Local growth of planar rhombus tilings

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Abstract. This paper is motivated by the issue of quasicrystal growth. It describes a simple local algorithm which appears to grow an infinite family of aperiodic tilings. Starting from a seed, tiles are added one by one at randomly chosen sites. A tile is added only if there is only one way to do this so that no forbidden local configuration is created. This algorithm rapidly grows large round-shaped patterns, up to a proportion of missing tiles which can be made arbitrarily small by taking a large enough seed.

1. Introduction

Since the discover of quasicrystals [1], a lot of attention has been drawn by the issue of whether local energetic interactions could explain the growth and stability of such aperiodic structures. Actually, the eventuality of a finite set of local configurations which characterize only aperiodic structures (*local rules*) has been considered in the early 60's by computer scientists [2]. The first example was found in 1964 [3]. Other cases were later found in tiling theory (see, *e.g.*, Sec. 5.7 of [4] and references therein), including the famous Penrose tilings [5].

Sadly, adding tiles one by one while taking care to observe some given local rules may not lead to a tiling characterized by these local rules. There are indeed arbitrarily large patterns which satisfy everywhere the local rules but can nevertheless not be extended further (*deceptions*) [6]. Does it means, as suggested by Penrose [7], that the growth of quasicrystals is an intrinsically non-local problem?

An interesting growth algorithm has been proposed for Penrose tilings [8]. Starting from a pattern (*seed*), tiles are added one by one at randomly chosen sites. The rule is to add a tile in a given site when there is *only one way* to do this so that local rules are satisfied (*forced tile*). This however eventually leads to a pattern with no more forced tiles (*dead pattern*). A special rule is then used to add a tile in some specific sites (*marginal sites*). Unfortunately, this special rule makes the growth non-local because the whole pattern has indeed to be examined to decide whether it is dead or not.

This non-locality problem led Socolar [9] to modify the algorithm: the special rule can now be used at any time, but only with a small probability ε. The algorithm becomes purely local, and since the probability that the special rule is used while the pattern is not dead is small (though it increases with the pattern size), the grown patterns look like thoses grown with the previous algorithm. The drawback is that the growth now stucks a very long time on each dead pattern (because ε is small and the marginal sites are quite scarce).

Another drawback, shared by both algorithms, is that the dead patterns the growth goes through have a very geometric shape, making such a mechanism questionable as a model of real quasicrystal growth (animated growths can be found on the web [10]).

In this paper, we modify the first algorithm [8] in a different way. We never use the special rule and, instead, slightly increase the radius of the local rules to determine forced tiles. This algorithm is thus local. It is moreover *deterministic*, in the sense that no choice is ever made and the patterns grown at the limit do not depend on the way sites to add tiles are chosen at random (whereas patterns grown by the Socolar's modified algorithm [9] depend on the marginal sites the special rule is applied in). An unavoidable consequence is that these limit patterns cannot *fully* cover the plane. Indeed, the information contained in the seed and the local rules never characterizes a unique aperiodic tiling (but a continuum [11]). Hence, none of the tiles which distinguish between these tilings can be added without making a choice. Fortunately enough, these missing tiles turn out to be quite sparse and appear to not prevent the local growth to "jump" over them. Moreover, the larger the initial seed is, the sparser they are (they form so-called *Conway worms* which are wider and wider spaced). We shall here consider not only Penrose tilings but all the *planar rhombus tilings with local rules*, concisely summarized Section 2. The algorithm is defined Section 3 and discussed Section 4.

2. Planar rhombus tilings with local rules

Pairwise non-collinear vectors $\vec{v}_1, \ldots, \vec{v}_n$ of the Euclidean plane define the rhombic **prototiles**:

$$T_{ij} = \{\lambda\vec{v}_i + \mu\vec{v}_j \mid 0 \leq \lambda, \mu \leq 1\}.$$

A **tile** is a translated prototile. A **rhombus tiling** is a covering of the plane by interior-disjoint tiles, with the intersection of two tiles, if not empty, being either a vertex or an entire edge. A **pattern** is a finite subset of a tiling.

