Melons are branched polymers

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melonic graphs ← theories of random tensors

random tensor

$T_{a_1 \ldots a_D}$

N^D $\mathbb{C}$-valued r.v.

transforms covariantly

$$T_{a_1 \ldots a_D} \to T'_{a'_1 \ldots a'_D} = {}^{(1)}U_{a'_1}{}^{a_1} \ldots {}^{(D)}U_{a'_D}{}^{a_D} \, T_{a_1 \ldots a_D}$$

contravariant counterpart

$$\overline{T}_{\bar{a}_1 \ldots \bar{a}_D}$$

trace-invariant observables $\mathrm{tr}_{\mathcal{B}}(T, \overline{T})$

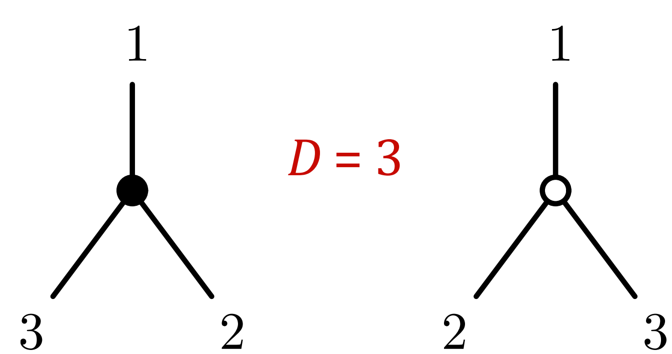
→ catalogued by *D*-colored graphs

D-colored graphs

bipartite

D-valent

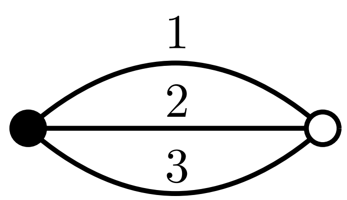
edge-colored



joint probability density

$$\mu = \frac{1}{Z(N,\{t\})}\left(\prod_a N^{D-1}\frac{\mathrm{d}T_a \mathrm{d}\overline{T}_a}{2\pi i}\right) e^{-N^{D-1}S_{\{t\}}(T,\overline{T})}$$

$$S_{\{t\}}(T,\overline{T}) = \mathrm{tr}_{\mathcal{B}_2}(T,\overline{T}) + \sum_{\mathcal{B}} t_{\mathcal{B}}\, \mathrm{tr}_{\mathcal{B}}(T,\overline{T})$$

$\mathcal{B}_2 =$ 

perturbed Gaussian

trace-invariant observables are
naturally arise as moments

2-point function

$$G_2(N,\{t\}) = \int \mu\, \frac{1}{N}\mathrm{tr}_{\mathcal{B}_2}(T,\overline{T})$$

expand w.r.t. coupling constants

1/N expansion

$$G_2(N,\{t\}) = \sum_{\mathcal{G}} A_{\mathcal{G}}(\{t\})\, N^{-\frac{2}{(D-1)!}\omega(\mathcal{G})}$$

→ catalogued by *(D+1)*-colored graphs

(D+1)-colored graphs

encode D-dimensional homology

topologically dual to abstract simplicial pseudomanifolds

Quantum gravity

tensor models capture a sum over weighted triangulations

topological interpretation → geometrical interpretation

quantum gravity = theory of random geometry

motivates investigation of geometrical observables:

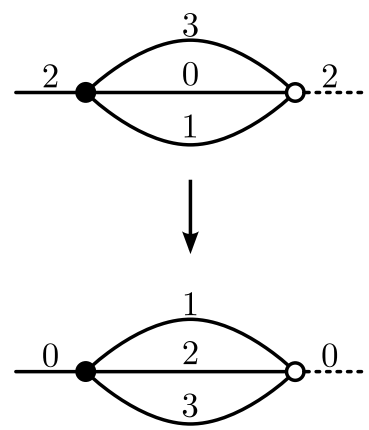
Hausdorff dimension   spectral dimension

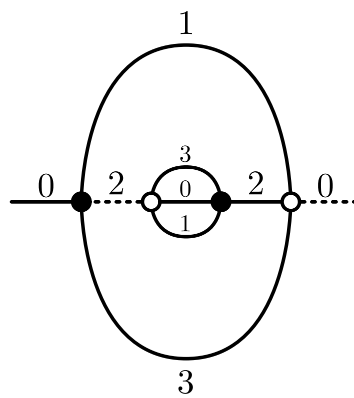
regime of interest

large-N limit	simple scaling
only leading order graphs survive	egalitarian weighting
rooted melonic graphs	$G_{2,\text{LO}}(g) = \sum_{\mathcal{M}} g^{p_{\mathcal{M}}}$

