

Deformations and coloured products.

From combinatorics of universal problems
to usual applications.

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Collaboration at various stages of the work
and in the framework of the Project

Evolution Equations in Combinatorics and Physics :

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CIP seminar 2021

Opening session

Goal of this talk

The goal of this talk is threefold

A bit of category theory: How to construct free objects w.r.t. a functor and two routes to reach the free algebra.

Representation theory: Categories of modules, semi-simplicity, isomorphism classes i.e. the framework of Kronecker coefficients

MRS factorisation: A local system of coordinates for Hausdorff groups

Bits and pieces of representation theory

and how bialgebras arise

Wikipedia says

Representation theory is a branch of mathematics that studies abstract algebraic structures by representing their elements as linear transformations of vector spaces, .../...

The success of representation theory has led to numerous generalizations. One of the most general is in category theory.

As our track is based on Combinatorial Physics and Experimental/Computational Mathematics, we will have a practical approach of the three main points of view

- Algebraic
- Geometric
- Combinatorial
- Categorical

Matters

1 Representation theory or theories

- 1 Geometric point of view
- 2 Combinatorial point of view (Ram and Barcelo manifesto)
- 3 Categorical point of view

2 From groups to algebras

Here is a bit of rep. theory of the symmetric group, deformations, Wedderburn and idempotents

3 Irreducible and indecomposable modules

4 Characters, central functions and shifts

5 Reductibility and invariant inner products

*Here stands **Joseph's result***

6 Commutative characters

*Here are time-ordered exponentials, iterated integrals, evolution equations and **Minh's results***

7 Lie groups Cartan theorem

*Here is **BTT***

Universal problem w.r.t. a functor

Free structures and objects

- 1 Let \mathcal{C}_{left} , \mathcal{C}_{right} be two categories and $F : \mathcal{C}_{right} \rightarrow \mathcal{C}_{left}$ a (covariant) functor between them

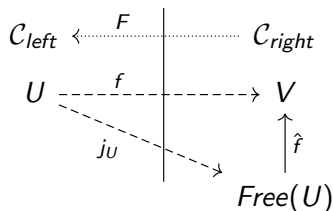


Figure: A solution of the universal problem w.r.t. the functor F is the datum, for each $U \in \mathcal{C}_{left}$, of a pair $(j_U, Free(U))$ (with $j_U \in Hom(U, F[Free(U)])$, $Free(U) \in \mathcal{C}_{right}$) such that, for all $f \in Hom(U, F[V])$ it exists a unique $\hat{f} \in Hom(Free(U), V)$ with $F[\hat{f}] \circ j_U = f$. Elements in $Hom(U, F[V])$ are called heteromorphisms their set is noted $Het_F(U, V)$.

$$(\forall f \in Hom(U, F[V])) (\exists! \hat{f} \in Hom(Free(U), V)) (F[\hat{f}] \circ j_U = f)$$

The pair $U \rightarrow \text{Free}(U)$ is, in fact, a functor.

Which, in turn, will prove to be left-adjoint to F

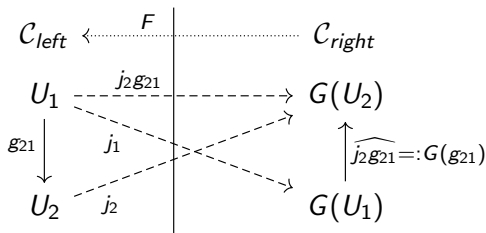


Figure: Making a free functor $G (= \text{Free})$ from F : for any morphism $g_{21} \in \text{Hom}(U_1, U_2)$, $G(g_{21})$ is the unique morphism in $\text{Hom}(G(U_1), G(U_2))$ such that

$$F[G(g_{21})] \circ j_1 = j_2 g_{21} \quad (**)$$

We now prove that G is a functor.

- If $U_1 = U_2$ and $g_{21} = \text{Id}_{U_1}$, then $j_1 = j_2 = j_2 g_{21}$ and $F[\text{Id}_{G(U_1)}] \circ j_1 = j_1 = j_2 g_{21}$ hence $G[\text{Id}_{U_1}] = \text{Id}_{G(U_1)}$
- **A remark:** $\text{Het}(?, ?)$ is intended to give a symmetric middle term/step to the adjunction chain $\text{Hom}(U, F[V]) =: \text{Het}_F(U, V) \simeq \text{Het}^G(U, V) := \text{Hom}(G(U), V) \simeq$ being constructed by a set of bijections.

Functor G from $Free/2$

- Let now $U_1 \xrightarrow{g_{21}} U_2 \xrightarrow{g_{32}} U_3$ be a chain of C_{left} -morphisms.
We have

$$F[G(g_{21})] \circ j_1 = j_2 \circ g_{21} \text{ and } F[G(g_{32})] \circ j_2 = j_3 \circ g_{32}$$

then

$$\begin{aligned} F[G(g_{32}) \circ G(g_{21})] \circ j_1 &\stackrel{(1)}{=} F[G(g_{32})] \circ F[G(g_{21})] \circ j_1 \stackrel{(2)}{=} \\ F[G(g_{32})] \circ j_2 \circ g_{21} &\stackrel{(3)}{=} j_3 \circ g_{32} \circ g_{21} \end{aligned}$$

(1) because F is a functor, (2) is Eq. (**)
(3) is Eq. (**)
applied to indices 21,
applied to indices 32.

Now, we know that $g \in Hom(U, U')$ being given, the solution
 $X \in Hom(G(U), G(U'))$ of

$$F[X] \circ j_1 = j_2 \circ g$$

is unique. Then $G(g_{32}) \circ G(g_{21}) = G(g_{32} \circ g_{21})$



Composition of functors F and G

Piling free structures.

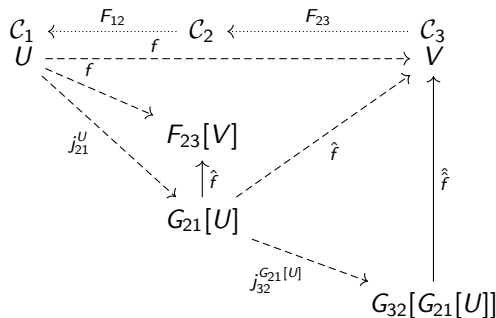
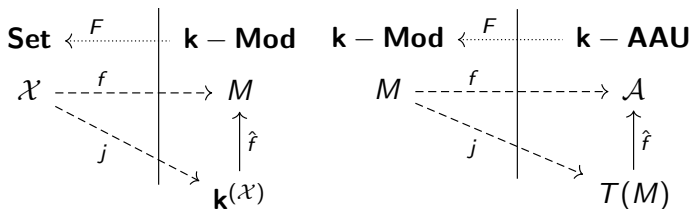
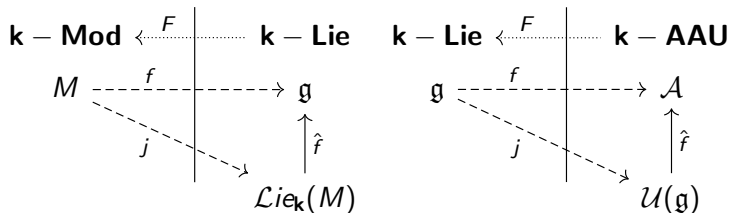


Figure: $[F_{12}[j_{32}^{G_{21}[U]}], G_{32}[G_{21}[U]]]$ is a solution of the universal problem for $F_{12}F_{23}$.

