Semigroup Bialgebras

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Perturbations of the shuffle product :

$$1 ⧢ w = w ⧢ 1 = w;$$
$$(au) ⧢ (bv) = a\Big(u ⧢ (bv)\Big) + b\Big((au) ⧢ v\Big)$$

Quasi-shuffle product :

$$\begin{cases} u \uplus 1 = 1 \uplus u = u; \\ y_i u \uplus y_j v = y_i\Big(u \uplus y_j v\Big) + y_j\Big(y_i u \uplus v\Big) + y_{i+j}\Big(u \uplus v\Big) \end{cases}$$

Goal : To give a general framework and a general theory to these operations.

Idea : Consider the dual law, for example

$$\Delta_{\uplus}(y_r) = y_r \otimes 1 + 1 \otimes y_r + \sum_{\substack{p+q=r \\ p,q \geq 1}} y_p \otimes y_q .$$

Roughly speaking : sum over the decomposition of the elements of a semigroup S.

$$\Delta(y_q) = \sum_{\substack{s,t \in S \\ s \cdot t = q}} y_s \otimes y_t .$$

Let $(S, \cdot)$ be a commutative semigroup.

- Alphabet

$$Y = \begin{cases} \{y_s\}_{s \in S \setminus \{\omega\}} & \text{in case } \omega \text{ is a zero of } S; \\ \{y_s\}_{s \in S} & \text{otherwise.} \end{cases}$$

- Application

$$\Delta_S : k\langle Y \rangle \to k\langle\langle Y \otimes Y \rangle\rangle$$

defined as a morphism of algebras given on the letters by

$$\Delta_S(y_s) = y_s \otimes 1_{Y^*} + 1_{Y^*} \otimes y_s + \sum_{s_1 \cdot s_2 = s} y_{s_1} \otimes y_{s_2}.$$

We say that S has the *finite decomposition property* if, $\forall s \in S \setminus \{\omega\}$,

$$\left| \{(s_1, s_1),\ s_1 \cdot s_2 = s\} \right| < \infty. \qquad \text{(D)}$$

Examples :

- the null semigroup (shuffle algebra) ;
- $\mathbb{N}^+$ (stuffle algebra) ;
- more generally, $\mathbb{N}^{(X)}$ for arbitrary X (finite or infinite).

A locally finite semigroup S is such that $\forall s \in S \setminus \{\omega\}$,

$$\left| \bigcup_{k \geq 1} D_k(s) \right| < \infty$$

where

$$D_k(s) = \{s_1, \ldots, s_k \in S \text{ such that } s_1 \cdot \ldots \cdot s_k = s\}.$$

Assume that S be a finite decomposition semigroup.

- The image of Δ_S is an element of $k\langle Y \otimes Y\rangle \simeq k\langle Y\rangle \otimes k\langle Y\rangle$.
- In fact, Δ_S defines a coassociative coproduct on $k\langle Y\rangle$.
- Define

$$\epsilon_Y(w) = \begin{cases} 1 \text{ if } w = 1_{Y^*}; \\ 0 \text{ otherwise.} \end{cases}$$

It is a counit for the bialgebra we are constructing.

Assume that S be $\mathbb{N}$-graded : there exists a function $\ell_S : S \to \mathbb{N}$ such that

$$S_m \cdot S_n \subset S_{m+n}, \forall m, n \in \mathbb{N}^*$$

(with $S_n = \{s \in S,\, \ell_S(s) = n\}$) ; assume also that $\ell_S^{-1}(0) = \emptyset$.

Then $\mathcal{B}$ is graded : define $|\cdot| : \mathcal{B} \to \mathbb{N}$ by

$$|y_s| = \ell_S(s); \qquad |y_{s_1} \dots y_{s_k}| = \sum_{i=1}^{k} |y_{s_i}|.$$

The product and the coproduct are compatible with the homogeneous components (defined by

$$\mathcal{B}_p = \mathrm{span}\left(y_{s_1} \dots y_{s_k},\, |y_{s_1} \dots y_{s_k}| = p\right), \quad p \in \mathbb{N}).$$

The letters y_m need not be primitive (for example for the law dual of the stuffle product).

