A Characterization of flip-accessibility for rhombus tilings of the whole plane

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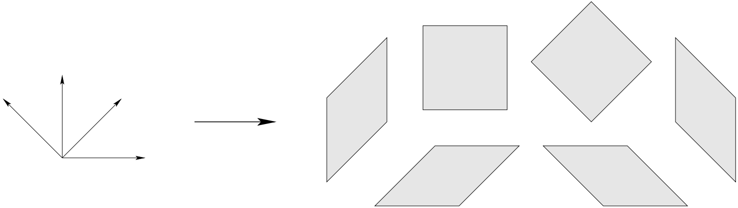
LATA'07 (Tarragona)

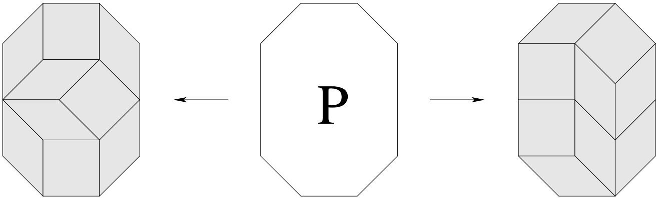
1. Rhombus tilings and flips
2. Stepped surfaces and shadows
3. Characterization of flip-accessibility

$\vec{v}_1, \ldots, \vec{v}_d$ in $\mathbb{R}^2 \rightsquigarrow$ rhombus tiles $T_{ij} = \{\lambda \vec{v}_i + \mu \vec{v}_j \mid \lambda, \mu \in [0,1]\}$.



In physics: models micro-arrangements of atoms ($\simeq$ stable).

Rhombus tilings of a polygon P: partitions by translated T_{ij}'s.



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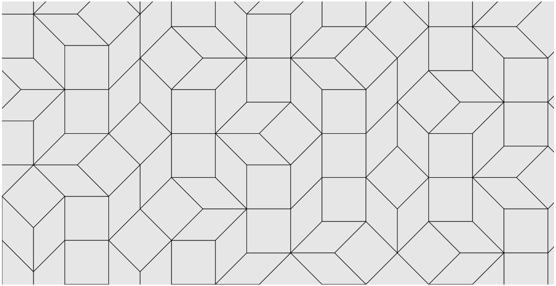
Rhombus tilings of the whole plane: $\mathbb{R}^2$ instead of a polygon P.



In physics: models (quasi)crystals, that is, (a)periodic material.

Flips

Flip: local exchange between the two possible tilings of a hexagon.

In physics: models local rearrangement of inter-atomic links.

Theorem (Kenyon, 1993)

If $\mathcal{T}$ and $\mathcal{T}'$ are rhombus tilings of a polygon P, then one can transform $\mathcal{T}$ into $\mathcal{T}'$ by performing a finite sequence of flips.

$\rightsquigarrow$ Notion of *flip-accessibility*.

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$\rightsquigarrow$ Notion of *flip-accessibility*.

Does not hold for tilings of $\mathbb{R}^2$ considering only *finite* sequences.

Extended notion of flip-accessibility for rhombus tilings of $\mathbb{R}^2$:

Definition

A tiling $\mathcal{T}'$ is *flip-accessible* from a tiling $\mathcal{T}$ if there is a sequence $(\mathcal{T}_n)_{n\geq 0}$ of tilings such that $\mathcal{T}_0 = \mathcal{T}$, $\mathcal{T}_{n+1}$ is obtained from $\mathcal{T}_n$ by performing a flip, and $\lim_{n\infty} d(\mathcal{T}_n, \mathcal{T}') = 0$,

where d distance over rhombus tilings of $\mathbb{R}^2$ defined by:

$$d(\mathcal{T}, \mathcal{T}') = \inf\{2^{-r} \mid \mathcal{T} \cap B(\vec{0}, r) = \mathcal{T}' \cap B(\vec{0}, r)\}.$$

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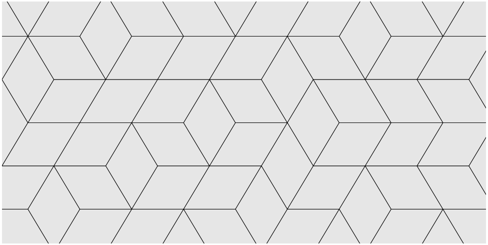
Does it make tilings of $\mathbb{R}^2$ flip-accessible? Not always. . .

