Philippe Flajolet and the Airy Function

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This chapter is devoted to Philippe Flajolet's works involving the Airy function, and gives bibliographical pointers to later developments of what Philippe was calling the "Airy phenomenon".

1. Historical backgrounds: the Airy function in Physics

The Airy function was introduced in 1838 by Sir George Biddell Airy (1801-1892), royal astronomer at Cambridge, in his analysis of the intensity of light in the neighbourhood of a caustic [1], which allows to explain diffraction in optics (and phenomena like rainbows).

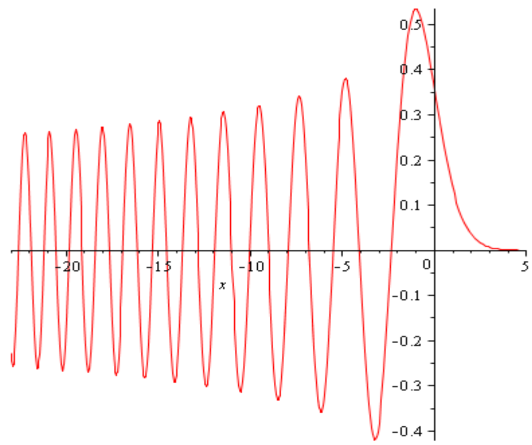
They are mainly 3 equivalent definitions of the Airy function $\mathrm{Ai(z)}$: as a solution of the differential equation $y'' - zy = 0$ (with $y(z) = 0$), as an integral, or as a power series (a variant of an ${}_2F_0(z)$ hypergeometric series), namely

$$\mathrm{Ai(z)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}(\mathrm{zt}+\mathrm{t}^3/3)}\,\mathrm{dt}$$

$$= \frac{1}{\pi 3^{2/3}} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{n+1}{3})}{n!} \sin\left(\frac{2(n+1)\pi}{3}\right) \left(3^{1/3} z\right)^n .$$

Accordingly, there are many equivalent ways to write the Airy function in terms of special functions [78], e.g. in terms of Bessel functions $I_\nu(z), K_\nu(z)$ at $\nu = 1/3$. The notation $\mathrm{Ai(z)}$ is due to J. C. P. Miller in 1946 [66], who was in charge of the BAASMTC (British Association for Advancement of Science, Mathematical Tables Committee), a committe founded in 1871 and initially managed by Cayley, Rayleigh, Kelvin, ... This notation $\mathrm{Ai(z)}$ was quickly popularised by the book "Methods of Mathematical Physics" of Jeffreys & Jeffreys [53].

```
> plot(AiryAi(x),x=-23..5);
```




```
> evalf(seq(AiryAiZeros(k),k=1..7));
-2.338107410, -4.087949444, -5.520559828, -6.786708090, -7.944133587, -9.022650853, -10.04017434
```

The Airy function $\mathrm{Ai(z)}$ oscillates on the real negative axis, where $\mathrm{Ai}(-x) \sim \frac{\sin\left(\frac{2}{3}x^{3/2}+\frac{1}{4}\pi\right)}{\sqrt{\pi}\,x^{1/4}}$ and where it has a discrete set of zeroes, while it decays exponential fast on the real positive axis, where $\mathrm{Ai}(x) \sim \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}\,x^{1/4}}$.

