rangle := \varphi(x^* y)$ is a non-degenerate hermitian scalar product[a] such that, identically

$$\langle x|a.y\rangle = \langle a^*.x|y\rangle$$

2. Prove that a complex finite-dimensional $*$-algebra admitting a FS is semi-simple.

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[a] I take the convention of semi-linearity on the left (see the link "Hilbert modules" below).

Exercises/2

Ex1: States and pre-states/2

Let G be a finite group, set $\mathcal{A} = \mathbb{C}[G]$ and, for $a = \sum_{g \in G} \alpha(g)\, g$, set $a^* = \sum_{g \in G} \overline{\alpha(g)}\, g^{-1}$

3. Prove that $(\mathcal{A}, *)$ is an $*$-algebra
4. Prove that $\varphi \in \mathcal{A}^*$ defined by $\varphi(a) = \alpha(1)$ is a FS on $\mathcal{A}$.

Links

1. Categorical framework(s)

   https://ncatlab.org/nlab/show/category
   https://en.wikipedia.org/wiki/Category_(mathematics)

2. Universal problems

   https://ncatlab.org/nlab/show/universal+construction
   https://en.wikipedia.org/wiki/Universal_property

3. Paolo Perrone, *Notes on Category Theory with examples from basic mathematics*, 181p (2020)
   arXiv:1912.10642 [math.CT]

   https://en.wikipedia.org/wiki/Abstract_nonsense

4. Heteromorphism

   https://ncatlab.org/nlab/show/heteromorphism

5. D. Ellerman, *MacLane, Bourbaki, and Adjoints: A Heteromorphic Retrospective*, David EllermanPhilosophy Department, University of California at Riverside

Links/2

6. https://en.wikipedia.org/wiki/Category_of_modules
7. https://ncatlab.org/nlab/show/Grothendieck+group
8. Traces and hilbertian operators
   https://hal.archives-ouvertes.fr/hal-01015295/document
9. State on a star-algebra
   https://ncatlab.org/nlab/show/state+on+a+star-algebra
10. Hilbert module
   https://ncatlab.org/nlab/show/Hilbert+module

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Thank you for your attention !