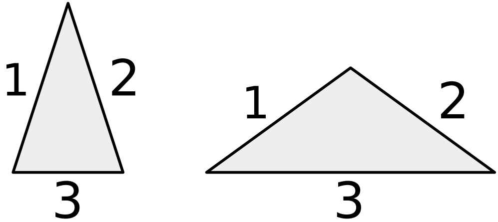
Combinatorial Substitutions and Sofic Tilings

Thomas Fernique & Nicolas Ollinger

Turku, December 16, 2010

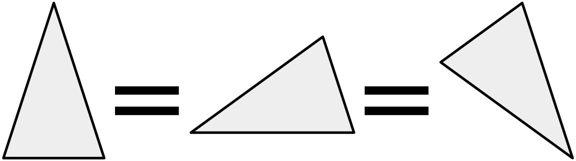
1. Sofic Tilings
2. Combinatorial Substitutions
3. Main result

Tiles and tilings



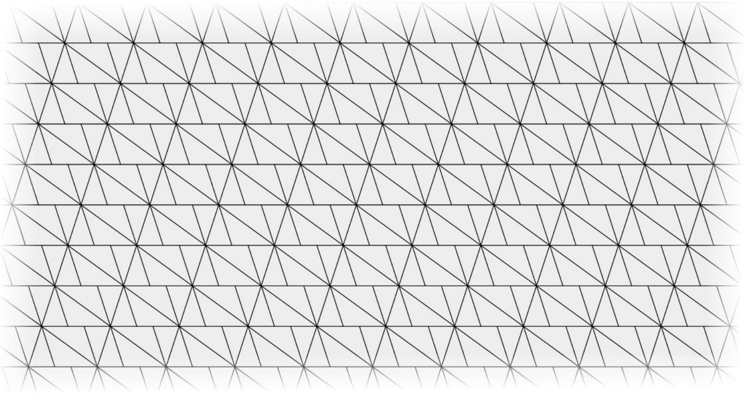
Tile: polytope of $\mathbb{R}^d$ with finitely many (numbered) facets.

Tiles and tilings



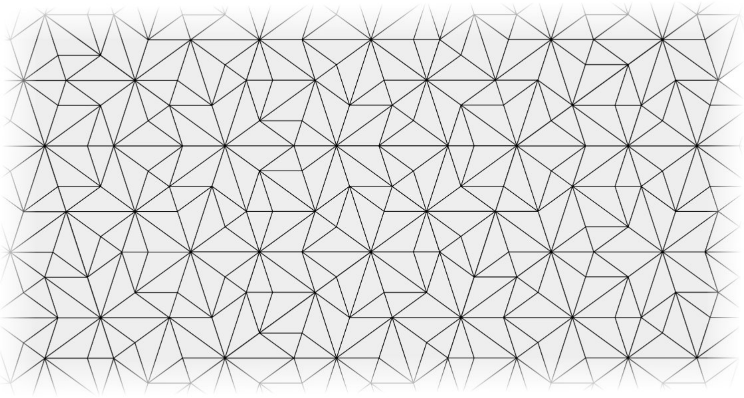
Tiles are here considered up to translations and rotations.

Tiles and tilings



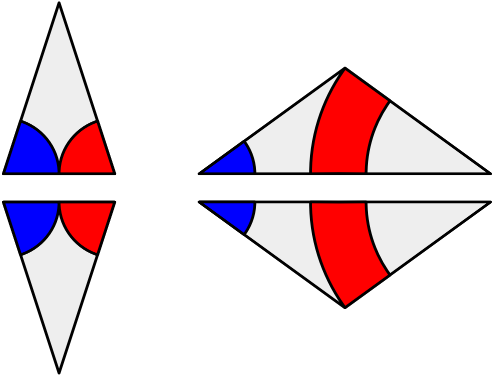
Tiling: covering of $\mathbb{R}^d$ by *facet-to-facet* tiles.

Tiles and tilings



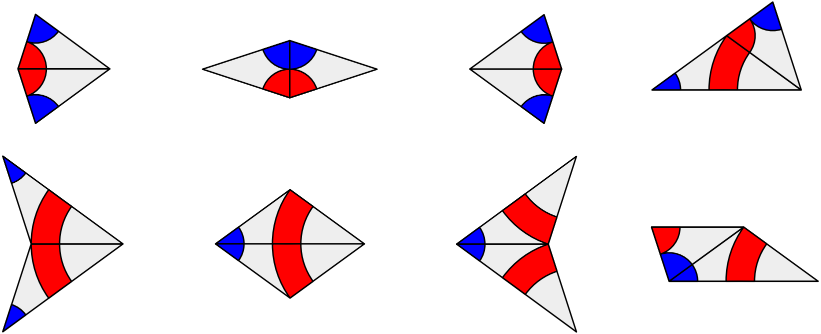
Tiling: covering of $\mathbb{R}^d$ by *facet-to-facet* tiles.