The Airy function has many application in physics (optics, quantum mechanics, electromagnetics, radiative transfer) [101]. Why are there still nowadays so many articles involving this function? It is mainly because it has an intimate link with quantum mechanics, via the Schrödinger equation $-\frac{\hbar}{2w}\psi''(x) + gx\psi(x) = E\psi(x)$. Indeed, if one looks for the level of energy E which are consistent with $\psi(0) = 0$, it implies that E has to be a zero of the Airy function! The equation is of the form $-a.y'' + b.x.y = E.y$, and the physics of the Schrödinger equation implies that $y(\pm\infty) < \infty$, up to a change of variable, one recognises the differential equation defining $\mathrm{Ai(z)}$, which in turn constraints E to belong to a discrete set of values $E = -a^{1/3}b^{2/3}\alpha_k$ (the $-\alpha_k$'s being the zeroes of $\mathrm{Ai(z)}$). This quantisation phenomena is thus typical in quantum mechanics, but it was a nice surprise when it was experimentally proven in 2001 that it holds for the four fundamental forces (i.e. *also* for gravitation), as it was observed (see [68]) that neutrons in Earth's gravitational field also exhibit such quantum states, at energy levels being nothing else than quotients of Airy zeroes, up to 4 significant digits! Other applications of the Airy function in physics are related to asymptotics expansions (Stokes phenomena, WKB method as initialy investigated by Harold Jeffreys in 1923).

In the 80's and the 90's, the Airy function popped up in an apparently unrelated field: combinatorics! It appears in fact in three families of limit laws:
• the area-Airy distributions (for the the area under Brownian motion),
• the Tracy–Widom distributions (for the largest eigenvalue of random matrices),
• the map-Airy distribution (for the size of the largest connected component in a map).

We present these three types of Airy distribution in the next three sections.

2. The area-Airy distributions: Brownian motion, linear probing hashing, additive parameters in grammars

This section is mainly dedicated to the random variable $\mathcal{B}$ giving the area below the Brownian excursion.

2.1. Area under a Brownian excursion. Motivated by a question of Philippe Flajolet, Guy Louchard made the first fundamental step for the birth of "Airy era" in combinatorics: in [61], he considered the area below the Brownian excursion $\mathcal{B} = \int_0^1 e(t)\,dt$ and he found a recurrence for the moments, and that the moment generating function $E\left[e^{-y\mathcal{B}}\right]$ of the area-Airy distribution $\mathcal{B}$ was given by

$$E\left[e^{-y\frac{\mathcal{B}}{\sqrt{8}}}\right] = \sqrt{2\pi}y\sum_{k=0}^{\infty}\exp\left(-\alpha_k y 2^{-1/3}\right) .$$

A formula for the density of $\mathcal{B}$ was given in 1991 by Takács [94]:

$$w(x) = \frac{d}{dx}\mathbb{P}\{\mathcal{B} \leq x\} = \frac{8\sqrt{3}}{x^2}\sum_{k=1}^{\infty} e^{-v_k}\, v_k^{2/3}\, U\left(\frac{-5}{6}, \frac{4}{3}; v_k\right) \qquad v_k = \frac{16\alpha_k^3}{27x^2}.$$

There, the quantities $-\alpha_k$ are the zeros of the Airy function $\mathrm{Ai(z)}$ and $U(a, b; z)$ is the confluent hypergeometric function. (Note that it is striking that this function U is related to the map-Airy distribution, it is unknown if this reflects the fact that $\mathcal{B}$ could be seen as a sum of properly weighted *map-Airy* distributed random variables).

The terminology "area-Airy" distributions (plural) is justified by the fact that similar results hold for the area below the Brownian bridge, Brownian motion, Brownian meander and some variants of them [97, 26, 50]: these area-Airy distributions have a distribution the moments of which are given by $\ln' \mathrm{Ai}(z)$, while their densities are given by a sum of Airy zeroes weighted by an hypergeometric.

The area-Airy distributions appeared since in lot of different contexts: connexity of random graphs [58, 41, 91, 92], area of polyominos [77, 30, 85, 82, 90], in statistical physics, for self-avoiding walks models [63, 84, 81, 87, 5, 79], inversion in trees [46, 72], internal path length in trees [95, 96], additive parameters of context-free grammars (conjecture proven to be true for Q-grammars [30, 83, 6], and some families of trees [34]), Area below discrete lattice paths [98, 69, 65, 99, 70, 9, 50, 57, 89, 88, 10], complexity cost of solving quadratic boolean systems, Gröbner basis computations [13, 12], and last but not least, this distribution is also appearing in the analysis of linear probing hashing [41].