New letters (primitive) obtained by projections $\mathcal{P}\mathrm{rim}(\mathcal{B})$ (set of primitive elements of $\mathcal{B}$).

Assume that S is graded *with finite fibers* which means that $S = \bigcup_{n\in\mathbb{N}} S_n$ with

$$\begin{cases} S_0 = \emptyset; \\ |S_k| < \infty, \qquad \forall\, k \in \mathbb{N}. \end{cases}$$

Denote by $\mathcal{B}_+$ the kernel of ϵ_Y. One has $\mathcal{B} = \mathcal{B}_+ \oplus k1_{Y^*}$.

Let I_+ denote the projector on $\mathcal{B}_+$ along k. If S is locally finite, I_+ is (locally) nilpotent.

Therefore, it is possible to define π_1 by :

$$\pi_1 = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} I_+^{*k} = \log_*(\mathrm{Id}_{\mathcal{B}})$$

where $*$ denotes the convolution product of $\mathcal{E}\mathrm{nd}(\mathcal{B})$.

Application / **Example** : Consider the bialgebra obtained with the previous setting for $S = (\mathbb{N}, +)$.

Then $\mathcal{B} = (k\langle Y\rangle, \mathrm{conc}, 1_{X^*}, \Delta_{\uplus}, \epsilon)$ with $Y = \{y_j\}_{j\geq 1}$.

We define, for $b \in \mathcal{B}_+$, $\Delta_+(b) = \Delta(b) - b \otimes 1 - 1 \otimes b$. One has

$$((I_+ \otimes I_+) \circ \Delta)(b) = (\Delta_+ \circ I_+)(b)$$

This implies that Δ_+ is a coassociative coproduct on $\mathcal{B}_+$ (since Δ is coassociative). Hence, on $\mathcal{B}_+$ and for all $k \geq 2$,

$$\Delta_+^{(k-1)} = I_+^{\otimes k} \circ \Delta^{(k-1)}.$$

This relation allows us to compute the convolution powers of $I_+(y_j)$:

$$I_+^{*k}(y_j) = \mu^{(k-1)} \circ (I_+^{\otimes k}) \circ \Delta^{(k-1)}(y_j) = \mu^{(k-1)} \circ \Delta_+^{(k-1)}(y_j)$$

and to prove that

$$\pi_1(y_j) = \sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{\substack{i_1+\cdots+i_n=j \\ i_1,\ldots,i_n>0}} y_{i_1} \ldots y_{i_n}.$$

We assume that it is possible to define a total order $<$ on the letters y_s :

$$y_{s_1} < \cdots < y_{s_k} < \ldots$$

If this is the case, it is possible to define

- Lyndon words $\mathfrak{L}\mathrm{yn}(Y)$;
- the standard factorization $\sigma(w)$ (a pair of Lyndon words ℓ_1 and ℓ_2 such that $w = \ell_1 \ell_2$ and ℓ_2 is of maximal length among all the factorizations of w).

We construct a basis of $k\langle Y\rangle$ as follows :

$$P_S(w) = \begin{cases} \pi_1(y_s) & \text{if } w = y_s; \\ [P_S(\ell_1), P_S(\ell_2)] & \text{if } w \in \mathfrak{L}\mathrm{yn}(Y) \text{ and } \sigma(w) = (\ell_1, \ell_2); \\ P_S(\ell_1)^{\alpha_1} \dots P_S(\ell_k)^{\alpha_k} & \text{if } \begin{cases} w = \ell_1^{\alpha_1} \dots \ell_k^{\alpha_k} \\ \ell_1 > \dots > \ell_k \end{cases} . \end{cases}$$

(we recall that $[P_1, P_2] = P_1 P_2 - P_2 P_1, \forall\, P_1,\ P_2 \in k\langle Y\rangle$).

- $(P_\ell)_{\ell \in \mathfrak{L}\mathrm{yn}(Y)}$ is a basis of $\mathrm{Prim}(k\langle Y\rangle)$;
- the PBW theorem ensures that $(P_w)_{w \in Y^*}$ is a basis of $k\langle Y\rangle$.