$p_{\mathcal{M}} = V_{\mathcal{M}}/2$

rooted melonic graphs



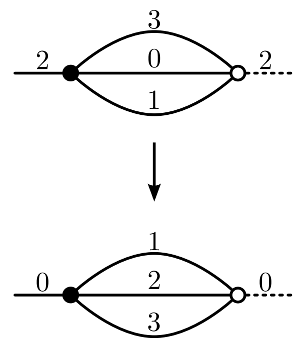


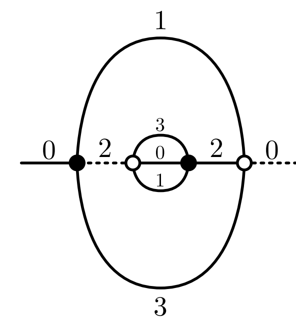
partition on melonic graphs

$$M_p = \{\text{melonic graphs with } 2p \text{ vertices}\}$$

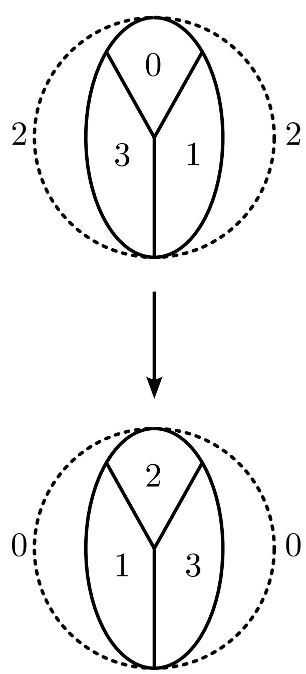
uniform distribution

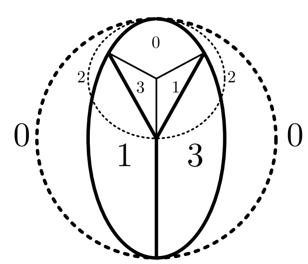
melonic graphs and counterparts



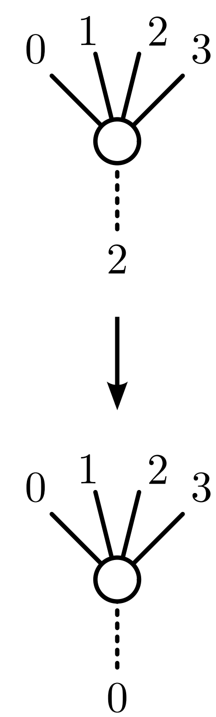


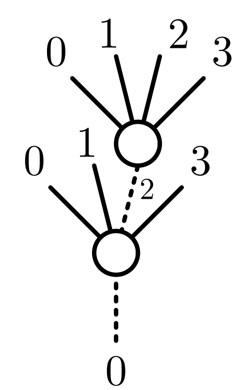
melonic D-balls





rooted colored (D+1)-ary trees

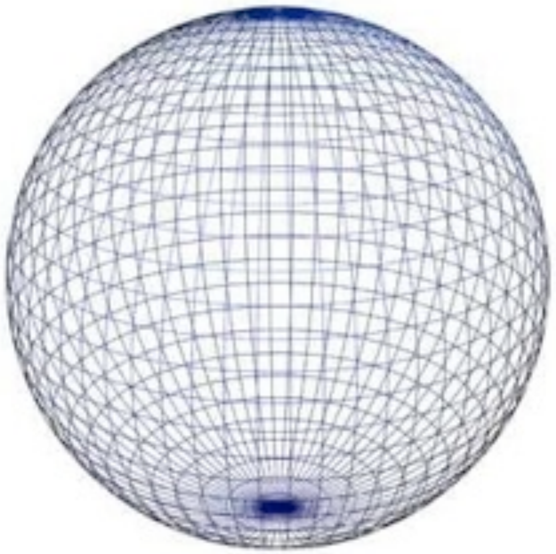




correspondences induce *partitions* and *uniform distributions*

LO 2-point function has finite radius of convergence

Hausdorff dimension



$$V \sim r^{d_H}$$

Branched polymers: $d_H = 2$

intelligent strategy: consult the experts [Albenque, Marckert: 08]