2.2. On the analysis of linear probing hashing. The Flajolet–Poblete–Viola article [41] is solving an old problem, which was in fact Knuth's first analysis of algorithm, in 1962. Don Knuth got so excited by the magical links with some parts of mathematics (like the Ramanujan Q function) that it convinced him to dedicate his life to the field he created, "analysis of algorithms" and to write *The Art of Computer Programming*.

The problem is nicely described in the article, and is equivalent to displacement in parking functions. It leads to the following functional equation:

$$\delta_z F(z,q) = \frac{F(z,q) - F(qz,q)}{1-q} F(z,q)\,.$$

It is a nice coincidence that progresses were done on this problem only 35 years later, in 1997, at the same time by Knuth and Flajolet. Indeed, Knuth found a close form solution [58], while Philippe and its coauthors got the limit law. (More on this story and on the work of Philippe related to hashing can be found in Volume IV, Chapter 4).

To this aim, Philippe developed a nice non-commutative operator approach, and by a "pumping moment" argument, he proves that when the parking has just one free slot, the law of $D_{n+1,n}/n^{3/2}$ converges to the area-Airy distribution.

These two tools (the operator point of view and the pumping moment method, thus leading to a recurrence allowing to identify the limiting distribution) will be at the heart of many later results involving the area-Airy distribution.

Note that by reversing the time, linear probing hashing can be seen as a fragmentation process: see the works of Rényi on “parking functions”, and the works of Bertoin, Pitman on the “additive coalescent”, so we can expect more results involving the Airy function for these processes!

2.3. Analytic variations on the Airy distribution. In the Flajolet–Louchard article [38], an essential rôle is played by what they called the “root zeta function” of the Airy function:

$$\Lambda(s) := \sum_{k=1}^{\infty} (\alpha_k)^{-s}, \qquad \text{where } -\alpha_k \text{ are the zeroes of } \mathrm{Ai(z)}.$$

This sum is a priori well-defined only for $\Re(s) > 3/2$, because of the growth of the $\alpha_k \sim \rho k^{2/3} \left(1 + \sum_{j=1}^{\infty} \frac{a_j}{k^j}\right)$. But there is a nice convergent-divergent trick which gives an analytic continuation of $\Lambda(s)$ on $\mathbb{C}$:

$$\Lambda(s) = \sum_{k \geq 1} (3\pi k/2)^{-2s/3} + \sum_{k\geq 1} \Big((\alpha_k)^{-s} - (3\pi k/2)^{-2s/3} \Big)$$

extends by analytic continuation to $\Re(s) > 0$ (and so on, by subtracting further asymptotics terms. This leads to an expression in terms of the Riemann ζ function, and thus an analytic continuation of Λ for $s \in \mathbb{C}$).

Using Mellin transforms (see Volume III, Chapter 4), the authors get that the moments of the Area-Airy distribution exist for any $s \in \mathbb{C}$ and satisfy

$$\mathbf{E}\left[\left(\frac{\mathcal{A}}{\sqrt{8}} \right)^s \right] = 3\sqrt{\pi} 2^{-s/2} \frac{\Gamma(\frac{3}{2}(1-s))}{\Gamma(-s)} \Lambda\Big(\frac{3}{2}(1-s)\Big),$$

Thus, moments of positive order are given by the expansion of $\mathrm{Ai(z)}$ at $+\infty$ or $-\infty$ while moments of negative order are given by the expansion of $\mathrm{Ai(z)}$ at 0. Accordingly, this leads to sexy evaluations, involving nice mathematical fundamental constants Philippe was in love with, like:

$$E\left[\left(\frac{\mathcal{A}}{\sqrt{8}}\right)^{-5/3}\right] \quad = \quad \frac{9\sqrt{\pi}2^{5/6}}{\Gamma(1/3)^7}\left(3^{1/3}\Gamma(1/3)^6 - 8\; 3^{5/6}\pi^3\right) .$$

Similar phenomena appear in appendix B of [8]. These two articles show that complex analysis gives full access to a lot of informations on the area-Airy distribution and the map-Airy distribution. Large deviations of the area-Airy distributions were later also investigated by Louchard and Janson [62], again motivated by a question of Philippe Flajolet.