Decorations



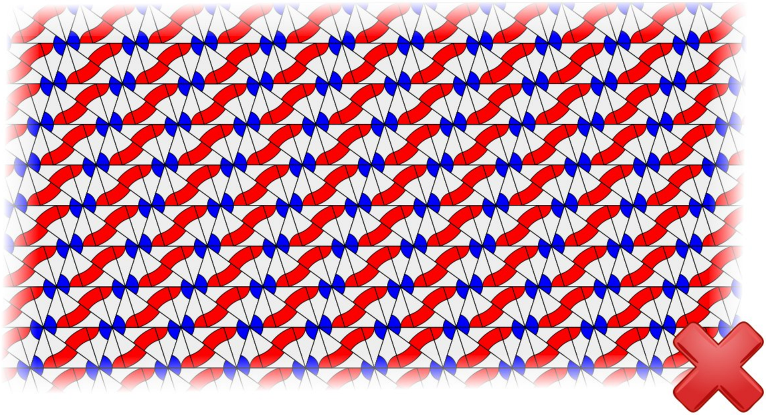
Decoration maps each point of tile boundaries to a color.

Decorations



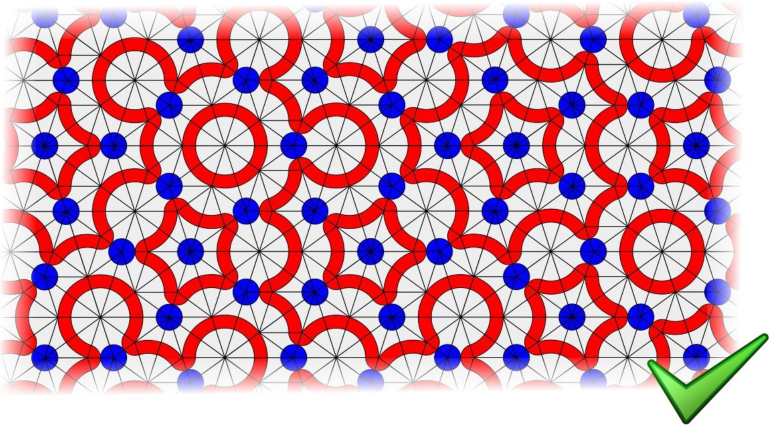
Decorated tiles *match* if decorations are equal over common facets.

Decorations



Decorated tiling: tiling by matching decorated tiles.

Decorations



Decorated tiling: tiling by matching decorated tiles.

Sofic tilings

Decorated tile set $\tau \rightsquigarrow$ set Λ_τ of decorated tilings.

Let π be the map which removes tile decorations.

Definition (Sofic tilings)

A set of tilings is said to be *sofic* if it can be written as $\pi(\Lambda_\tau)$, where τ is a finite decorated tile set.

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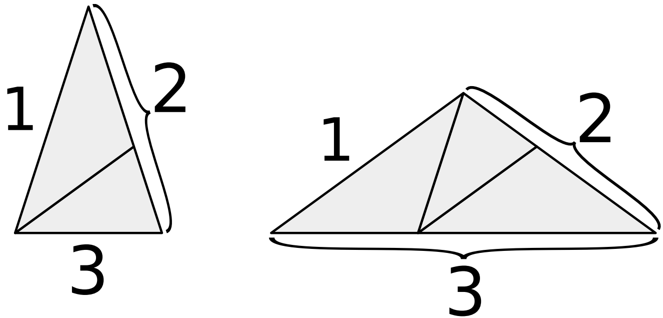
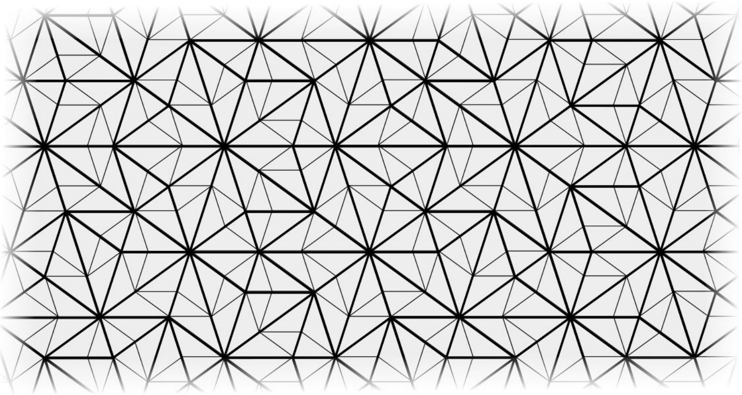
What (interesting) properties on tilings can be enforced by soficity?