$(\vec{e}_1, \ldots, \vec{e}_d)$ canonical basis of $\mathbb{R}^d$ $\rightsquigarrow$ d-dim. viewpoint of tilings:

tile $\vec{x} + T_{ij}$ $\rightsquigarrow$ unit face $(\vec{x}, t_{ij}) = \{\lambda\vec{e}_i + \mu\vec{e}_j \mid \lambda, \mu \in [0,1]\}$;

tiling $\rightsquigarrow$ sort of rugged surface made of unit faces.



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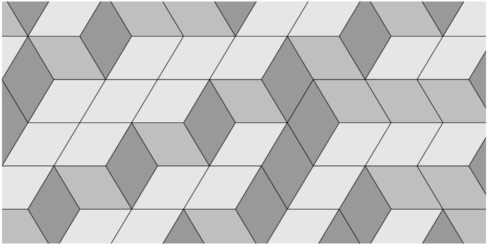


Especially natural when $d = 3$, using coloured tiles.

More precisely, let $\Psi : \mathbb{R}^d \to \mathbb{R}^2$ be defined by:

$$\Psi \ : \ (x_1, \ldots, x_d) = \sum_{i=1}^{i=d} x_i \vec{v}_i.$$

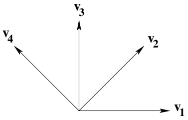
Definition

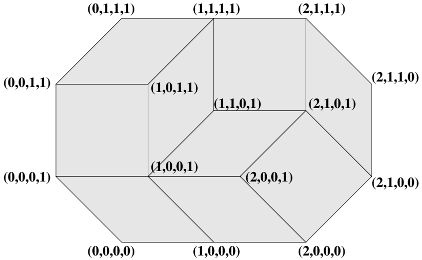
Stepped surface $\mathcal{S}$: set of unit faces homeomorphic by Ψ to $\mathbb{R}^2$.

Then, $\{\Psi(\vec{x}, t_{ij}) \mid (\vec{x}, t_{ij}) \in \mathcal{S}\}$ is a rhombus tiling of $\mathbb{R}^2$.

Conversely, following principles introduced by Thurston (1990), we define a *height function* h from the vertices of a tiling $\mathcal{T}$ to $\mathbb{Z}^d$ by:

$$(\vec{x}, \vec{x} + \vec{v}_i) \text{ edge of } \mathcal{T} \;\Leftrightarrow\; h(\vec{x} + \vec{v}_i) - h(\vec{x}) = \vec{e}_i.$$

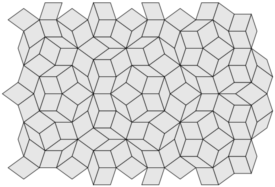




$\{(h(\vec{x}), t_{ij}) \mid \vec{x} + T_{ij} \in \mathcal{T}\}$ homeo. to $\mathcal{T}$ by $\Psi \rightsquigarrow$ stepped surface.

Projection π_i: removes the i-th entry of a real vector.

k-shadows of a stepped surface: images by k such projections.



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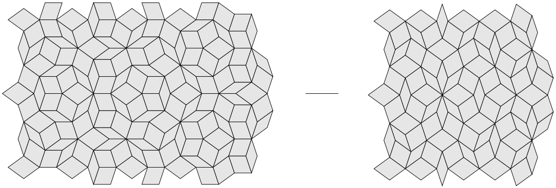

Shadows

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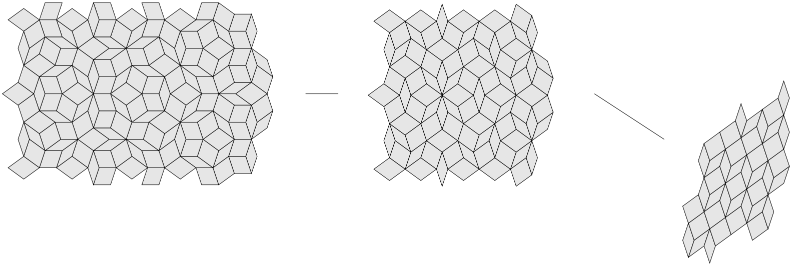
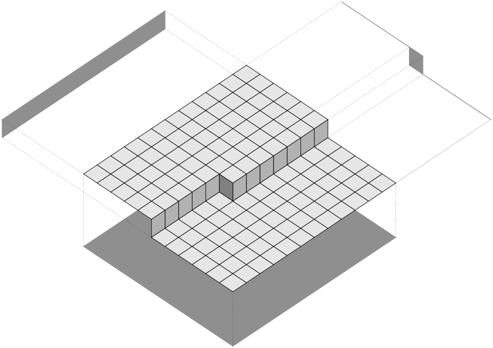


$k \leq d - 3 \rightsquigarrow d$-dim. stepped surface charac. by its k-shadows.