It is nice that a slightly modified root zeta function of the Airy function, namely $\Lambda_n(s) := \sum_{k\neq n}(\alpha_k - \alpha_n)^{-s}$, appears in the context of evaluation of quantum-mechanical sum rules and perturbation theory calculations for the Stark effect [14], and leads to tables of relations reminiscent of tables for multi zeta values [42].

2.4. Hachage, arbres, chemins & graphes. Philippe (together with Philippe Chassaing) wrote a survey (for the Gazette of the French Mathematical Society), on “Hachage, arbres, chemins & graphes” [25], which is a very nice overview of the links between the fundamental structures involving the area-Airy distribution and which concludes this chapter on “Philippe Flajolet and the Airy function”.

3. Random matrices, Airy kernel and the Tracy–Widom distributions

Let us start with a problem initially considered by Erdős (in 1935) and Ulam (in 1961): what is the expected length L_n of the longest increasing subsequence of a random permutation of size n? E.g. the permutation **2 3** 6 **4 5** 1 **7** has a longest increasing subsequence of length 5. This highly nontrivial problem have a nice link with algebraic combinatorics, as this length is the width of an associated Young tableau (this gives en passant a linear time algorithm to compute it). A sequel of articles gave the formula, the average, the variance, and finally the limit law of L_n [45, 102, 60, 4].

Although Philippe Flajolet did not publish on the subject (see however pages 227, 532, 598, 716, 752 of the Flajolet–Sedgewick book “Analytic combinatorics”), we give here a short discussion of few results obtained via a determinantal process involving an Airy kernel. It is a part of what Philippe Flajolet was calling the “Airy phenomena”, i.e. the apparition of the Airy function in several combinatorial problems, as a reflect of some analytical phenomena (conjecturally) involving some coalescing saddle points.

In the 90’s, the Airy function also appeared in statistical mechanics, in link with random matrix theory, where some determinantal process involve the following “Airy kernel”: $\frac{\mathrm{Ai(x)Ai'(y)-Ai'(x)Ai(y)}}{x-y}$. The Airy function is finally related to the distribution F_2 of the largest eigenvalue of some random hermitian matrices, a distribution given by Tracy & Widom in 1993 [100]:

$$F_2(s) := \exp\left(-\int_s^\infty (x-s)q(x)^2 dx\right)$$

where q is the solution of the Painlevé II equation $q'' = sq + 2q^3$ satisfying $q(s) \sim \mathrm{Ai}(s)$ as $s \to \infty$.

With respect to the initial Erdős–Ulam problem, Baik, Deift and Johansson [4] proved in 1999 that the length of the longest increasing subsequence in a permutation of size n follows this Tracy–Widom distribution:

$$\mathbb{P}(\frac{L_n - 2\sqrt{n}}{n^{1/6}} \leq x) \to F_2(x)\,.$$

There exists two related distributions $F_1(x)$ and $F_4(x)$. It is striking that some recent results in physics show that $F_2(s)$ is the law for gaps in superconductors. Many other statistics involve this type of Airy distribution [93, 28, 54, 55].

4. The map-Airy distribution: coalescing saddle points, connectivity in graphs and maps

4.1. First cycle in an evolving graph. Since the pioneering work on random graphs by Erdős & Rényi in 1959, many results obtained via probabilist tools showed some fascinating transition phases when the proportion of edges is increasing (see the numerous works of Bollobás' school [21]).