Proof: In fact, $Het_{F_{12}F_{23}}(U, V) = Hom(U, F_{12}F_{23}[V]) = Het_{F_{12}}(U, F_{23}[V])$, hence existence of $\hat{f} \in Hom(G_{21}[U], F_{23}[V]) = Het_{F_{23}}(G_{21}[U], V)$, hence again existence of \hat{f} . Uniqueness of \hat{f} is left to the reader.

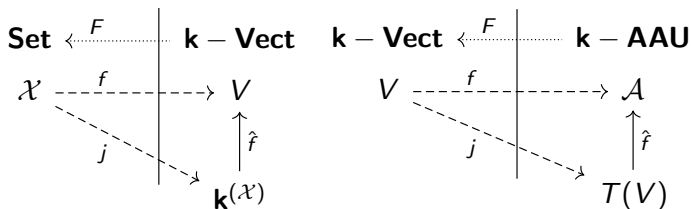
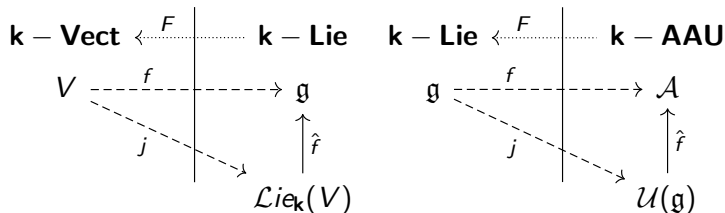
First example: $T = UL$.



$$T(M) = \mathcal{U}(\mathcal{L}ie_{\mathbf{k}}(M))$$

$$\mathbf{k}\langle \mathcal{X} \rangle = T(\mathbf{k}\langle \mathcal{X} \rangle)$$

First example: $T = UL$, \mathbf{k} field.



$$T(V) = \mathcal{U}(\text{Lie}_{\mathbf{k}}(V))$$

$$\mathbf{k}\langle \mathcal{X} \rangle = T(\mathbf{k}\langle \mathcal{X} \rangle)$$

An immediate (and although rich) example/1

Piling free structures/2

- 1 First, $\mathcal{C}_1 = \mathbf{Set}$ (sets and maps) and $\mathcal{C}_2 = \mathbf{Mon}$ (monoids and morphisms) gives you the triple $(\mathcal{X}, j_{21}, \mathcal{X}^*)$

Usually \mathcal{X} , a set, is seen as an *alphabet* that is to say a *set of non commuting variables*. Let us introduce the ring \mathbf{k} of coefficients

- 2 With $\mathcal{C}_2 = \mathbf{Mon}$ (monoids and morphisms) and $\mathcal{C}_3 = \mathbf{k-AAU}$ (\mathbf{k} -associative algebras with unit), one gets $\mathbf{k}[M]$ the algebra of a monoid M , we get the triple $(M, j_{32}, \mathbf{k}[M])$ and,
- 3 by transitivity of free objects with $\mathcal{C}_1 = \mathbf{Set}$ (sets and maps) and \mathcal{C}_3 as above, we get the triple $(\mathcal{X}, j_{31}, \mathbf{k}\langle\mathcal{X}\rangle)$, $\mathbf{k}\langle\mathcal{X}\rangle = \mathbf{k}[\mathcal{X}^*]$ being the algebra of noncommutative polynomials.
- 4 we immediately obtain that $\mathbf{k}\langle\mathcal{X}\rangle = \mathbf{k}[\mathcal{X}^*]$ is free with $\{w\}_{w \in \mathcal{X}^*}$ (this will be useful for the principal pairing)

An immediate (and although rich) example/2

- 5 let us observe here that $\mathbf{k}\langle\mathcal{X}\rangle$ can be reached, instead of

$$[\mathbf{Set}] \longrightarrow [\mathbf{Mon}] \longrightarrow [\mathbf{k} - \mathbf{AAU}]$$

by another path, and this will provide a host of other very interesting (combinatorial) bases.

- 6 the preceding route amounts to the formula $\mathbf{k}\langle\mathcal{X}\rangle = \mathbf{k}[\mathcal{X}^*]$, but it can be also proved that $\mathbf{k}\langle\mathcal{X}\rangle = \mathcal{U}(\mathcal{L}ie_{\mathbf{k}}\langle\mathcal{X}\rangle)$

$$[\mathbf{Set}] \longrightarrow [\mathbf{k} - \mathbf{Lie}] \longrightarrow [\mathbf{k} - \mathbf{AAU}]$$

An immediate (and although rich) example/3

Piling free structures and dual bases

- 7 from the first (obvious) way (sets to monoids to \mathbf{k} -AAU) we got the basis $\{w\}_{w \in \mathcal{X}^*}$ which provides the fine grading of $\mathbf{k}\langle \mathcal{X} \rangle$. indeed to each word $w \in \mathcal{X}^*$, we can associate the family

$$\beta(w) = (|w|_x)_{x \in \mathcal{X}} \in \mathbb{N}^{(\mathcal{X})}$$

- 8 therefore, due to this partitioning of the basis (of words), we get

$$\mathbf{k}\langle \mathcal{X} \rangle = \bigoplus_{\alpha \in \mathbb{N}^{(\mathcal{X})}} \mathbf{k}_\alpha \langle \mathcal{X} \rangle \quad (1)$$

where $\mathbf{k}_\alpha \langle \mathcal{X} \rangle := \text{span}_{\mathbf{k}}\{w \mid \beta(w) = \alpha\}$.

