Higher order asymptotics from multivariate generating functions

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(joint with Robin Pemantle, Alex Raichev)

Rutgers, 2009-11-19

References

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- E. Bender, survey on Asymptotic Enumeration, SIAM Review 16:485-515, 1974.
- L. Hörmander, The Analysis of Linear Partial Differential Operators (Ch 7), Springer, 1983.

Notation

- Boldface denotes a multi-index: $\mathbf{z} = (z_1, \ldots, z_d)$, $\mathbf{r} = (r_1, \ldots, r_d)$, $\mathbf{z}^{\mathbf{r}} = z_1^{r_1} \ldots z_d^{r_d}$, $d\mathbf{z} = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_d$.

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- If the series converges in a neighbourhood of $\mathbf{0} \in \mathbf{C}^d$, then F defines an analytic function there.

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- the directions $\overline{\mathbf{r}} := \mathbf{r}/|\mathbf{r}|$ of interest for which we seek asymptotics of $a_{\mathbf{r}}$ are generic, so nothing changes qualitatively in a small neighbourhood;
- $F = G/H$ with G, H entire functions but F is not itself entire. Key examples: rational function that is not a polynomial.

$d = 1$: analysis is easy

- Consider the Cauchy integral representation

$$a_r = \int_C \omega := \frac{1}{2\pi i} \int_C z^{-r} F(z) \, \frac{dz}{z}$$

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- Thus $a_r = \int_{C'} \omega - \sum_{c \neq 0} \operatorname{Res}(\omega, c)$ and the integral is exponentially smaller than the residues.
- Note that if $c \neq 0$, then $\operatorname{Res}(\omega, c) = c^{-r} \operatorname{Res}(F, c)$ and so asymptotics are dominated by the pole with smallest modulus. This is positive real (Vivanti-Pringsheim).

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- The integral is $O((1+\varepsilon)^{-r})$ while the residue equals $-e^{-1}$.
- Thus $[z^r]F(z) \sim e^{-1}$ as $r \to \infty$.
- Since there are no more poles, we can push C to ∞ in this case, so the error in the approximation decays faster than any exponential.

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- (Flajolet/Sedgewick 2009) "Roughly, we regard here a bivariate GF as a collection of univariate GFs . . . ."

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- Other collaborators: Yuliy Baryshnikov, Wil Brady, Andrew Bressler, Timothy DeVries, Manuel Lladser, Alexander Raichev, Mark Ward, ...

Cauchy integral representation

- Let U be the open polydisc of convergence, $\partial\,\mathrm{U}$ its boundary, C a product of circles centred at $\mathbf{0}$, inside U. Then

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- Good general idea: saddle point method: using analyticity, we deform the contour C to minimize the maximum modulus of the integrand. Usually we minimize only the factor $|z|^{-|r|}$.
- The other main idea is residue theory. The Leray residue formula and reduces dimension of the integral by 1; we still need to integrate the residue form.

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- The analysis depends on the direction $\overline{\mathbf{r}}$ as a parameter. If done right the dependence is as uniform as possible.

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- When $\mathbf{z}^*(\overline{\mathbf{r}})$ is a smooth point (simple pole) of $\mathcal{V}$,

$$a_{\mathbf{r}} \sim \mathbf{z}^*(\overline{\mathbf{r}})^{-\mathbf{r}} \sum_{q \geq 0} b_q(\mathbf{z}^*)|\mathbf{r}|^{-(d-1)/2-q}$$

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- Leading term can be expressed in terms of outward normal to, and Gaussian curvature of, $\mathcal{V}$ in appropriate coordinates.

$d = 2$, smooth point, explicit leading term

- Suppose that $F = G/H$ has a simple pole at $P = (z^*, w^*)$ and $F(z,w)$ is otherwise analytic for $|z| \leq |z^*|, |w| \leq |w^*|$. Define

$$Q(z,w) = -A^2B - AB^2 - A^2z^2H_{zz} - B^2w^2H_{ww} + ABH_{zw}$$

where $A = wH_w, B = zH_z$, all computed at P. Then when $s \to \infty$ with $r/s = B/A$,

$$a_{rs} = (z^*)^{-r}(w^*)^{-s}\left[\frac{G(z^*,w^*)}{\sqrt{2\pi}}\sqrt{\frac{-A}{sQ(z^*,w^*)}} + O((r+s)^{-3/2})\right].$$

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- This simplest case already covers Pascal, Catalan, Motzkin, Schröder, . . . triangles, generalized Dyck paths, ordered forests, sums of IID random variables, Lagrange inversion, transfer matrix method, ...