Let $\vec{e}_1, \ldots, \vec{e}_n$ be the canonical basis of $\mathbb{R}^n$. The **lift** of a rhombus tiling is defined as follows: an arbitrary vertex is first mapped onto a point of $\mathbb{Z}^n$, then each tile T_{ij} is mapped onto the 2-dimensional face of a unit hypercube of $\mathbb{Z}^n$ generated by $\vec{e}_i$ and $\vec{e}_j$, with two tiles adjacent along an edge $\vec{v}_i$ being mapped onto two faces adjacent along an edge $\vec{e}_i$. The lift of a rhombus tilings is thus a "stepped" surface in $\mathbb{R}^n$. A rhombus tiling is said to be **planar** if there is a plane $E \subset \mathbb{R}^n$, called the **slope**, such that the tiling can be lifted into the tube $E + [0,1]^n$.

An r**-map** of center x of a rhombus tiling $\mathcal{T}$ is the pattern formed by all the tiles of $\mathcal{T}$ connected by a path of at most r edges to a vertex x of $\mathcal{T}$. The r**-atlas** of $\mathcal{T}$ is the set of its r-maps. A planar rhombus tiling $\mathcal{T}$ is said to admit **local rules** of radius r if any tiling whose r-atlas is a subset of the one of $\mathcal{T}$ is planar with the same slope as $\mathcal{T}$.

Penrose tilings, for example, can be defined as planar rhombus tilings for $n = 5$ [12] and admit local rules of radius 0. Another famous example are Ammann-Beenker tilings, which can be defined as planar rhombus tilings for $n = 4$ [13] but do not admit local rules [14].

The **shadow** of a rhombus tiling $\mathcal{T}$ is obtained by "shrinking" all the $\vec{v}_i$'s but three to $\vec{0}$, then removing all the tiles whose interior becames empty. This is a rhombus tilings with $n = 3$, and its lift is the orthogonal projection of the lift of $\mathcal{T}$ on the space generated by the three $\vec{e}_i$'s whose indices are those of the non-shrinked $\vec{v}_i$'s. A non-zero vector $\vec{p} \in \mathbb{Z}^3$ is a **subperiod** if $\mathcal{T}$ has a $\vec{p}$-periodic shadow, *i.e.*, a shadow whose lift is invariant under a translation by $\vec{p}$.

Subperiods seem to play an important role w.r.t. local rules. A necessary condition for a rhombus tiling to admit local rules is to have only periodic shadows [15]. This is, for example, the case of Penrose tilings. This is not sufficient: Ammann-Beenker tilings do not admit local rules but have only periodic shadows. A sufficient condition has been proven for $n = 4$ (and conjectured to be true for any n): if the subperiods of a planar rhombus tiling characterize its slope (*i.e.*, any planar rhombus tiling with the same subperiods must have this slope), then it admits local rules [16]. The point is that checking that a slope is enforced by subperiods just amounts to some basic algebra. This easily yields arbitrarily many examples, for $n = 4$, to test our local growth algorithm.

3. Local growth

The local growth is defined by Algo. 1, below. Fig. 1 illustrates one step on a toy example. Fig. 2 shows two larger patterns obtained in the case of *golden octagonal tilings*. These are the planar aperiodic octagonal tilings with a slope directed by $(-1, 0, \varphi, \varphi)$ and $(0, 1, \varphi, 1)$, where φ is the golden ratio. This slope is enforced by subperiods and admits local rules of radius 3 [16].

Algorithm 1: Local growth

Data: An r-atlas $\mathcal{A}$ and a pattern $\mathcal{P}$ (the seed)
while *true* **do**
- $x \leftarrow$ a random vertex of $\mathcal{P}$ not completly surrounded by tiles;
- $\mathcal{P}_x \leftarrow$ the pattern formed by the tiles of $\mathcal{P}$ within distance r of x;
- $\mathcal{A}_x \leftarrow$ the maps in $\mathcal{A}$ which, once centered in x, contain $\mathcal{P}_x$;
- $\mathcal{F}_x \leftarrow$ the tiles shared by all the maps in $\mathcal{A}_x$ (that is, the forced tiles);
- $\mathcal{P} \leftarrow \mathcal{P} \cup \mathcal{F}_x$;

