Roots $x_k(y)$ of a formal power series $f(x,y) = \sum_{n=0}^{\infty} a_n(y)\, x^n$, with applications to graph enumeration and q-series

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Many problems in combinatorics, statistical mechanics, number theory and analysis give rise to power series (whether formal or convergent) of the form

$$f(x,y) \;=\; \sum_{n=0}^{\infty} a_n(y)\, x^n \;,$$

where $\{a_n(y)\}$ are formal power series or analytic functions satisfying $a_n(0) \neq 0$ for $n = 0,1$ and $a_n(0) = 0$ for $n \geq 2$. Furthermore, an important role is played in some of these problems by the roots $x_k(y)$ of $f(x,y)$ — especially the "leading root" $x_0(y)$, i.e. the root that is of order y^0 when $y \to 0$. Among the interesting series $f(x,y)$ of this type are the "partial theta function"

$$\Theta_0(x,y) \;=\; \sum_{n=0}^{\infty} x^n\, y^{n(n-1)/2} \;,$$

which arises in the theory of q-series and in particular in Ramanujan's "lost" notebook; and the "deformed exponential function"

$$F(x,y) \;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!}\, y^{n(n-1)/2} \;,$$

which arises in the enumeration of connected graphs.

In this series of seminars I will describe recent (and mostly unpublished) work concerning this area of combinatorics/analysis. In addition to explaining my theorems, I will also describe some conjectures that I have verified numerically to high order but have not yet succeeded in proving — my hope is that one of you will succeed where I have not!

Some background material can be found in

Some variants of the exponential formula, with application to the multivariate Tutte polynomial (alias Potts model), Alexander D. Scott and Alan D. Sokal
http://arxiv.org/abs/0803.1477

A ridiculously simple and explicit implicit function theorem, Alan D. Sokal
http://arxiv.org/abs/0902.0069

and a taste of what I will discuss can be found in

Some wonderful conjectures (but almost no theorems) at the boundary between analysis, combinatorics and probability,
http://ipht.cea.fr/statcomb2009/misc/Sokal_20091109.pdf