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- Solving, and using the smooth point formula above we obtain (uniformly for $r/s, s/r$ away from 0)

$$a_{rs} \sim \left[\frac{\Delta - s}{r}\right]^{-r} \left[\frac{\Delta - r}{s}\right]^{-s} \sqrt{\frac{rs}{2\pi\Delta(r+s-\Delta)^2}}$$

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- Extracting the diagonal ("central Delannoy numbers") is now trivial:

$$a_{rr} \sim (3+2\sqrt{2})^r \frac{1}{4\sqrt{2}(3-2\sqrt{2})} r^{-1/2}.$$

Extensions, jargon, applications

Check out the following in the references — no time here!

- higher order poles ("multiple points", e.g. queueing networks);
- other nonsmooth points ("cone points", e.g. tilings);
- non-generic directions ("Airy phenomena", e.g. maps);
- periodicity ("torality", e.g. quantum random walks);
- (Gaussian) limit laws follow directly from the analysis;

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- An efficient algorithm for computing symbolically/numerically?
- Higher order terms are useful for many reasons (e.g. better approximations for smaller indices, cancellation of lower terms).
- There are many "formulae" in the literature for asymptotic expansions, but higher order terms are universally acknowledged to be hard to compute.

Explicit integral: Delannoy numbers

- The integral of the residue turns out to be

$$\int_{-\varepsilon}^{\varepsilon} \exp\left[ir\theta - s\log\left(\frac{1+z^*e^{i\theta}}{1+z^*}\frac{1-z^*}{1-z^*e^{i\theta}}\right)\right]\frac{1}{1-z^*e^{i\theta}}\,d\theta.$$

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- Note that the argument $g(\theta)$ of the exponential has Maclaurin expansion

$$i\left(\frac{r(z^*)^2+2sz^*-r}{(z^*)^2-1}\right)\theta+\frac{sz^*(1+(z^*)^2)}{(1-(z^*)^2)^2}\theta^2+\dots$$

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- Recall that $\mathrm{crit}(\overline{(r,s)})$ is defined by $1-z-w-zw=0, s(1+w)z = r(1+z)w$. Eliminating w yields $rz^2+2sz-r=0$.
- Thus $g(0)=0$, and $g'(0)=0$ because (z^*, w^*) is a critical point for direction $\overline{(r,s)}$.

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- The above ideas reduce the problem to large-λ analysis of integrals of the form

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  - typically $\det g''(\mathbf{0}) \neq 0$ and there are no other stationary points of the phase on D.
- Difficulties in analysis: interplay between exponential and oscillatory decay, nonsmooth boundary of simplex.

Low-dimensional examples of F-L integrals

- A typical smooth point example looks like

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- Multiple point with $n = 2, d = 1$ gives an integral like

$$\int_{-1}^{1}\int_{0}^{1}\int_{-x}^{x} e^{-\lambda(z^2+2izy)}\, dy\, dx\, dz.$$

  Simplex corners now intrude, continuum of critical points.

Asymptotics from F-L integrals

- This is a classical topic with many applications in physics, treated by many authors. However many of our applications to generating function asymptotics do not fit into the standard framework. In some cases, we need to extend what is known.

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- Assume that there is a single stationary point that is quadratically nondegenerate (this holds in our applications to mvGFs, under our standard assumptions). The integral then has an asymptotic expansion of the form

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- If $u(\mathbf{0}) \neq 0$ then the leading term is given by $b_0 = u(\mathbf{0})$. This is fine, but how to compute the higher order terms?

Explicit series: Hörmander's formula

We want the coefficents b_q from above. Define

$$L_q(u,g) := \sum_{l=0}^{2q} \frac{\mathcal{H}^{q+l}(u\underline{g}^l)(\mathbf{0})}{(-1)^q 2^{q+l} l!(q+l)!},$$

$$\underline{g}(\theta) = g(\theta) - \frac{1}{2}\theta g''(\mathbf{0})\theta^T$$

$$\mathcal{H} = -\sum_{a,b} (g''(\mathbf{0})^{-1})_{a,b}\partial_a\partial_b.$$

Then $b_q = L_q(u,g)$.