An immediate (and although rich) example/4

Graded bases through free Lie algebra

- 9 each $\mathbf{k}_\alpha\langle\mathcal{X}\rangle$ is free of dimension $\frac{|\alpha|!}{\alpha!}$; for example with two letters a, b , we have $\mathbf{k}\langle\mathcal{X}\rangle = \bigoplus_{(p,q)\in\mathbb{N}^2} \mathbf{k}_{(p,q)}\langle\mathcal{X}\rangle$ and $\dim(\mathbf{k}_{(p,q)}\langle\mathcal{X}\rangle) = \frac{(p+q)!}{p!q!} = \binom{p+q}{p}$.
- 10 this fine grading is a grading of algebra as

$$\mathbf{k}_\alpha\langle\mathcal{X}\rangle\mathbf{k}_\beta\langle\mathcal{X}\rangle \subset \mathbf{k}_{\alpha+\beta}\langle\mathcal{X}\rangle ; 1_{\mathcal{X}^*} \in \mathbf{k}_0\langle\mathcal{X}\rangle \quad (2)$$

- 11 now through the second route (sets-Lie-AAU), we can construct many finely homogeneous bases of $\mathbf{k}\langle\mathcal{X}\rangle$ using the following scheme
- pick any finely homogeneous basis of $\mathcal{L}ie_{\mathbf{k}}\langle\mathcal{X}\rangle$, $(P_i)_{i\in I}$ (we will construct at least one)
 - (Totally) order I and form the PBW basis (of $\mathbf{k}\langle\mathcal{X}\rangle$). it is finely homogeneous (due to eq. 2).
 - use this for MRS factorisation (unfolded below after semi-simplicity)

Semi-simple categories of modules/1

Next steps

- 1 Semi-simple categories of modules
- 2 Link with non-degenerate bilinear forms + examples

About pronunciation

Here are examples of pronunciation

<https://www.linguee.fr/anglais-francais/traduction/semi.html>

including: “semi-detached house”, “semi-public”, “semiconductor”. On this ground, I think that, until there is evidence to the contrary, the “i” of “semi-simple” should sound as in “fish” and not as in “file”.

Semisimple categories of modules/2

- Semi-simple categories of modules, see in general

<https://ncatlab.org/nlab/show/semisimple+category>

Definition

Let R be a ring. We note $R\text{-Mod}$ the category of R -modules (whatever the size) the arrows being that of R -linear mappings between objects.

Remarks

- 1 This is a category with direct sums (coproducts) and products.
- 2 Subcategory of finite length modules (ex. finite dim when R is a \mathbf{k} algebra) admit (finite) decompositions (Krull) in indecomposables. Another example will be subcategory of semi-simple modules (see below).
- 3 In the preceding case (finite dim when R is a \mathbf{k} algebra) it is a subcategory
- 4 Link with non-degenerate bilinear forms + examples

Semisimple categories of modules/3

Definition: Simple and semi-simple modules

- 1 A module $M \in R\text{-Mod}$ is said simple if it is not (0) and if its set of submodules is $\{(0), M\}$
- 2 A module $M \in R\text{-Mod}$ is said semi-simple iff $M = \bigoplus_{i \in I} M_i$ where M_i are simple submodules of M .

Proposition [A]

Let $M \in R\text{-Mod}$

- 1 If M is such that $M = \sum_{i \in I} M_i$ where M_i are simple submodules of M and $N \subset_{\text{submod}} M$, then it exists $J \subset I$ such that $M = (\bigoplus_{i \in J} M_i) \oplus N$.
- 2 In particular a submodule or a quotient of a semi-simple module is semi-simple.

Proof

- A.1) Let $\mathfrak{S} \subset 2^I$ defined on 2^I

$$\mathfrak{S} = \{J \subset I \mid (\bigoplus_{i \in J} M_i) \oplus N \text{ is well defined}\} \quad (3)$$

The set of \mathfrak{S} is non-empty and of finite character. Then, by Tukey-Teichmüller theorem it admits at least a maximal element for inclusion. Let J_0 be such an element. If $J_0 = I$ we are done, otherwise let $i \in I \setminus J_0$ and set $T = ((\bigoplus_{i \in J_0} M_i) \oplus N)$. We cannot have $M_i \cap T = (0)$ otherwise we would get $J_0 \cup \{i\} \in \mathfrak{S}$ and $i \in J_0$, a contradiction. Remains $M_i \subset T$ because M_i is simple. Hence $(\forall i \in I \setminus J_0)(M_i \subset T)$ and this entails $M = T$.

Remark that, setting N to (0) , one obtains that if a module is a sum (direct or not) of simple submodules, then it is semi-simple.

- A.2) We suppose $M = \bigoplus_{i \in I} M_i$ to be semi-simple. Let $f : M \rightarrow Q$, . Setting $N = \ker(f)$ in the preceding situation, we get a subfamily $(M_i)_{i \in J}$ such that $M = (\bigoplus_{i \in J} M_i) \oplus N$. Then, by f , $(\bigoplus_{i \in J} M_i) \simeq Q$ and we are done. Now, if N is any submodule of M , by (A.1), it is direct summand and we can write $M = N \oplus N_1$ with projectors p_N, p_{N_1} . From $p_N : M \rightarrow N$ we are done.

Case when R_s itself is semi-simple

Any ring R can be considered as a $R - R$ bimodule by the left and right actions (for $a, b \in R$), $\lambda_a(m) = a.m$, $\rho_b(m) = m.b$. these two actions commute. By definition R_s is the left-module defined by the action $\lambda_a(m)$. We have the following

Proposition [B]

If R_s is semi-simple, all R -module is so.

Proof.

We suppose that R_s is semi-simple. Let M be a R -module, then for all $x \in M$ the (principal) R -submodule $R.x$ is a semi-simple image (that of $t \rightarrow t.x$), hence semisimple. The result is then a consequence of Proposition [A].1 in view of the fact that $M = \sum_{x \in M} R.x$. □

A sufficient condition for R_S to be semi-simple

Proposition [C]

Under the preceding conditions

- 1 If R_S is semi-simple then every left ideal is direct summand of R_S within the lattice of left ideals.
- 2 The converse is true in the case when this lattice^a satisfies ACC+DCC chain conditions.

https://en.wikipedia.org/wiki/Ascending_chain_condition

^aThe lattice of left ideals.

For hilbertian traces, see Dieudonné XV.6 [4].

In the category of modules, ACC is Noetherian, DCC is Artinian.

Next steps: Frobenius characteristics, characters, case of finite groups, the symmetric group, Kronecker, Littlewood-Richardson and Clebsch–Gordan coefficients.