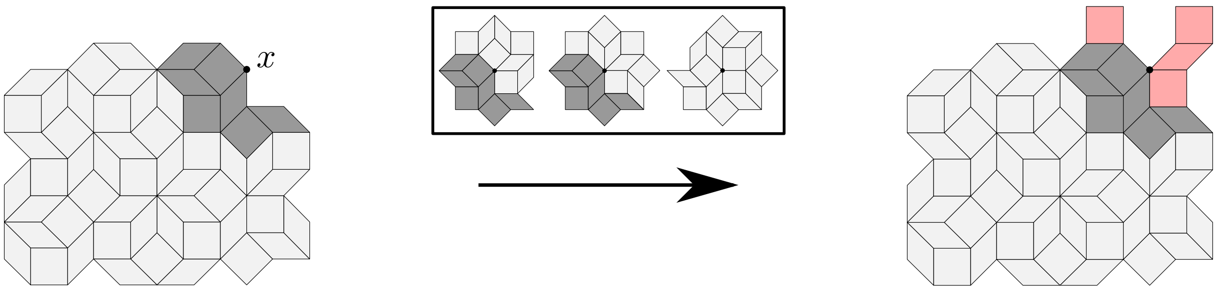

Figure 1. A pattern $\mathcal{P}$ (left) and, for $r = 1$, a pattern $\mathcal{P}_x$ (dark). The 1-atlas $\mathcal{A}$ (framed) has two maps which contain $\mathcal{P}_x$. These two maps have 4 other tiles in common, added to $\mathcal{P}$ (right).

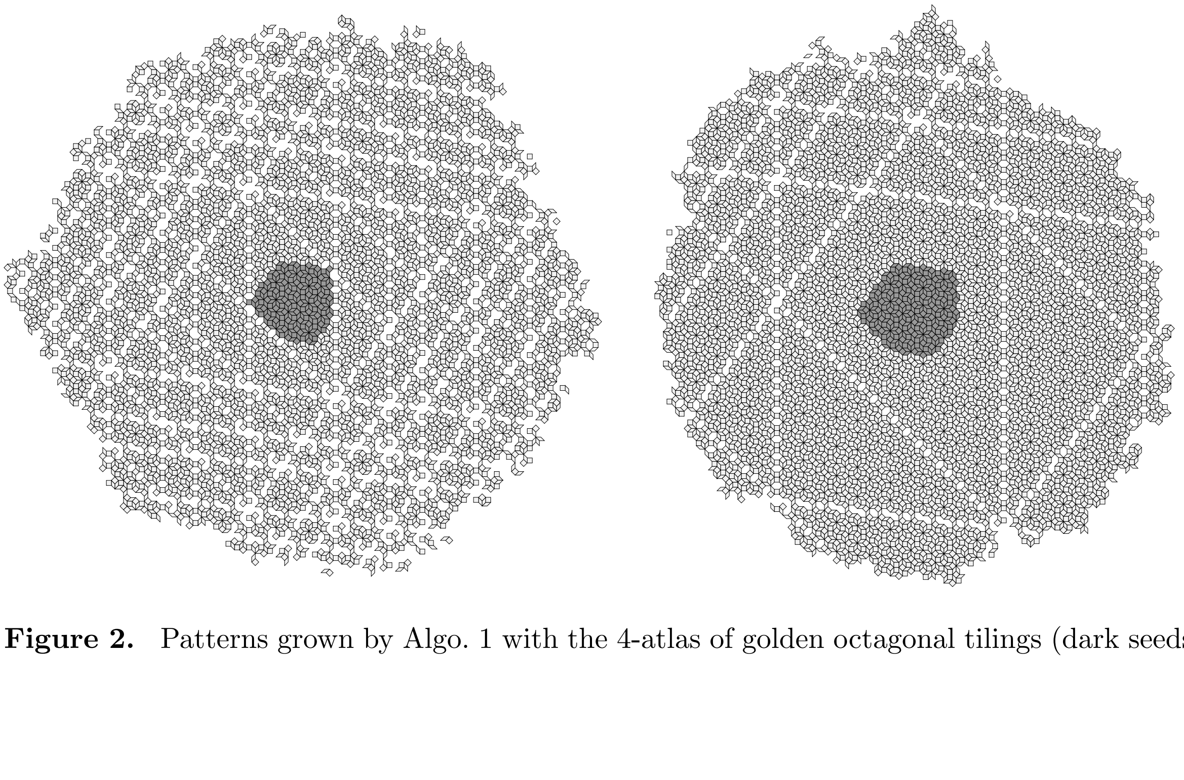

Figure 2. Patterns grown by Algo. 1 with the 4-atlas of golden octagonal tilings (dark seeds).

4. Discussion

Simulations on various planar octagonal tilings with local rules show that, for large enough seed and atlas, the patterns grown by Algo. 1 eventually cover almost all the plane [17].

The growth appears to be enforced by the atlas (which may need a slightly larger radius than the one of local rules). A key role may be played by subperiods: Fig. 3 explain how they, so to speak, "conspire" to force tiles. We tested a modified algorithm which purely relies on this subperiod conspiration: the growth indeed looks the same.