Macro-tiles and macro-tilings



Macro-tile: finite partial tiling with (numbered) *macro-facets*.

Macro-tiles and macro-tilings



Macro-tiling: *macro-facet-to-macro-facet* tiling by macro-tiles.

Combinatorial substitution

Definition (Combinatorial substitution)

A *combinatorial substitution* is a finite set of pairs tile/macro-tile.

Let $\sigma = \{(P_i, Q_i)\}_i$ be a combinatorial substitution.

Preimage under σ of a tiling by the P_i's: macro-tiling by the Q_i's with the same *combinatorial structure*.

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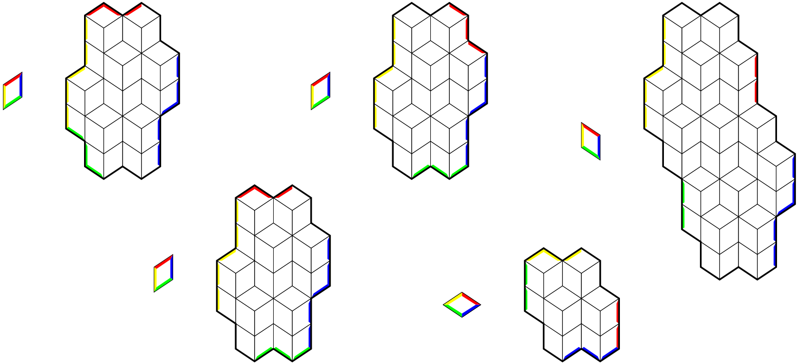
Definition (Limit set)

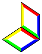
The *limit set* of a combinatorial substitution σ is the set of tilings which admit an infinite sequence of preimages under σ.

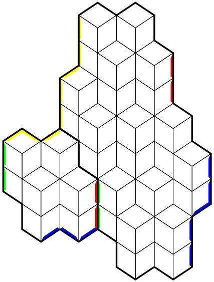
Example



These pairs define the so-called *Rauzy combinatorial substitution*.

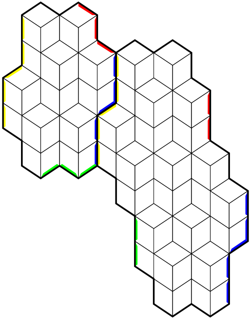
Example





Tiles match in a tiling as macro-tiles in its image (and conversely).

Example



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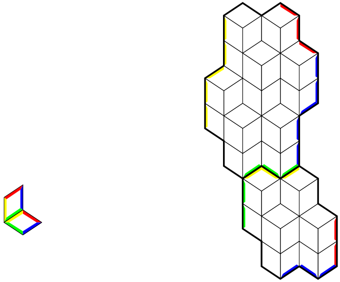

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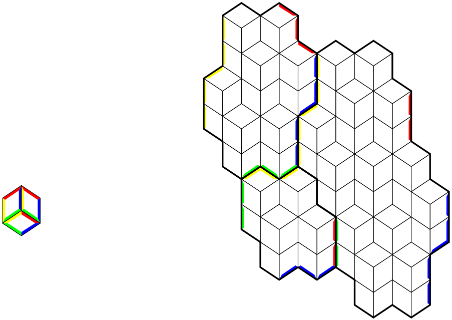

Tiles match in a tiling as macro-tiles in its image (and conversely).

Main result

Our main result is a constructive proof of the following result:

Theorem (Fernique-Ollinger)

The limit set of a (good) *combinatorial substitution is sofic.*

This extends (and simplifies?) previous similar results:

- Shahar Mozes in 1990 (rectangular substitutions);
- Chaim Goodman-Strauss in 1998 (homothetic substitutions).

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Complete detailed proof: abstract.
Here: sketch of the main part.