Proof of Proposition [C]

- 1) In fact this is true of every semi-simple module by Proposition [A].1.
- 2) As in (1), this converse is true for every module satisfying the same conditions (i.e. every submodule is direct summand + ACC + DCC). Let M be such a module, we build the following double sequence

① (Init.) $C_0 = ((0), M)$

② (Running) $C_n = (\oplus_{i=1}^n N_i, Q_n)$ with N_i simple submodules of M and $\oplus_{i=1}^n N_i \oplus Q_n = M$

③ (Halt) $Q_n = (0)$ (then we are done)

④ (Next Step) Suppose $C_n = (\oplus_{i=1}^n N_i, Q_n)$ with $Q_n \neq (0)$ (non-halting step) then we choose a minimal submodule Q_{min} of M among those such that $(0) \subsetneq Q \subset Q_n$ (it is possible because M satisfies DCC). We set $N_{n+1} = Q_{min}$ and remark that the family $(N_i)_{1 \leq i \leq n+1}$ is in direct sum and, by hypothesis, it exists Q_{n+1} such that $\oplus_{i=1}^{n+1} N_i \oplus Q_{n+1} = M$ then set $C_{n+1} = (\oplus_{i=1}^{n+1} N_i, Q_{n+1})$

Proof of Proposition [C]/2 and first applications

- **Proof that this algorithm halts)** unless $M = (0)$ there is at least one step. Let $n + 1$ be any valid rank of a step. By construction $\bigoplus_{i=1}^n N_i \subsetneq \bigoplus_{i=1}^{n+1} N_i$, a strictly increasing sequence of submodules. By ACC this sequence must be finite.
- **Semi-simplicity)** Let m is the last index of the sequence C_n . We have $Q_m = (0)$ and then $M = \bigoplus_{i=1}^m N_i$. CQFD

Applications

- 1 **Applies to** Every finite dimensional $*$ -algebra which admits a SPS (then is semi-simple). See exercise below (todo)
- 2 **and in particular to** $\mathcal{A} = \mathbb{C}[G]$ where G is a finite group. With

$$\left(\sum_{g \in G} \alpha(g) g \right)^* := \left(\sum_{g \in G} \overline{\alpha(g)} g^{-1} \right)$$

and $\varphi(Q) = \langle 1_G | Q \rangle$

An example: Non-degenerate states in star-algebras

- Let \mathcal{A} be an $*$ -algebra ($x \rightarrow x^*$ is semi-linear, involutive and an anti-automorphism)
- $\mathcal{C}_+(\mathcal{A})$, generated by elements of the form $\sum_{i \in F} x_i x_i^*$ (F finite) is an hermitian (self-dual) convex cone
- $\text{State}(\mathcal{A})$ is the set of linear forms $f \in \mathcal{A}^*$ such that $z \in \mathcal{C}_+(\mathcal{A}) \implies f(z) \geq 0$
- A non degenerate state (SPS) is such that

$$z \in \mathcal{C}_+(\mathcal{A}) \text{ and } f(z) = 0 \implies z = 0$$

Proposition [D]

A finite dimensional star-algebra with a non-degenerate state is semi-simple.

Applications cont'd

Unfolding 1-2

- 1 Let φ be one of these SPS (strictly positive state) states and set

$$\langle x|y \rangle = \varphi(x^* y)$$

it is a non-degenerate hermitian form such that, identically $\langle x|a.y \rangle = \langle a^*.x|y \rangle$. Let \mathcal{J} be a left-ideal of the algebra \mathcal{A} , then it is easy to prove that \mathcal{J}^\perp is a left ideal

- 2 In particular with the preceding setting ($\mathcal{A} = \mathbb{C}[G]$ where G is a finite group, star-structure and state) we have the result.

- 3 We decompose \mathcal{A} into minimal left ideals $\mathcal{A} = \bigoplus_{j \in F} \mathcal{J}_j$ and then $1_{\mathcal{A}} = \sum_{i \in F} p_i$
- 4 One can prove that $\mathcal{J}_i = \mathcal{A} p_j$ and $p_j p_i = p_i p_j = \delta_{ij} p_i$ (complete orthogonal family of minimal idempotents)
- 5 The lemma $\text{Hom}_{\mathcal{A}}(\mathcal{A}.e, \mathcal{A}.f) \simeq e.\mathcal{A}.f$ (sandwich) gives Wedderburn's decomposition.

Construction of the matrix units.

- 1 For e, f idempotents then $eaf \rightarrow (x \rightarrow x e a f)$ is an iso of \mathbf{k} -spaces between $e \cdot \mathcal{A} \cdot f$ and $\text{Hom}_{\mathcal{A}}(\mathcal{A} \cdot e, \mathcal{A} \cdot f)$ the inverse being $f \rightarrow f(e)$ (note that $f(e) \in e \cdot \mathcal{A} \cdot f$).
- 2 Return to $1_{\mathcal{A}} = \sum_{i \in F} p_i$ (each p_i is minimal) and set $i \sim j \iff e_i \cdot \mathcal{A} \cdot e_j \neq (0)$ (block equivalence)
- 3 Take a block C , order $C = \{i_1 < i_2 < \dots < i_m\}$ totally.
- 4 For $1 \leq j < m$ choose $a_{ij} \in e_{i_j} \cdot \mathcal{A} \cdot e_{i_{j+1}} \setminus (0)$

Faithful state

The fact that A be a star-algebra of finite dimension, sum of matrix algebras is by no means sufficient to imply that the projectors on the blocks are $*$ -invariant nor $A \simeq \mathbb{C}$ as shows the following counterexample. Take $B = \mathbb{C}^{n \times n}$ (algebra of complex square matrices of dimension $n > 0$) and $A = B \oplus B$ with the anti-automorphism $(X, Y)^* = (Y^*, X^*)$. Then (A, \star) is easily checked to be a star algebra. It is of finite dimension, sum of matrix algebras but $\dim_{\mathbb{C}} = 2n^2 \neq 1$. Indeed, the existence of a faithful state is crucial as there is none over A .

Elements of the form a^*a need not have a positive spectrum as shows the following counterexample. Taking $n = 1$ in the above we get

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^* \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} \bar{b} & 0 \\ 0 & \bar{a} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} \bar{b}a & 0 \\ 0 & \bar{a}b \end{pmatrix} \quad (4)$$

We recall a few notable facts:

- 1 We start with a finite dimensional $*$ -algebra \mathcal{A} (a \mathbf{k} – **AAU** and remark that e^* is neutral so that $e^* = e$).
- 2 Now, \mathcal{A} is equipped with a SPS φ as in (30). With φ , we build the following 2-form

$$g(x, y) = \langle x|y \rangle = \varphi(x^*y). \quad (5)$$

- 3 One checks (see below) at once that $(x, y) \rightarrow \langle x|y \rangle$ a positive definite hermitian form (inner product) therefore (\mathcal{A}, g) is an Hilbert space. We have $|\langle x|y \rangle| \leq \|x\| \cdot \|y\|$ and $\varphi(x^*) = \varphi(x)$.
- 4 This inner product satisfies identically $\varphi(x^*(a.y)) = \langle x|a.y \rangle = \langle a^*.x|y \rangle = \varphi((a^*.x)^*y)$ and from that, we get that \mathcal{A} is semi-simple.