The grown patterns appear to depend on the seed. Given a seed which lifts in a tube $E + [0,1]^n$, the slope E can always be slightly shifted so that the tube still contains that lift (while the direction of E is fixed by the atlas). The intersection of all the planar tilings with these shifted slopes appears to be the limit pattern (missing tiles form so-called *Conway worms*). The larger the seed is, the smaller the allowed shift on E is and the sparser are the missing tiles.

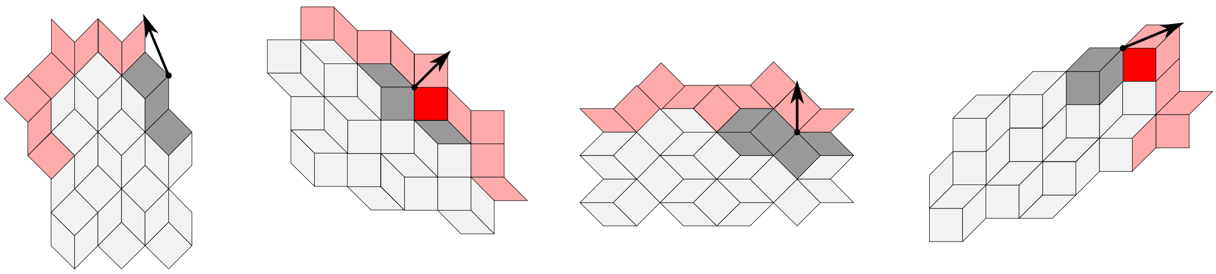

Figure 3. The four shadows of a pattern of a golden octagonal tiling (actually the one Fig. 1) are periodic (each subperiod is depicted). In each shadow, tiles are forced in the direction of the subperiod according only to the local environment (some red tiles have been added). Two neighboor tiles in the shadow obtained by shrinking $\vec{v}_i$ may be separated by an unknown number of edges $\vec{v}_i$ in the original pattern. The position in the original pattern of a tile forced in a shadow is therefore unclear. However, if such a tile is next to a known location in *two* shadows (*e.g.*, here, the black point and the plain red tile in the second and fourth shadow), since no $\vec{v}_i$ can have been shrinked in both shadow, then the position of the tile in the pattern can be deduced from its position in the shadows. The subperiods *conspire* to force tiles in some (most of) sites.

5. References

[1] Shechtman D, Blech I, Gratias D and Cahn J 1984 *Phys. Rev. Let.* **53** 1951–1953
[2] Wang H 1961 *Bell Systems technical journal* **40** 1–41
[3] Berger R 1964 *The Undecidability of the Domino Problem* Ph.D. thesis Harvard University
[4] Baake M and Grimm U 2013 *Aperiodic Order* vol 149 (Cambridge University Press)
[5] Penrose R 1978 *Eureka* **39**
[6] Dworkin S and Shieh J I 1995 *Communications in Mathematical Physics* **168** 337–352
[7] Penrose R 1989 *Introduction to the mathematics of quasicrystals* ed Jarić M (Academic press) pp 53–79
[8] Onoda G, Steinhardt P, DiVincenzo D and Socolar J 1988 *Phys. Rev. Lett.* **60** 2653
[9] Socolar J 1991 *Quasicrystals: the State of the art* ed DiVincenzo D and Steinhardt P (World Scientific) pp 213–238
[10] Korvola T 1997 Penrose XScreenSaver `www.youtube.com/watch?v=atlkrWkbYHk` (accessed: 2019-05-20)
[11] Dolbilin N 1995 *Discrete & Computational Geometry* **13** 405–414
[12] De Bruijn N 1981 *Nederl. Akad. Wetensch. Indag. Math.* **43** 39–52
[13] Beenker F 1982 Algebraic theory of non periodic tilings of the plane by two simple building blocks: a square and a rhombus Tech. Rep. TH Report 82-WSK-04 Technische Hogeschool Eindhoven
[14] Bédaride N and Fernique T 2015 *Discrete & Computational Geometry* **54** 980–992
[15] Levitov L 1988 *Communications in Mathematical Physics* **119** 627–666
[16] Bédaride N and Fernique T 2015 *Communications in Mathematical Physics* **335** 1099–1120
[17] Galanov I 2019 Growing aperiodic tilings `www.youtube.com/watch?v=XbyGrIRlmGw` (accessed: 2019-05-20)