Self-simulation

Decorated macro-tile and *decorated macro-tiling*: straightforward.

Definition (Self-simulation)

A decorated tile set τ *self-simulates* if there are τ-macro-tiles s.t.

1. any τ-tiling is also a macro-tiling by these τ-macro-tiles;
2. each τ-macro-tile is *combinatorially equivalent* to a τ-tile.

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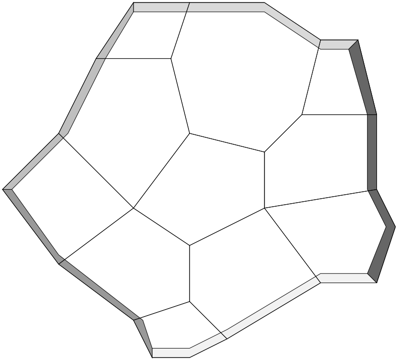
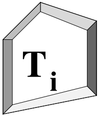
Proposition

If τ self-simulates, then $\pi(\Lambda_\tau)$ is a subset of the limit set of the combinatorial substitution with pairs τ-macro-tile/equivalent τ-tile.

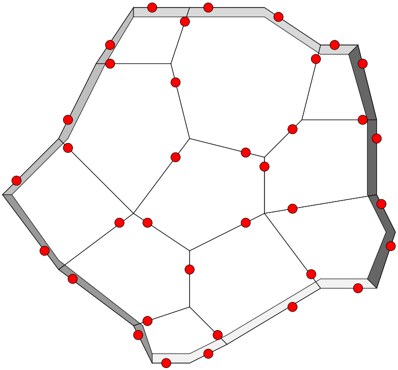
A self-simulating decorated tile set τ

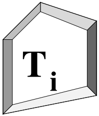




Fix a set of macro-tiles and let $T_1, \ldots, T_n$ be all their tiles.

A self-simulating decorated tile set τ





To enforce τ-tilings to be τ-macro-tilings: decorations specify tile neighbors within macro-tiles and mark macro-facets.

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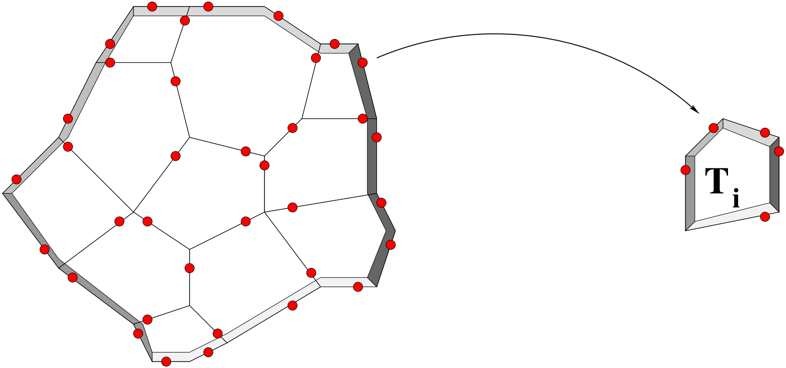

This yields so-called macro-indices on tile facets.

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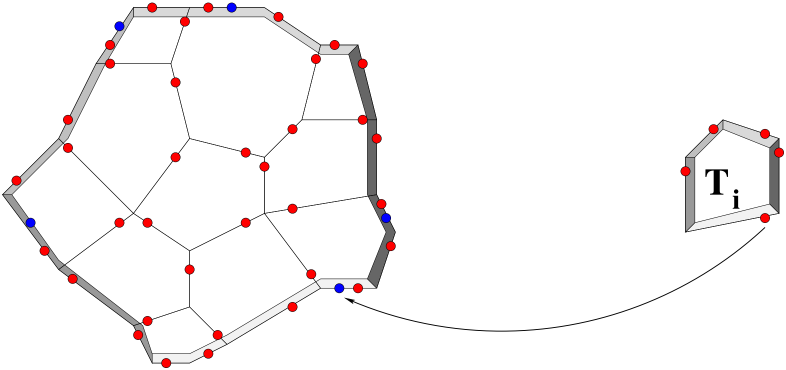
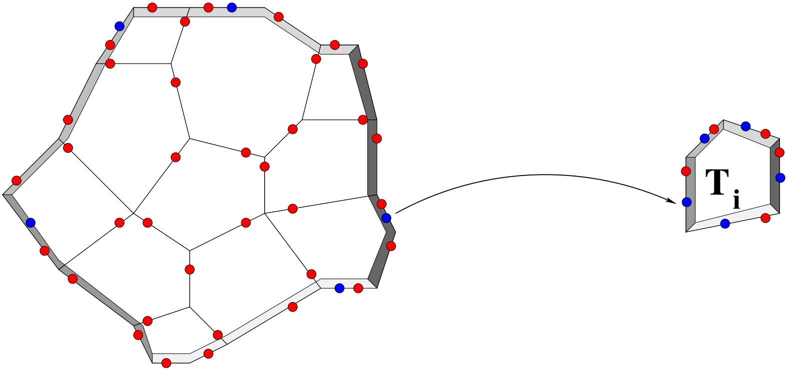