- 5 Consider a finite group $H \subset \text{Aut}_{\mathbb{C}}(\mathcal{A})$ of linear automorphisms of the $*$ -algebra \mathcal{A} (i.e. automorphisms of algebra which commute with the $*$ -involution) such that $(\forall (h, a) \in H \times \mathcal{A})(\varphi(h.a) = \varphi(a))$. In other words, φ does not see the action of H . We check that H is a group of isometries for $g(x, y)$:

$$\begin{aligned}\langle h.a|h.b \rangle &= \varphi((h.a)^*h.b) = \varphi((h.a^*)(h.b)) = \varphi(h.(a^*.b)) \\ &= \varphi(a^*.b) = \langle a|b \rangle\end{aligned}\quad (6)$$

- 6 Now, we form an algebra $\kappa(H, \mathcal{A})$ of orbits (with multiplicities) linearly generated by the vectors $H.a$, for all $a \in \mathcal{A}$:

$$\kappa(H, \mathcal{A}) = \text{Span}_{\mathbb{C}}\{H.a\}_{a \in \mathcal{A}} \quad (7)$$

where

$$H.a := \sum_{h \in H} h.a \quad (8)$$

- 7 We can check (see below) that $\kappa(H, \mathcal{A})$ is a subalgebra of \mathcal{A} which is, moreover $*$ -closed.
- 8 As a consequence, if \mathcal{A} is a C^* -algebra then $\kappa(H, \mathcal{A})$ is a C^* -algebra.
- 9 In the next slide, we prove the points 3, 7 and 8, all remaining facts are easily achieved.

Proofs

- 10 Proof of 3. – Linearity on the right is straightforward. To show hermitian symmetry, we first compute $g(x + y, x + y) = g(x, x) + [g(x, y) + g(y, x)] + g(y, y)$ which proves that

$$\Im(g(y, x)) = -\Im(g(x, y)). \quad (9)$$

Then, from,

$$\begin{aligned} g(x + iy, x + iy) &= g(x, x) + [g(x, iy) + g(iy, x)] + g(iy, iy) = \\ &g(x, x) + i[g(x, y) - g(y, x)] + g(y, y) \end{aligned} \quad (10)$$

we get $i[g(x, y) - g(y, x)] \in \mathbb{R}$ meaning $\Re([g(x, y) - g(y, x)]) = 0$. Then $\Re(g(y, x)) = \Re(g(x, y))$ with (9) shows

$$g(y, x) = \overline{g(x, y)} \quad (11)$$

therefore, with $y = e$, we get $\varphi(x^*) = g(x, e) = \overline{g(e, x)} = \overline{\varphi(x)}$. The inequality $|g(x, y)| \leq \|x\| \cdot \|y\|$ is a consequence of Cauchy-Schwartz theorem.

- 11 It suffices to consider that

$$\begin{aligned} (H.a)(H.b) &= \sum_{u,v \in H} (u.a)(v.b) = \sum_{u,v \in H} u.((a)(u^{-1}v.b)) \\ &\stackrel{w=u^{-1}v}{=} \sum_{u \in H} u. \left(\sum_{w \in H} (a)(w.b) \right) = \sum_{w \in H} H.(a(w.b)) \end{aligned} \quad (12)$$

Moreover, $h.a^* = (h.a)^*$ implies $(H.a)^* = H.a^*$, so $\kappa(H, \mathcal{A})$ is $*$ -closed.

- 12 This is the consequence of the general fact that an $*$ -closed subalgebra of a C^* -algebra is a C^* -algebra.

A remark

Remark. –

- 13 The reader should be aware that (\mathcal{A}, g) is not necessarily a C^* -algebra¹. In fact, one has the following equivalent conditions:
- (\mathcal{A}, g) is a C^* -algebra (i.e. for $\|x\| = \sqrt{g(x, x)}$)
 - $\dim_{\mathbb{C}}(\mathcal{A}) = 1$
- 14 However, we can make \mathcal{A} a C^* -algebra in the following way. Let $\rho : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{A})$ be the left-regular representation (i.e. $\rho(a) = (\xi \rightarrow a.\xi)$). This representation is into \mathcal{A} , if $\rho(a) = 0$, in particular $\rho(a)[a^*] = aa^* = 0$ and $\varphi(aa^*) = 0$ hence $a = 0$. Moreover

$$\langle x | \rho(a)[y] \rangle = \langle x | a.y \rangle = \langle a^*.x | y \rangle = \langle \rho(a^*).x | y \rangle \quad (13)$$

which entails that $\rho(a^*) = \rho(a)^*$ and, from this \mathcal{A} is a C^* -algebra (for the new norm $\|a\|_g = \sup_{\|\xi\|=1} \|a.\xi\|$).

¹See discussion in <https://math.stackexchange.com/questions/3964927>.

Universal problem without functor: Coproducts

All here is stated within the same category \mathcal{C} .

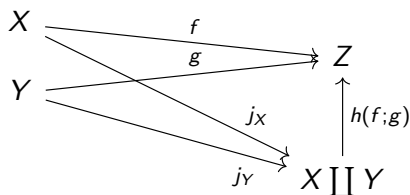


Figure: Coproduct $(j_X, j_Y; X \amalg Y)$.

$$\begin{aligned} & (\forall (f, g) \in \text{Hom}(X, Z) \times \text{Hom}(Y, Z)) \\ & (\exists! h(f; g) \in \text{Hom}(X \amalg Y, Z)) \\ & (h(f; g) \circ j_X = f \text{ and } h(f; g) \circ j_Y = g) \end{aligned} \quad (14)$$

Coproducts: Sets

All here is stated within the category **Set**.

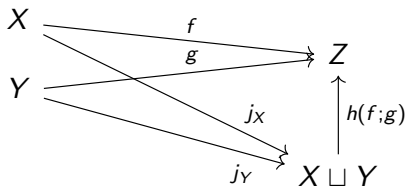


Figure: Coproduct $(j_X, j_Y; X \sqcup Y)$.

$$\begin{aligned} & (\forall (f, g) \in \text{Hom}(X, Z) \times \text{Hom}(Y, Z)) \\ & (\exists! h(f; g) \in \text{Hom}(X \sqcup Y, Z)) \\ & (h(f; g) \circ j_X = f \text{ and } h(f; g) \circ j_Y = g) \end{aligned} \tag{15}$$

Coproducts: Vector Spaces

All here is stated within the same category $\mathbf{k} - \mathbf{Vect}$.