After the works of E. M. Wright (of Hardy & Wright fame) and T. R. Walsh, the Flajolet–Knuth–Pittel article [37] is the second important step for the introduction of analytical methods in the study of random graphs. It strongly relies on identities related to the Lambert function (the Cayley tree function) and on the saddle point method [1]; the authors prove that the first cycle has length $\sim Kn^{1/6}$ where

$$K = \frac{1}{\sqrt{8\pi i}} \int_{-\infty}^{\infty} \int \exp\left((\mu+2s)\frac{(\mu-s)^2}{6}\right) \frac{ds}{s} d\mu \approx 2.0337\,.$$

When Philippe was fed up of some administrative work, he was joining the coffee room, by asking "give me an integral!", as this was like a playtime for him. The above evaluation (up to four digits) also reflects his love for numerical evaluation of mathematical constants, in the spirit of [35], a book he liked a lot.

The Flajolet–Knuth–Pittel article is thus the direct predecessor of the major analytical work on random graphs: the " giant paper on the giant component" [52], which relies also strongly on the Wright coefficients (that we already encountered in our Section on the area-Airy distributions!).

1. The saddle point method was one of the three main asympotical methods of Philippe, together with singularity analysis and Mellin tranform techniques, a preliminary version of Flajolet–Sedgewick's "Analytic Combinatorics" had a chapter dedicated to the saddle point method. There is however no specific chapter on this method in these complete works, but the union of this chapter and Chapter 2 of Volume I plays this rôle.

In the 2000's, similar approaches were used to analysed graphs avoiding some pattern, hypergraphs, connectivity in some class of graphs, planar graphs [80, 76, 47, 29], and transition phases in satisfaction problems (k-SAT problems) and other NP-complete problems [49, 67, 11]. We also refer the reader to Volume IV, Chapter 5 to get more on Philippe's work on Random Graphs, Mappings, Maps.

4.2. Random maps, coalescing saddles, singularity analysis, and Airy phenomena.

4.2.1. *Random maps.* Initially motivated by the question of uniform random generation of some families of connected maps, the Banderier–Flajolet–Schaeffer–Soria article [8] gives also some universal results for asymptotics involving coalescing saddle points and for compositions of functions singular at the same time.

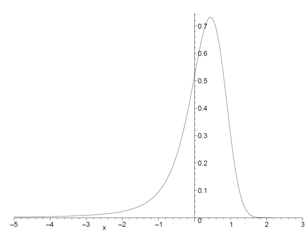
A map is a planar graph drawn on the sphere. Maps were intensively studied by Tutte, who wanted to refute the 4 colour theorem, he then found a way to enumerate maps (sequence of articles in the 60's). Gilles Schaeffer's Phd thesis extended these results of Tutte, and gave a bijective explanation for many variants of maps, and tackled the question of uniform random generation of connected maps. It is easy and fast to generate a (non-connected) map of size N, and to extract its largest connected component, so just do some rejection and pray so that you get a connected map of the wanted size n. Or... use analytic combinatorics and find the optimal value of N so that you do on average the least number of rejections!

The fastest algorithm consists in choosing $N = 3n - (3n)^{2/3}x_0$, where x_0 is the location of the peak of $\mathcal{A}(x)$, the density of the map-Airy distribution.

This density satisfies:

$\mathcal{A}(x) = 2\exp\left(-\frac{2}{3}x^3\right)\left(x\mathrm{Ai}(\mathrm{x}^2) - \mathrm{Ai}'(\mathrm{x}^2)\right)$

Here is a plot of $\mathcal{A}(x)$:



This idea of tuning a rejection algorithm to do some faster uniform random generation of combinatorial structures was puzzling Philippe and he then got the idea of what he called the Boltzmann method [31], this method was a true revolution (improving by several order of magnitudes the state of the art), and we urge the reader to have a look on it! The key idea consists in giving a *value* to the variable of the generating function, this confers a probabilist weight to the associated combinatorial objects. Then you just need a pinch of symbolic method, analytic method, limit laws, automatisation to get a uniform random generator for your favourite combinatorial structure. This was really "the cherry on the cake" of 30 years of work of Philippe, coming as a rape fruit in 2001. A later evolution of this idea lead to what Philippe called the Buffon machines [39], a very efficient way to simulate exactly many distributions (even involving transcendental numbers). All of this is presented in PFAC Volume VI, Chapter 3.