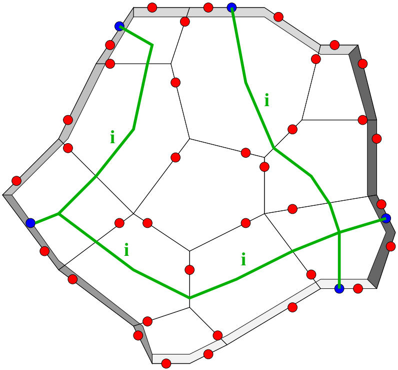
The macro-indices of facets of a τ-tile must then be encoded on the corresponding macro-facets of its simulating τ-macro-tile.

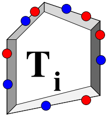
A self-simulating decorated tile set τ



This yields so-called neighbor-indices on tile facets.

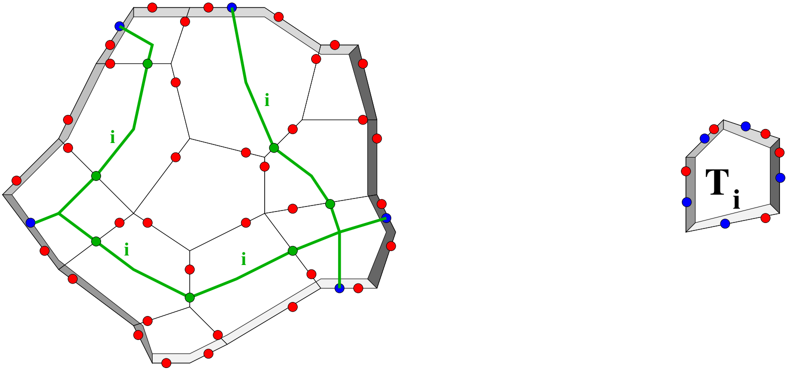
A self-simulating decorated tile set τ





We force these neighbor-indices to come from the same tile T_i, called parent-tile, by carrying its index i between macro-facets, where it is converted into the suitable neighbor-index.

A self-simulating decorated tile set τ



Such tile indices are encoded on facets by so-called parent-index.

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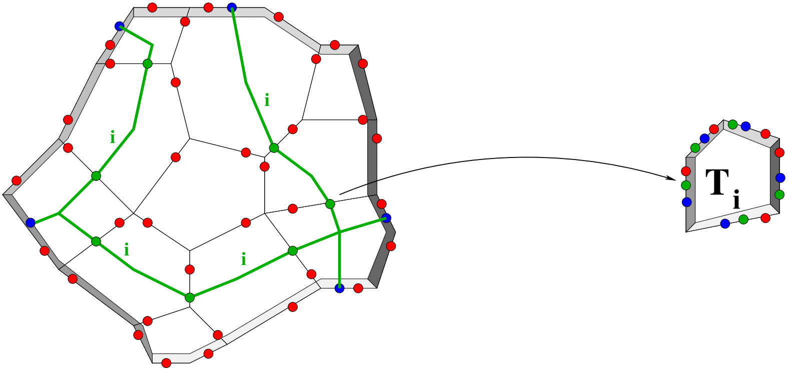
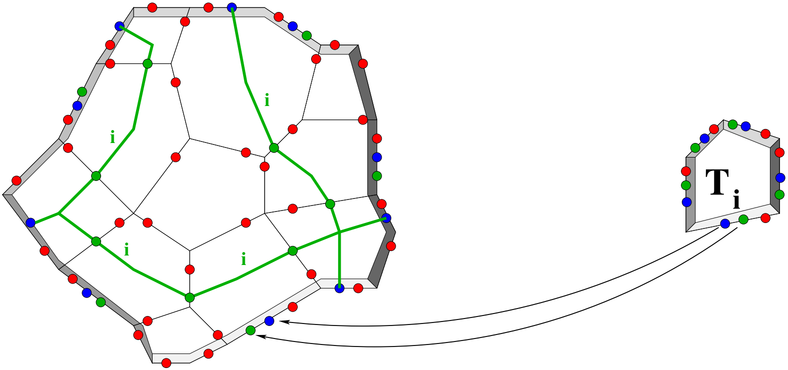