$(d-2)$-shadows (or *shadows*) are subsets of $\mathbb{R}^2$:



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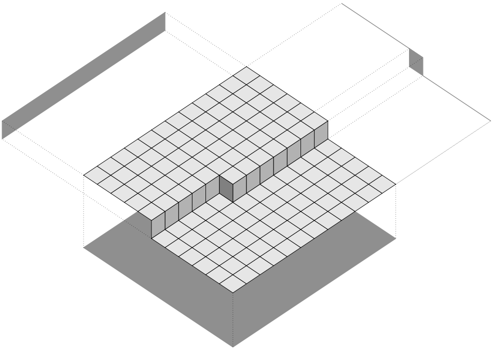
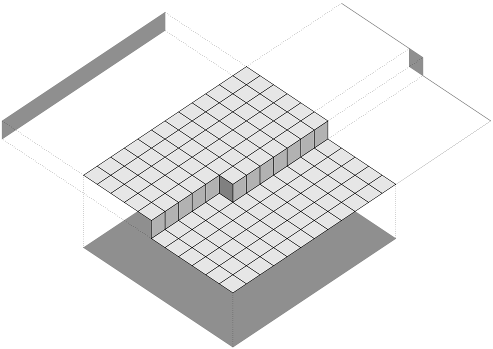


Note: different stepped surfaces can have identical shadows.

Note also: by performing flips, shadows never increase. . .



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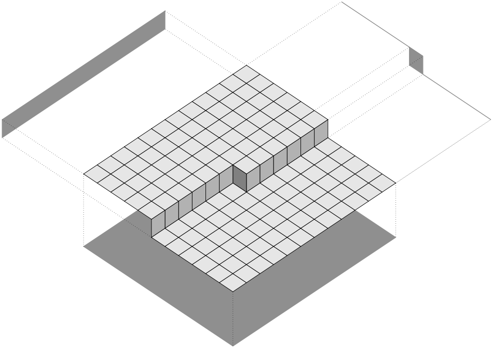

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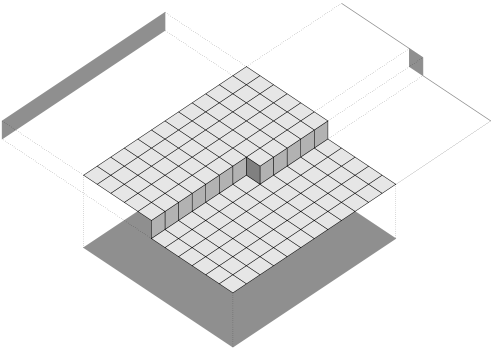

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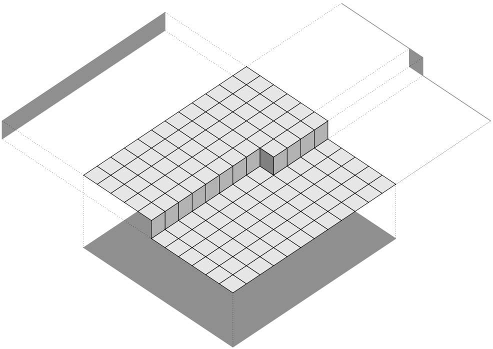

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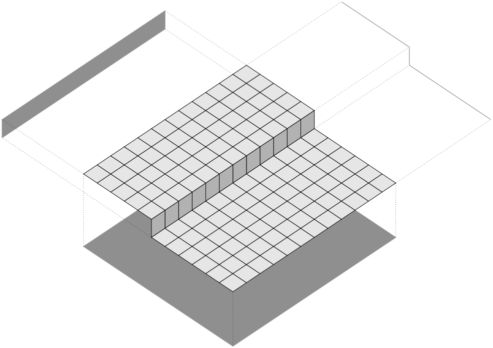


but can decrease. Thus, flip-accessibility $\Rightarrow$ inclusion of shadows.