Consequence of Hörmander for our mvGF application

- $$a_{\mathbf{r}} \sim \mathbf{z}^{*-\mathbf{r}} \left[(2\pi)^{(n-d)/2} (\det M(\mathbf{z}^*))^{-1/2} \sum_{0 \le q} c_q r_d^{(n-d)/2-q} \right],$$

  where M is a certain nonsingular matrix

  $$c_q = \sum_{\substack{0 \le j \le \min\{n-1,q\} \\ \max\{0,q-n\} \le k \le q \\ j+k \le q}} L_k(\widetilde{u}_j, \widetilde{g}) \binom{n-1}{j} (-1)^{q-j-k} \begin{bmatrix} n-j \\ n+k-q \end{bmatrix}$$

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  - $\left[\begin{matrix} a \\ b \end{matrix} \right]$ denotes the unsigned Stirling number of the first kind;
  - the functions $\widetilde{u}_j$ involve derivatives up to order j of G;
  - $\widetilde{g}$ gives a local parametrization of $\mathcal{V}$ eliminating z_d .

Delannoy example: next term in the expansion

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- $$a_{2n,3n} = \left(c_1^{-3} c_2^{-2}\right)^n \left(b_0 n^{-1/2} + b_1 n^{-3/2} + O\left(n^{-5/2}\right)\right)$$

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  as $n \to \infty$, where

  $$c_1^{-3}c_2^{-2} \approx 71.16220050$$
  $$b_0 = \frac{13^{3/4}\sqrt{3}}{156\sqrt{\pi}}(5+\sqrt{13}) \approx 0.36906$$
  $$b_1 = -(5/1898208)13^{3/4}\sqrt{3}(79\sqrt{13}+767)/\sqrt{\pi} \approx -0.018536$$

Delannoy example: improved numerics

Here E_1, E_2 denote the relative error when using the 1- and 2-term approximations A_1, A_2.

n	1	2	4	8	16
$a_{2n,3n}$	25	1289	$4.673 \cdot 10^6$	$8.528 \cdot 10^{13}$	$3.978 \cdot 10^{28}$
A_1	26.263	1321.542	$4.732 \cdot 10^6$	$8.581 \cdot 10^{13}$	$3.990 \cdot 10^{28}$
A_2	24.944	1288.355	$4.673 \cdot 10^6$	$8.527 \cdot 10^{13}$	$3.978 \cdot 10^{28}$
E_1	-5%	-2.5%	-1.3%	-0.6%	-0.3%
E_2	0.2%	0.05%	0.01%	0.003%	0.0007%

Example: cancellation in variance computation

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- W counts words over a d-ary alphabet X, where x_j marks occurrences of letter j of X and y marks snaps (occurrences of nonoverlapping pairs of duplicate letters).

Example: variance computation II

- The coefficient $[x_1^n \dots x_d^n, y^s] W(x, y)$ equals the number of words with n occurrences of each letter and s snaps.

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- However the first order terms cancel out in the computation of the variance. So we require at least a 2-term expansion for the mean and second moment.
- The answer is (for $d = 3$):

$$E[\psi_n] = \frac{3}{4}n - \frac{15}{32} + O(\frac{1}{n})$$
$$E[\psi_n^2] = \frac{9}{16}n^2 - \frac{27}{64}n + O(1)$$
$$V[\psi_n] = \frac{9}{32}n + O(1)$$

Application: algebraic functions

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- A little-known result by Safonov (2000) shows the converse. Every algebraic function in d variables is the "generalized diagonal" of a rational function in $d+1$ variables. When $d=1$ this is the usual leading diagonal.
- The construction is algorithmic but quite involved and uses a sequence of blowups to resolve singularities.

Example: Narayana numbers

- The GF for the Narayana numbers (enumerating Dyck paths by length and number of peaks) is

$$F(z,w) = \frac{1}{2}\left(1 + z(w-1) - \sqrt{1 - 2z(w+1) + z^2(w-1)^2}\right).$$

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$$[z^n w^k]F(z,w) = [t^n z^n w^k]\frac{t^2(z(w-1)+2)+t}{(z(w-1)-1)t - zw + 1}.$$

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- Interestingly the whole process commutes with the specialization $w = 1$, which gives an analogous result for the (shifted) Catalan numbers C_n, agreeing with what is known from other methods:

$$C_n = 4^n\left[\frac{1}{4\sqrt{\pi}}n^{-3/2} + \frac{3}{32\sqrt{\pi}}n^{-5/2} + O(n^{-7/2})\right].$$

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- Contributing points can lie at infinity (more topology!)
- Plenty of stimulus for further research, even if Safonov proves to be less effective than other approaches (such as directly resolving the Cauchy integral).

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- However the error reduces quickly with the number of terms, so not many terms are needed in practice it seems.

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- Find and classify contributing singularities algorithmically.
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- Patch together asymptotics in different regimes: uniformity, phase transitions.