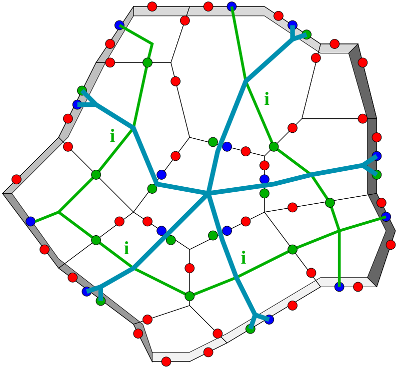
This yields, once again, a new index on each tile facets. . .

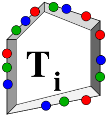
A self-simulating decorated tile set τ



But the trick is that the neighbor-indices and parent-indices of facets of a τ-tile can be encoded on the corresponding big enough macro-facets of the equivalent τ-macro-tile without any new index!

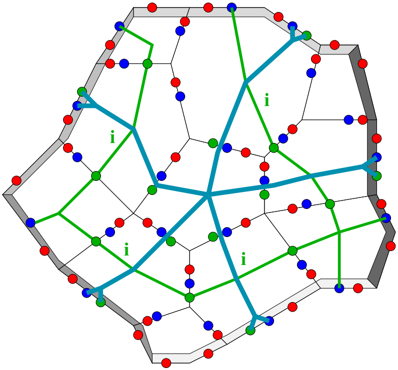
A self-simulating decorated tile set τ

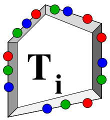




In big enough macro-tiles, we can then carry these pairs of neighbor/parent indices up to a central tile along a star-like network.

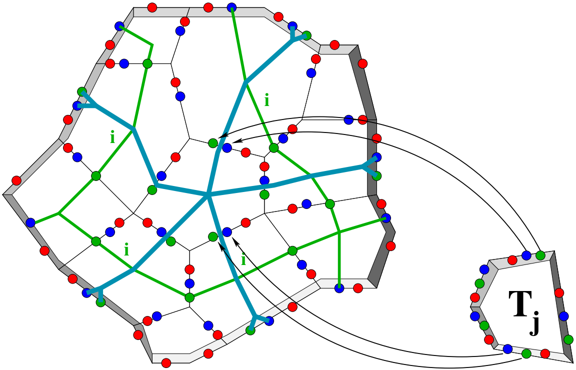
A self-simulating decorated tile set τ





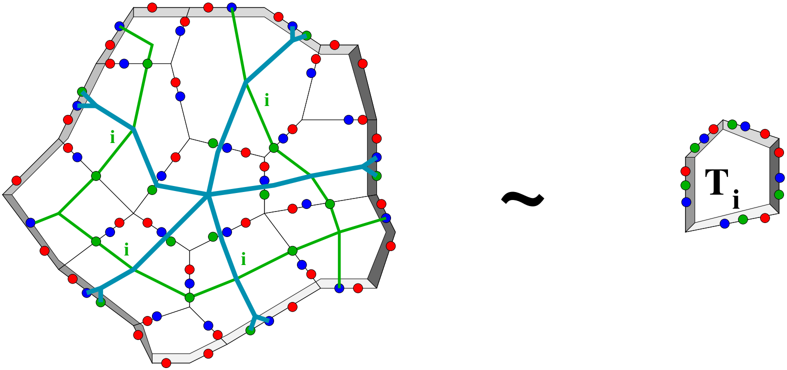
On internal facets not crossed by this network, we copy the macro-index on the neighbor-index (this redundancy is later used).