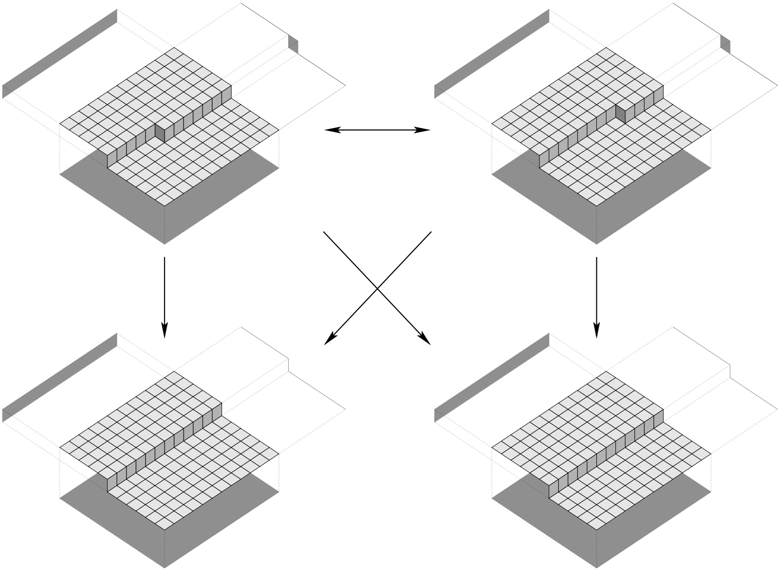
Main result: the previous necessary condition is sufficient:

Theorem

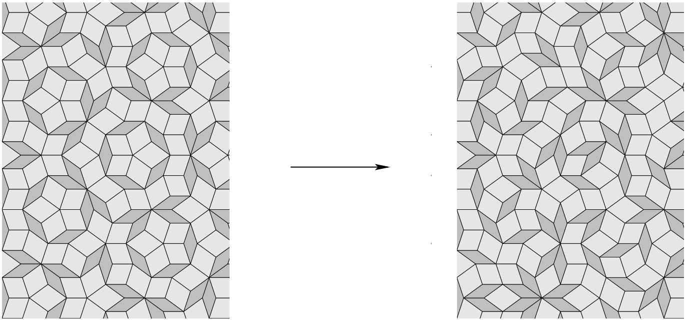
A tiling $\mathcal{T}'$ is flip-accessible from a tilings $\mathcal{T}$ iff the shadows of the former are (respectively) included in the shadows of the latter.

Thus, flip-accessibility does not always hold (and is not symmetric).

Characterization

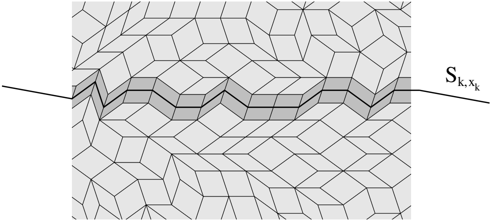


Ex: slice of $\mathbb{R}^d \rightsquigarrow$ *canonical projection tiling* (Penrose *etc.*).
Shadows equal to $\mathbb{R}^2 \Rightarrow$ “source” for flip-accessibility:



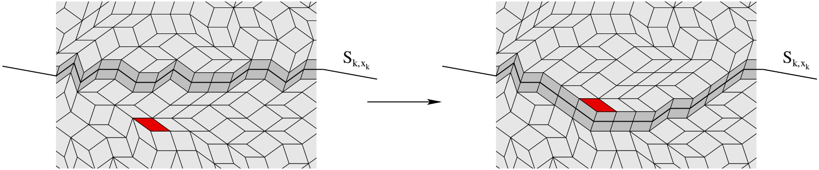
Main tool: *de Bruijn sections* of a stepped surface $\mathcal{S}$:

$$S_{i,k} = \{((x_1, \ldots, x_d), t_{ij}) \in \mathcal{S} \mid x_i = k\}.$$

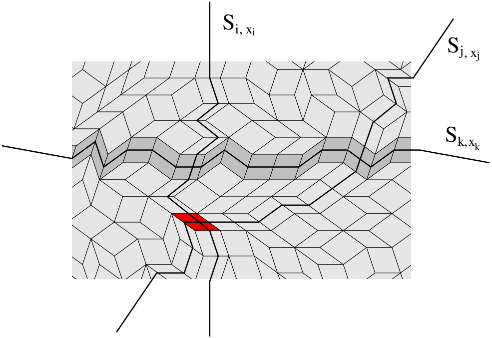


Idea: $\mathcal{S} \xrightarrow{flips} \mathcal{S}'$ by moving unit face $S_{i,x_i} \cap S_{j,x_j}$ to $S'_{i,x_i} \cap S'_{j,x_j}$.