Step 1: a scaling limit for random trees

[Aldous: 91]

trees having n vertices
w/ uniform distribution

Continuum Random Tree
(CRT)

critical GW process

$\mu = 1$

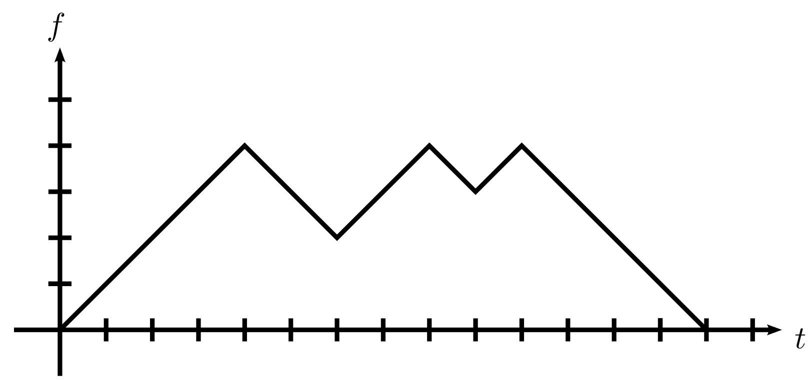
σ

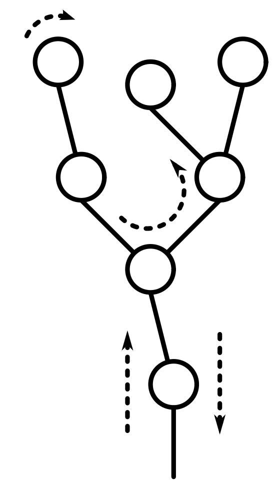
$$\left(T_n, \frac{d_{T_n}}{\sqrt{n/\sigma}}\right) \xrightarrow[n\to\infty]{} (\mathcal{T}_{2e}, d_{2e})$$

converge under distribution
w.r.t.
Gromov-Hausdorff topology

Aside: Continuum Random Tree

Real tree coded by f:





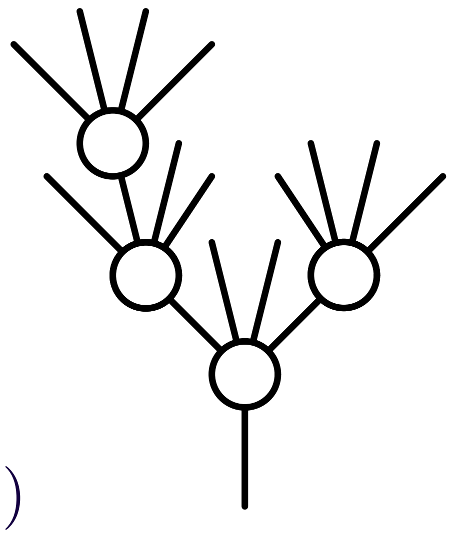
CRT: $\mathcal{T}_e$

coded by the Brownian excursion e

$$d\mu_e = \frac{1}{Z}\left[dq(t)\right]\Bigg|_{\substack{q(0)=q(1)=0 \\ q(t)>0}} e^{-\frac{1}{2}\int_0^1 [\dot{q}(t)]^2 dt}$$

Step 2: (D+1)-ary trees are GW trees

(D+1)-ary trees with *p* **internal** vertices: T_p



GW process offspring distribution

offspring distribution: ξ

$$\xi_0 = D/(D+1)$$

$$\xi_{D+1} = 1/(D+1)$$

critical GW process

induces uniform distribution on T_p

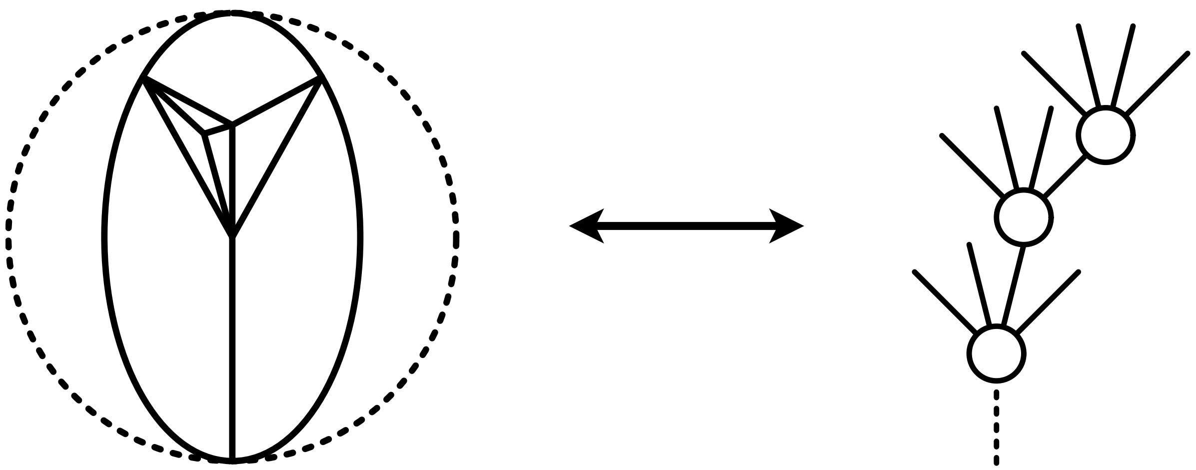
Step 3: Defoliated (D+1)-ary trees are GW trees