4.2.2. *Coalescing saddle points.* The first asymptotic key tool in the Banderier–Flajolet–Schaeffer–Soria article [8] is a variant of the saddle point method. Indeed, applying the Cauchy formula on an expression coming from the work of Tutte, the probability that a map of size n has a kernel of size k is given as an integral:

$$Pr(X_n = k) = \frac{M_{n,k}}{M_n} = C_k \frac{1}{2i\pi} \int_\gamma F(z) G(z)^k H(z)^n dz\,.$$

For a specific ratio n/k, a local expansion (and a lot of technical details) leads to $\int_\Gamma \exp(n(a_0 + a_1(z-\tau) + a_2(z-\tau)^2 + a_3(z-\tau)^3 + \dots))dz$; a wonderful phenomena is that, for many families of maps, one has here a double saddle (in fact two coalescing saddle points) so $a_1 = 0$ and $a_2 = 0$, and the integral is then clearly just an avatar of the Airy function.

4.2.3. *Singularity analysis.* The Apendix A of [8] is dedicated to singularity analysis of critical schemes of the type $[z^n u^k] f(ug(z))$. For a specific ratio n/k, such composition schemes are shown to lead to some limiting distributions which are stable distributions. Stable distributions are typically the limit laws of sums of iid random variables (with infinite variance), however, for all our combinatorial problems, they pop up in contexts where they are not expressed as sum of random variables, or those variables are dependent. This suggests that Lévy theory of stable distributions / the generalisation of the Gnedenko–Kolmogorov generalised central limit theorem to the case of dependent variable may exist.

It is nice that complex analysis gives the distribution of these limit laws via some Hankel contour integration: for any parameter $\lambda \in (0, 2)$, define the entire function

$$G(x,\lambda) := \begin{cases} \dfrac{1}{\pi} \displaystyle\sum_{k\geq 1} (-1)^{k-1} x^k \frac{\Gamma(1+\lambda k)}{\Gamma(1+k)} \sin(\pi k \lambda) & (0 < \lambda < 1) \\ \dfrac{1}{\pi x} \displaystyle\sum_{k\geq 1} (-1)^{k-1} x^k \frac{\Gamma(1+k/\lambda)}{\Gamma(1+k)} \sin(\pi k/\lambda) & (1 < \lambda < 2) \end{cases}$$

Note that the "symmetry" $\lambda < 1$ vs $\lambda > 1$ is explained by Zolotarev reciprocity law, and that a parameter (the "skewness") of the stable distribution is here equal to 0. Philippe and its coautors proved that the coefficient of z^n in a large power $g(z)^k$ of a fixed algebraic–logarithmic function $g(z)$ with singular exponent λ admits asymptotic estimates involving this stable distribution density $G(x, \lambda)$, as detailed in Theorem 11 and 12 of [8]. This covers Zipf laws, Cauchy distribution ($\lambda = 1$), Rayleigh distribution ($\lambda = 1/2$), our map-Airy / Holtsmark distribution ($\lambda = 3/2$), while for $\lambda \geq 2$ one has a Gaussian distribution.

Due to the ubiquity of such composition scheme, many later articles are in fact related to it, while tackling different topics: random maps [56, 20, 19, 17, 75, 22, 44, 33, 15], planar graphs [64, 18, 23, 71, 16], limiting objects in probability theory [86, 3, 103, 27, 48], percolation [2], statistical mechanics [32], asymptotics of bivariate meromorphic functions [73, 59, 74]...

4.3. Airy phenomena and analytic combinatorics of connected graphs. In [43], Flajolet, Salvy and Schaeffer show that it is possible to make analytic sense of the divergent series that expresses the generating function of connected graphs. This builds on works by E.M. Wright in the 70/80's, the Knuth–Flajolet–Pittel article [37] and the Janson–Knuth– Łuczak–Pittel "giant paper on the giant component" [52].