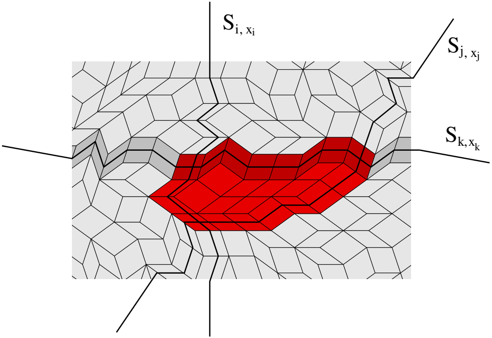
Moving $S_{i,x_i} \cap S_{j,x_j}$ by $\vec{e}_k \Leftrightarrow$ moving it over S_{k,x_k}:



Moving $(\vec{x}, t_{ij}) = S_{i,x_i} \cap S_{j,x_j}$ by $\vec{e}_k$



Moving $(\vec{x}, t_{ij}) = S_{i,x_i} \cap S_{j,x_j}$ by $\vec{e}_k \rightsquigarrow$ *de Bruijn triangle* $F_k(\vec{x}, t_{ij})$:



Sketch of the proof

Moving $(\vec{x}, t_{ij}) = S_{i,x_i} \cap S_{j,x_j}$ by $\vec{e}_k$ $\rightsquigarrow$ *de Bruijn triangle* $F_k(\vec{x}, t_{ij})$:
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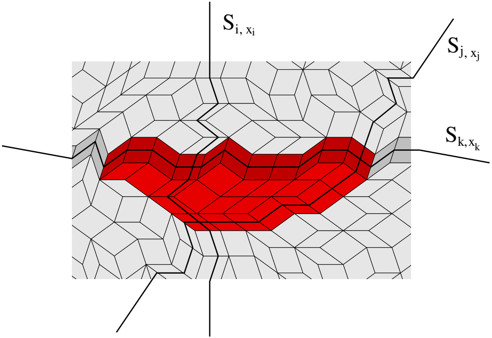

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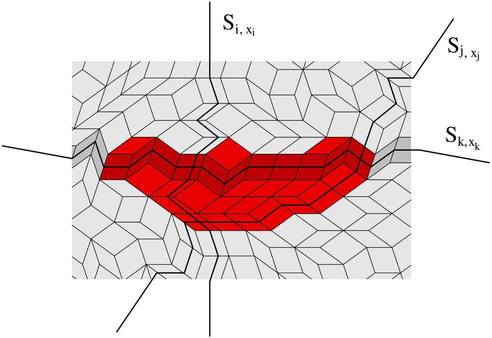

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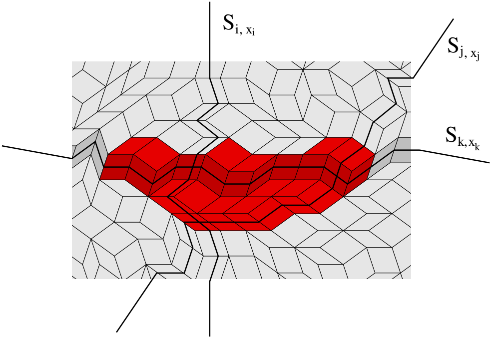

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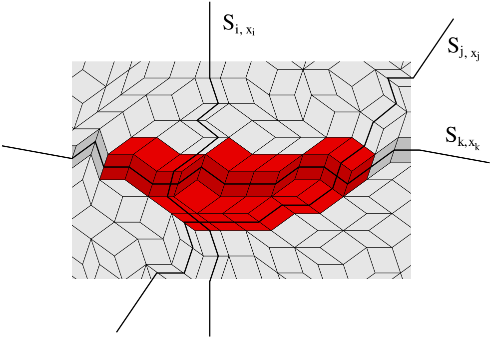

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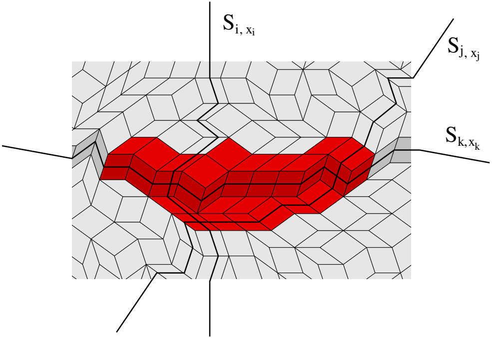