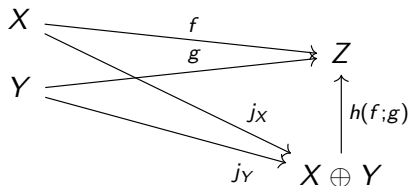


Figure: Coproduct $(j_X, j_Y; X \oplus Y)$ here $h(f; g) = f \oplus g$.

$$\begin{aligned} & (\forall (f, g) \in \text{Hom}(X, Z) \times \text{Hom}(Y, Z)) \\ & (\exists! h(f; g) \in \text{Hom}(X \oplus Y, Z)) \\ & (h(f; g) \circ j_X = f \text{ and } h(f; g) \circ j_Y = g) \end{aligned} \quad (16)$$

Coproducts: $\mathbf{k} - \mathbf{CAAU}$

All here is stated within the same category $\mathbf{k} - \mathbf{CAAU}$.

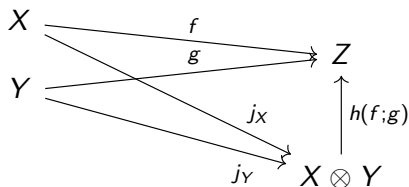


Figure: Coproduct $(j_X, j_Y; X \otimes Y)$ here $h(f; g) = f \otimes g$.

$$\begin{aligned} & (\forall (f, g) \in \text{Hom}(X, Z) \times \text{Hom}(Y, Z)) \\ & (\exists! h(f; g) \in \text{Hom}(X \otimes Y, Z)) \\ & (h(f; g) \circ j_X = f \text{ and } h(f; g) \circ j_Y = g) \end{aligned} \quad (17)$$

Coproducts: Augmented \mathbf{k} – AAU

All here is stated within the same category *Augmented \mathbf{k} – AAU*.

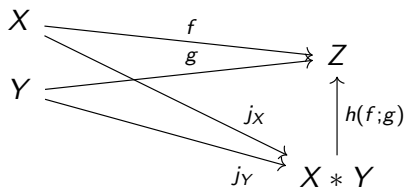


Figure: Coproduct $(j_X, j_Y; X * Y)$ here $h(f; g) = f * g$.

$$\begin{aligned} & (\forall (f, g) \in \text{Hom}(X, Z) \times \text{Hom}(Y, Z)) \\ & (\exists! h(f; g) \in \text{Hom}(X * Y, Z)) \\ & (h(f; g) \circ j_X = f \text{ and } h(f; g) \circ j_Y = g) \end{aligned} \quad (18)$$

Non-coproducts: Tensor products of \mathbf{k} – **AAU**

Tensor product of \mathbf{k} – **AAU** described as in [14] Prop. 11.4.1.

- 15 All here is stated within the same category \mathbf{k} – **AAU**.

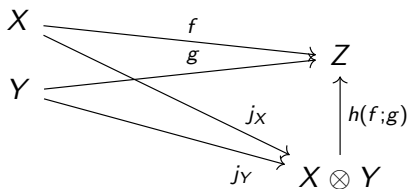


Figure: Tensor product $(j_X, j_Y; X \otimes Y)$, the condition is that $f(X)$ commutes pairwise elementarily with $g(Y)$ within Z .

$$\begin{aligned} & (\forall (f, g) \in \text{Hom}(X, Z) \times \text{Hom}(Y, Z)) \\ & ((\forall (x, y) \in X \times Y)(f(x)g(y) = g(y)f(x))) \\ & (\exists! h(f; g) \in \text{Hom}(X \otimes Y, Z)) \\ & (h(f; g) \circ j_X = f \text{ and } h(f; g) \circ j_Y = g) \end{aligned} \tag{19}$$

Non-coproduct: the same, but categorized

was [14] Prop. 11.4.1.

16 We construct a derived category $\mathcal{C}_2(X, Y)$ as follows

1 Objects: Pairs (f, g) of morphisms of $\mathbf{k} - \mathbf{AAU}$ such that

1 $\text{codom}(f) = \text{codom}(g)$

2 $(\forall(x, y) \in X \times Y)(f(x)g(y) = g(y)f(x))$

2 Arrows: $h \in \text{Hom}[(f_1, g_1), (f_2, g_2)]$ is such that

1 $h \in \text{Arr}(\mathbf{k} - \mathbf{AAU})$

2 $f_2 = hf_1$ and $g_2 = hg_1$

17 **Little training.** –

1) Make precise what $(j_X, j_Y, X \otimes Y)$ is in the case above
(Hint: Use [14] Prop. 11.4.1).

2) Prove that this triple is a solution of one universal problem from $\mathcal{C}_2(X, Y)$ as an initial element.

Non-coproducts: Coloured tensor products of $\mathbf{k} - \mathbf{grAAU}$

Deformation theory: Colour (or commutation) factor, see [7].

Some bibliographical elements

Colour factors were introduced by R. Ree [19], and the theory was developed or used in [8, 9]. In fact, some of them (“Commutation factors”, with values in $\{-1, 1\}$ and an [anti]symmetry condition) are already considered in [1]. See Section 10 (Dérivations) of Chapter III and §4.7.

Deformed and shifted laws

- We begin by a very general version of the “shifting lemma” (similar but more general than the one given and needed in [9])
- **Shifting lemma.** – Let $\mathcal{A} = \bigoplus_{\alpha \in \mathcal{D}} \mathcal{A}_\alpha$ be an algebra decomposed (as a vector space) over a semigroup \mathcal{D} (\mathcal{A} is then graded as a vector space, not as an algebra). Let $\alpha \mapsto T_\alpha$ be a semigroup homomorphism $\mathcal{D} \rightarrow \text{End}^{\text{gr}}(\mathcal{A})$. We suppose that the shifted law defined, for $x \in \mathcal{A}_\alpha$, by $x \bar{*} y = x * T_\alpha(y)$ is \mathcal{D} -graded. Then, if the law $*$ is associative, so is the law $\bar{*}$.

Non-coproducts: Coloured tensor products of $\mathbf{k} - \mathbf{grAAU}$

Tensor product of $\mathbf{k} - \mathbf{grAAU}$ with commutation factor (deformation).

- 18 All here is stated within the same category $\mathbf{k} - \mathbf{grAAU}$.

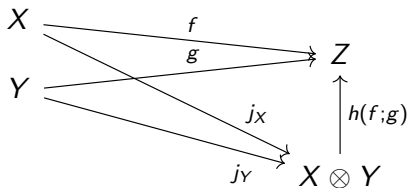


Figure: Tensor product $(j_X, j_Y; X \otimes Y)$, the condition is that $f(X)$ commutes pairwise elementarily with $g(Y)$ with commutation factor χ and within Z .

$$\begin{aligned} & (\forall (f, g) \in \text{Hom}^{\text{gr}}(X, Z) \times \text{Hom}^{\text{gr}}(Y, Z)) \\ & ((\forall (x, y) \in X_p \times Y_q)(g(y)f(x) = \chi(q, p)f(x)g(y))) \\ & (\exists! h(f; g) \in \text{Hom}(X \otimes Y, Z)) \\ & (h(f; g) \circ j_X = f \text{ and } h(f; g) \circ j_Y = g) \end{aligned} \tag{20}$$

Continuation: An immediate (and although rich) example/5

Words and Lyndon words, details.