The enumeration of connected graphs by excess (of number of edges over number of vertices) derives from a simple saddle-point analysis. Furthermore, a refined analysis based on coalescent saddle points yields complete asymptotic expansions for the number of graphs of fixed excess, through an explicit connection with Airy functions. The amazing part of this work relates asymptotics at -1 and at 0.

Note that another analysis of a divergent generating function was presented by Svante Janson [51], at the occasion of conference celebrating Philippe's 60th birthday.

5. Anecdotes

We have seen at the beginning of our introduction that the Airy function was introduced by an astronomer, and while Philippe was working on the Airy function in 1998, he got amused to discover on the web a photography of his own grandfather, who was also an astronom.

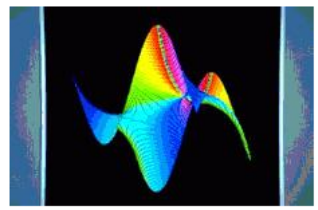
Photography of Philippe Flajolet (1885-1948), meteorologist and astronom at the Observatory of Lyon and grandfather of Philippe. Philippe's mother was deeply appreciating her father-in-law, who died while she was pregnant, she then decided to call her unborn son "Philippe". In 1998, Philippe Flajolet got amazed when he discovered the photography of its grandfather on internet, and he then wrote the webpage algo.inria.fr/flajolet/numbers.html where he was making a pun by saying he had "Erdös number 2" but "Flajolet number 0". However Doron Zeilberger later argued [104] that Philippe should have Flajolet number 7, with an amazing path via the posthumous presentation by Hadamard of Jean Merlin's work, and then via any student of Hadamard.

As a second anecdote, Philippe Flajolet and Bruno Salvy were working in August 1997 on a Maple session for analysing connectivity in random graphs (this later lead to the article [43]). Then an INRIA communications manager knocked at the door and ask them what can of logo INRIA could use for the Algorithms Project (the team whose the fearless leader was Philippe...), Philippe then pinpointed the nice monkey saddle on the screen, and began to explain to the guy why it would be a very nice illustration. More than a logo, this monkey saddle (which is a kind of signature of the Airy phenomena) stayed until the end the blazon of the Algorithms Project, appearing on all its webpages. So let us end this introduction with the own words of Philippe:

Algorithms Project's Logo

Welcome!	Research Topics	People	Publications	Seminars	Software	On-Line Applications	Jobs & Internships

Our logo shows the behaviour in the complex plane of the generating function of connected graphs counted according to number of nodes and edges. In critical regions, two saddle points coalesce giving rise to a so-called "monkey saddle" (a saddle that you'd use if you had three legs!)



The fine analysis of this coalescence is crucial to the understanding of connectivity in random graphs. This problem has applications in the design of communication networks and it relates to a famous series of problems initiated by Erdös and Renyi in the late 1950's. See the paper [Janson, Knuth, Luczak, Pittel: The birth of the giant component. Random Structures Algorithms 4 (1993), no. 3, 231--358]. As said by Alan Frieze in his review [**MR**94h:05070]:

> *"This paper and its predecessor [**MR**90d:05184] mark the entry of generating functions into the general theory of random graphs in a significant way. Previously, their use had mainly been restricted to the study of random trees and mappings. Most of the major results in the area, starting with the pioneering papers of P. Erdös and A. Renyi [**MR**22#10924] have been proved without significant use of generating functions. However, at the early stages of the evolution of a random graph we find that it is usually not too far from being a forest, and this allows them an entry..."*

The icon was generated by Maple code like this:

```
f:=u^3/3-3*u+1;
plots[complexplot3d]([Re(f),argument(f)],
                u=-4-3*I..4+3*I,style=patchcontour,
                        contours=30,numpoints=50*50);
```

Last modified: September 17, 2003. For problems or questions involving this site, please contact Virginie Collette.

http://algo.inria.fr/logo.html

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