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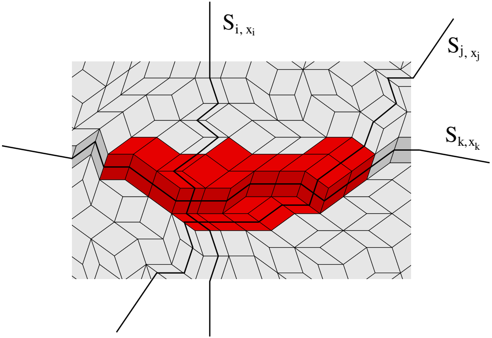

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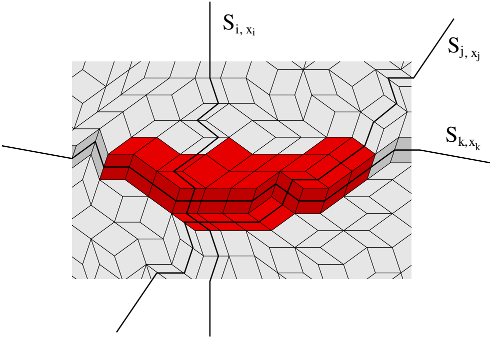

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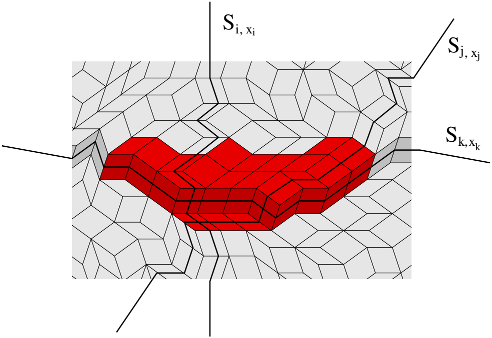

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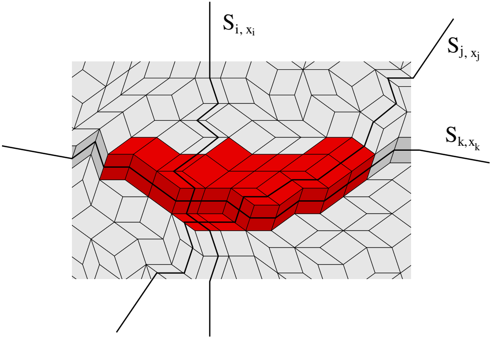

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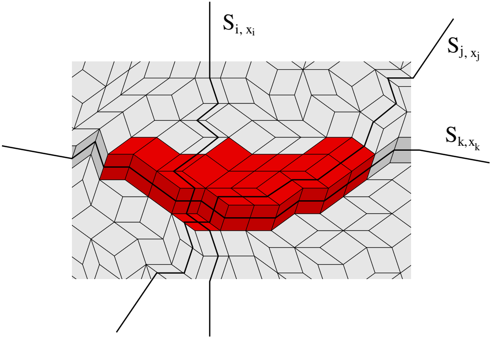

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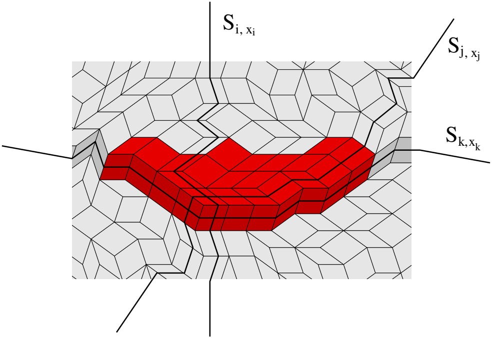

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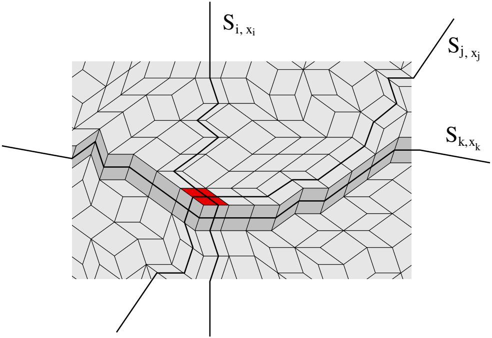

Some motivations for studying flip-accessibility in rhombus tilings:

Physics:
Model growth of (quasi)crystals, in particular study defects.

Combinatorics:
Minimal "complexity" of aperiodic tilings (*c.f.* Nivat or Pleasants).

Discrete geometry and number theory:
Recognize discrete plane by performing multi-dim. continued fraction expansions of stepped surfaces.