Algebraic structure

- Concatenation: This law is noted *conc*
- With the empty word as neutral, the set of words is the free monoid $(X^*, \text{conc}, 1_{X^*})$
- The pairing between series and polynomials is defined by

$$\langle S|P \rangle = \sum_{w \in X^*} \langle S|w \rangle \langle P|w \rangle$$

Coding by words gives access to a welter of structures, studies, relations and results (algebra, geometry, topology, probability, combinatorics on words, on polynomials and series). We will use in particular their complete factorisation by **Lyndon words**.

An immediate (and although rich) example/6

Words and classes

Example with $\mathcal{X} = \{a, b\}$, $a < b$, in red Lyndon words ($= \mathcal{Lyn}\mathcal{X}$).

<i>Length</i>	<i>words</i>
0	$1_{\mathcal{X}^*}$
1	a, b
2	aa, ab, ba, bb
3	$aaa, aab, aba, abb, baa, bab, bba, bbb$
4	$a^4, a^3b, a^2ba, a^2b^2, aba^2, abab, ab^2a, ab^3, ba^3, ba^2b, baba, babb, b^2a^2, b^2ab, b^3a, b^4$

Two properties of Lyndon words

- 1 All $\ell \in \mathcal{Lyn}\mathcal{X} \setminus \mathcal{X}$ factorises (not uniquely in general) as $\ell = \ell_1\ell_2$, $\ell_1 \prec \ell_2$, $\ell_i \in \mathcal{Lyn}\mathcal{X}$
(ex. $a^3ba^2bab = a^3b|a^2bab = a^3ba^2b|ab$), the one with the longest right factor will be called standard $\sigma(\ell) = (\ell_1, \ell_2)$.
- 2 Every word $w \in \mathcal{X}^*$ factorises uniquely $w = \ell_1^{i_1} \dots \ell_k^{i_k}$ with $\ell_1 \succ \dots \succ \ell_k$, ($\ell_i \in \mathcal{Lyn}\mathcal{X}$)

An immediate (and although rich) example/7

Shuffle product(s)

Non deformed case

Coming from the route where $\mathbf{k}\langle\mathcal{X}\rangle = \mathcal{U}(\mathcal{L}ie_{\mathbf{k}}\langle\mathcal{X}\rangle)$, we have a structure of coalgebra on $\mathbf{k}\langle\mathcal{X}\rangle$ its comultiplication is given by its value on letters

$$\Delta_{\text{III}}(x) = x \otimes 1_{\mathcal{X}^*} + 1_{\mathcal{X}^*} \otimes x \quad (21)$$

Then shuffle product is defined as a dual law, for each $w \in \mathcal{X}^*$ by

$$\langle P_{\text{III}} Q | w \rangle = \langle P \otimes Q | \Delta_{\text{III}}(w) \rangle \quad (22)$$

We get the following recursion for shuffle products

$$w_{\text{III}} 1_{\mathcal{X}^*} = 1_{\mathcal{X}^*}_{\text{III}} w = w \quad \text{for any word } w \in \mathcal{X}^*; \quad (23)$$

$$au_{\text{III}} bv = a(u_{\text{III}} bv) + b(au_{\text{III}} v) \quad (24)$$

An immediate (and although rich) example/8

Two bases in duality/1: Combinatorial constructions

Lyndon basis

$$\begin{aligned}P_x &= x && \text{for } x \in X, \\P_\ell &= [P_s, P_r] && \text{for } \ell \in \mathcal{Lyn}\mathcal{X} \setminus \mathcal{X} \text{ and } \sigma(\ell) = (s, r), \\P_w &= P_{\ell_1}^{i_1} \dots P_{\ell_k}^{i_k} && \text{for } w = \ell_1^{i_1} \dots \ell_k^{i_k}, \ell_1 \succ \dots \succ \ell_k, (\ell_i \in \mathcal{Lyn}\mathcal{X}).\end{aligned}$$

where \succ stands for the lexicographic (strict) ordering defined from $x_0 \prec x_1$.

Triangular property

Indeed $\{P_w\}_{w \in X^*}$ is lower unitriangular w.r.t. words (this property, joined with the fact that this family is finely homogeneous, implies that $\{P_w\}_{w \in X^*}$ is a basis of $\mathbf{k}\langle \mathcal{X} \rangle$)

$$P_w = w + \sum_{v \succ w, \beta(v) = \beta(w)} c_v v \text{ with } c_v \in \mathbb{Z} \quad (25)$$

Dual bases

Construction of $(S_w)_{w \in \mathcal{X}^\alpha}$

For each multidegree α , let \mathcal{X}^α be the (finite) set of words with multidegree α and T_α be the lower unitriangular matrix of $\{P_w\}_{\beta(w)=\alpha}$ w.r.t. words of \mathcal{X}^α then, the matrix *transpose* (T^{-1}) defines a family $(S_w)_{w \in \mathcal{X}^\alpha}$ such that

- 1 $S_w = w + \sum_{v \prec w, \beta(v)=\beta(w)} d_v v$ with $d_v \in \mathbb{Z}$
- 2 For all $u, v \in \mathcal{X}^\alpha$, $\langle S_u | P_v \rangle = \delta_{u,v}$.
- 3 The quantification of the preceding property can be extended to all $u, v \in \mathcal{X}^*$ due to the fact that the decomposition (1) is, in fact, orthogonal.

Schützenberger's basis (\mathbf{k} is a \mathbb{Q} -algebra)

M. -P. Schützenberger proved that, when \mathbf{k} is a \mathbb{Q} -algebra, the basis $(S_w)_{w \in \mathcal{X}^*}$ can be computed recursively as follows

$$\begin{aligned}
 S_x &= x && \text{for } x \in \mathcal{X}, \\
 S_l &= xS_u, && \text{for } l = xu \in \mathcal{Lyn}\mathcal{X} \setminus \mathcal{X}, \\
 S_w &= \frac{S_{l_1}^{\text{III } i_1} \text{III} \dots \text{III } S_{l_k}^{\text{III } i_k}}{i_1! \dots i_k!} && \text{for } w = l_1^{i_1} \dots l_k^{i_k}, l_1 \succ \dots \succ l_k.
 \end{aligned}$$

Triangular properties (recall)

$$P_w = w + \sum_{v \succ w, \beta(v) = \beta(w)} c_v v \quad \text{and} \quad S_w = w + \sum_{v \prec w, \beta(v) = \beta(w)} d_v v. \quad (26)$$

We recall that the bases $\{S_w\}_{w \in \mathcal{X}^*}$ and $\{P_w\}_{w \in \mathcal{X}^*}$ are lower and upper triangular respectively and that they are (finely) graded (all the monomials have the same partial degrees).