A self-simulating decorated tile set τ



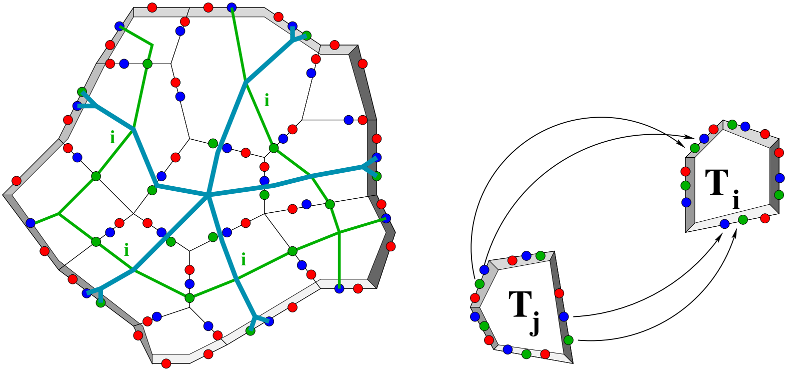
The pairs on a central τ-tile can be those of any non-central τ-tile (from which the central τ-tile is said to derive).

A self-simulating decorated tile set τ



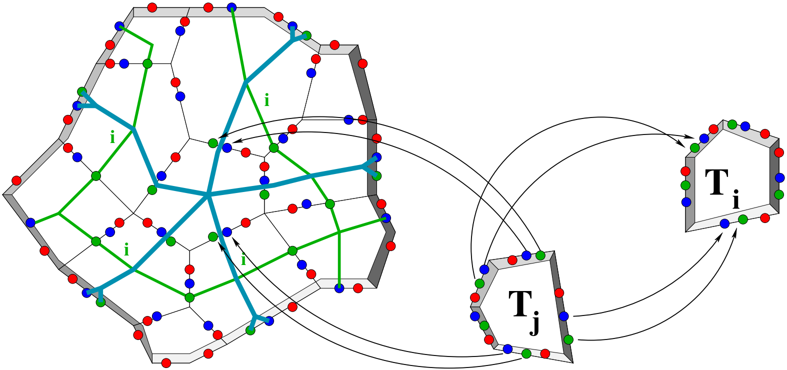
The τ-macro-tile with parent-index i is combinatorially equivalent to T_i endowed with the pairs of the central τ-tile. But is it a τ-tile?

A self-simulating decorated tile set τ



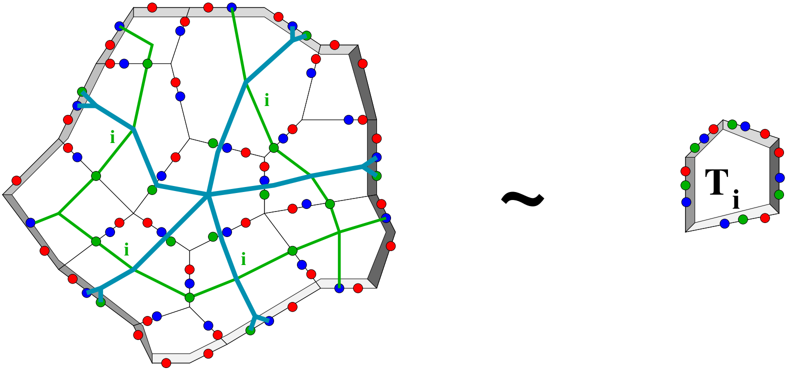
If T_i is a central tile, then its pairs can be derived from any non-central τ-tile (as for any central tile)...

A self-simulating decorated tile set τ



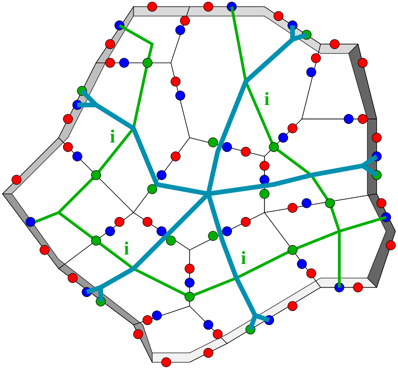
. . . in particular from the non-central τ-tile from which are also derived the pairs of the central τ-tile of our τ-macro-tile.

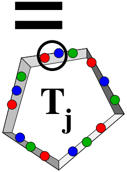
A self-simulating decorated tile set τ



In this case, the equivalent decorated T_i is a derived central τ-tile.

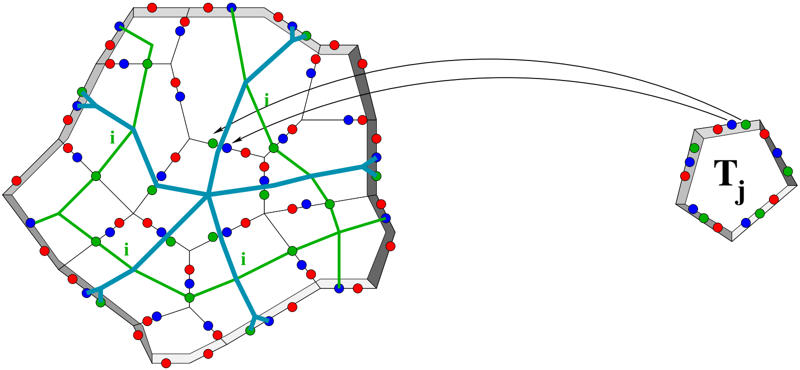
A self-simulating decorated tile set τ





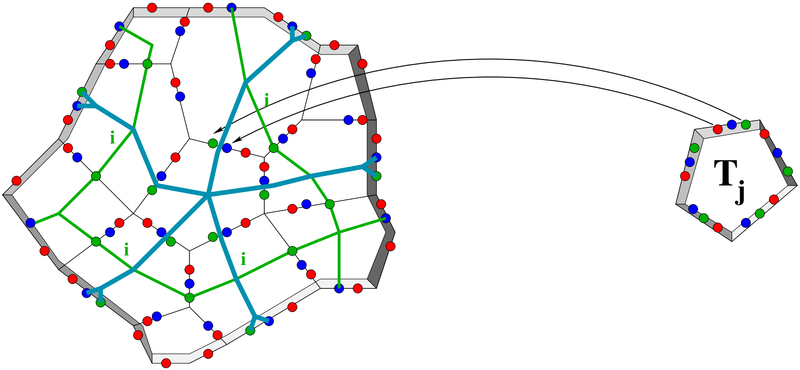
Otherwise, consider the non-central τ-tile from which derives our central τ-tile; at least one facet is internal and not crossed by a network: its neighbor and macro indices are equal (by redundancy).

A self-simulating decorated tile set τ



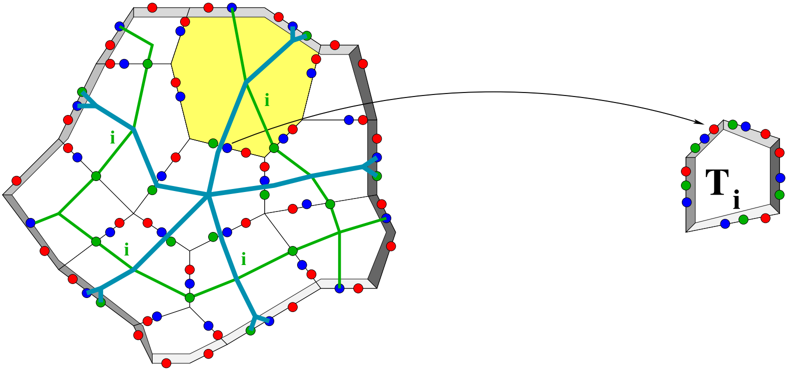
Thus, by copying the neighbor and parent indices (derivation)...

A self-simulating decorated tile set τ



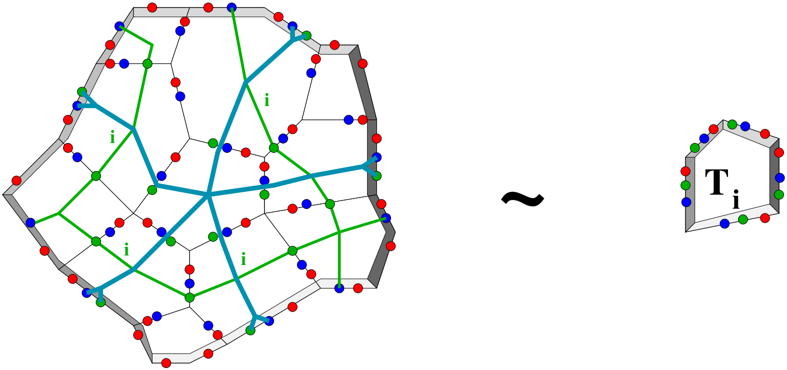
...one copies a macro-index on our central τ-tile, and thus on the whole corresponding network branch.

A self-simulating decorated tile set τ



A tile on this k-th branch which also knows the parent-index i can then force this macro-index to be the one on the k-th facet of a decorated T_i (recall that all the decorated T_i have the same one).

A self-simulating decorated tile set τ



In this case, the equivalent decorated T_i is the non-central τ-tile from which derives the central τ-tile of our τ-macro-tile.

The End