Table of these bases

Example (First values)

Let $X = \{x_0, x_1\}$ with $x_0 < x_1$.

I	P_I	S_I
x_0	x_0	x_0
x_1	x_1	x_1
$x_0 x_1$	$[x_0, x_1]$	$x_0 x_1$
$x_0^2 x_1$	$[x_0, [x_0, x_1]]$	$x_0^2 x_1$
$x_0 x_1^2$	$[[x_0, x_1], x_1]$	$x_0 x_1^2$
$x_0^3 x_1$	$[x_0, [x_0, [x_0, x_1]]]$	$x_0^3 x_1$
$x_0^2 x_1^2$	$[x_0, [[x_0, x_1], x_1]]$	$x_0^2 x_1^2$
$x_0 x_1^3$	$[[[x_0, x_1], x_1], x_1]$	$x_0 x_1^3$
$x_0^4 x_1$	$[x_0, [x_0, [x_0, [x_0, x_1]]]]$	$x_0^4 x_1$
$x_0^3 x_1^2$	$[x_0, [x_0, [[x_0, x_1], x_1]]]$	$x_0^3 x_1^2$
$x_0^2 x_1 x_0 x_1$	$[[x_0, [x_0, x_1]], [x_0, x_1]]$	$2x_0^3 x_1^2 + x_0^2 x_1 x_0 x_1$
$x_0^2 x_1^3$	$[x_0, [[[x_0, x_1], x_1], x_1]]$	$x_0^2 x_1^3$
$x_0 x_1 x_0 x_1^2$	$[[x_0, x_1], [[x_0, x_1], x_1]]$	$3x_0^2 x_1^3 + x_0 x_1 x_0 x_1^2$
$x_0 x_1^4$	$[[[[[x_0, x_1], x_1], x_1], x_1]$	$x_0 x_1^4$

Factorisation of the diagonal as a resolution of identity.

Resolution of identity as an infinite product

Now we are in the position of writing the principal factorisation of the diagonal series. In here, series multiply by shuffle on the left and concatenation on the right.

$$\mathcal{D}_X := \sum_{w \in X^*} w \otimes w = \sum_{w \in X^*} S_w \otimes P_w = \prod_{I \in \mathcal{L}_{yn} X} \exp(S_I \otimes P_I) \quad (27)$$

Application to factorisation of characters

If we have a shuffle-character $\chi : (\mathbf{k}\langle X \rangle, \text{III}, 1_{X^*}) \rightarrow \mathcal{A}$, we act on the left

$$\chi = \sum_{w \in X^*} \chi(w) \otimes w = \prod_{I \in \mathcal{L}_{yn} X} \exp(\chi(S_I) \otimes P_I) \quad (28)$$

But with a conc-character $\chi : (\mathbf{k}\langle X \rangle, \text{conc}, 1_{X^*}) \rightarrow \mathcal{A}$, we act on the right

$$\chi = \sum_{w \in X^*} w \otimes \chi(w) = \prod_{I \in \mathcal{L}_{yn} X} \exp(S_I \otimes \chi(P_I)) \quad (29)$$

Conclusion

- 1 The values of iterated integrals (standard of regularized) are shuffle-characters, then we have factorisations and they constitute multiplicative regularizations.
- 2 The values of matrix representations of the free monoid (as the transitions of rational series for instance) are conc-characters and we get useful factorizations of them.
- 3 In the next talk (friday morning ?), we will see the deformed case through CQMM and applications to harmonic sums.

Thank you for your attention.

Exercises/1

Ex1: States and pre-states

Let \mathcal{A} be a complex finite-dimensional $*$ -algebra. A SPS (Strictly Positive State) is a linear form $\varphi \in \mathcal{A}^*$ such that

$$(\forall x \in \mathcal{A} \setminus \{0\})(\varphi(x^*x) > 0) \quad (30)$$

- 1 Prove that the bilinear form $\langle x|y \rangle := \frac{1}{2}(\varphi(x^*y) + \overline{\varphi(y^*x)})$ is a non-degenerate hermitian scalar product^a such that, identically

$$\langle x|a.y \rangle = \langle a^*.x|y \rangle$$

- 2 Prove that a complex finite-dimensional $*$ -algebra admitting a SPS is semi-simple.

^aI take the convention of semi-linearity on the left (see the link “Hilbert modules” below).

Ex1: States and pre-states/2

Let G be a finite group, set $\mathcal{A} = \mathbb{C}[G]$ and, for $a = \sum_{g \in G} \alpha(g) g$, set

$$a^* = \sum_{g \in G} \overline{\alpha(g)} g^{-1}$$

- 3 Prove that $(\mathcal{A}, *)$ is an $*$ -algebra
- 4 Prove that $\varphi \in \mathcal{A}^*$ defined by $\varphi(a) = \alpha(1)$ is a SPS on \mathcal{A} .

Links

1 Categorical framework(s)

<https://ncatlab.org/nlab/show/category>

[https://en.wikipedia.org/wiki/Category_\(mathematics\)](https://en.wikipedia.org/wiki/Category_(mathematics))

2 Universal problems

<https://ncatlab.org/nlab/show/universal+construction>

https://en.wikipedia.org/wiki/Universal_property

3 Paolo Perrone, *Notes on Category Theory with examples from basic mathematics*, 181p (2020)

arXiv:1912.10642 [math.CT]

https://en.wikipedia.org/wiki/Abstract_nonsense

4 Heteromorphism

<https://ncatlab.org/nlab/show/heteromorphism>

5 D. Ellerman, *MacLane, Bourbaki, and Adjoints: A Heteromorphic Retrospective*, David Ellerman Philosophy Department, University of California at Riverside

Links/2

- 6 https://en.wikipedia.org/wiki/Category_of_modules
- 7 <https://ncatlab.org/nlab/show/Grothendieck+group>
- 8 Traces and hilbertian operators
<https://hal.archives-ouvertes.fr/hal-01015295/document>
- 9 State on a star-algebra
<https://ncatlab.org/nlab/show/state+on+a+star-algebra>
- 10 Hilbert module
<https://ncatlab.org/nlab/show/Hilbert+module>

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[GriRei20] DARIJ GRINBERG, VICTOR REINER, *Hopf algebras in Combinatorics*, version of 27 July 2020,
arxiv1409.8356

See also

<http://www.cip.ifi.lmu.de/~grinberg/algebra/HopfComb-sols>.
for a version that gets updated.