$$\sigma = D/(D+1)$$

Step 4: *(D+1)*-ary trees converges to the CRT

$$\left(T_p, \frac{d_{T_p}}{\sqrt{\frac{(D+1)p}{D}}}\right) \xrightarrow[p\to\infty]{} (\mathcal{T}_{2e}, d_{2e})$$

Step 5: Passage function (depth) - definition



$$w = 0u_1 \ldots u_n$$

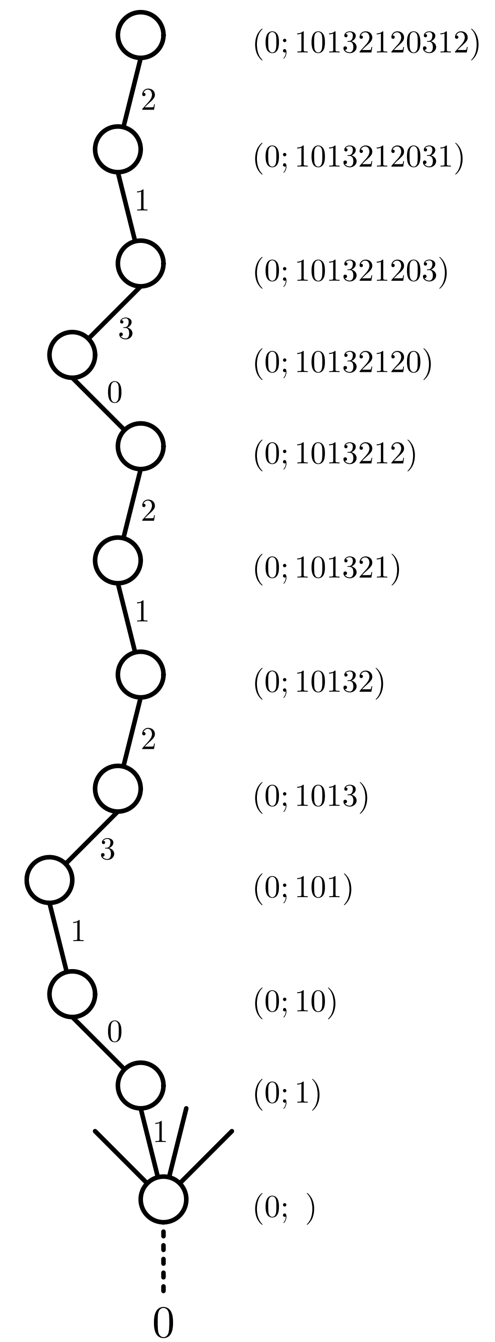
tree distance = #letters

$$w = \tau_0\tau_1 \ldots \tau_k \quad \Longrightarrow \quad \Lambda(w) = k$$

depth in D-ball = #subwords - 1

$$w = (0; 10132120312) = (0)(1)(013)(2120)(312)$$

words on trees



Step 6: Passage function (depth) - large *n* behaviour

average ratio behaves as:

$$\frac{1}{n}\Lambda(w) \xrightarrow[n\to\infty]{} \Lambda_\Delta$$

$$\Lambda_\Delta^{-1} = (D+1) \sum_{0\leq r\leq D} (-1)^{D-r} \binom{D}{r} \frac{r}{(D+1-r)^2}$$

Step 7: metric convergence

$$\left| \frac{d_{m_p}}{\Lambda_\Delta \sqrt{\frac{(D+1)p}{D}}} - \frac{d_{T_p}}{\sqrt{\frac{(D+1)p}{D}}} \right| \xrightarrow[p\to\infty]{(prob.)} 0$$

[Marckert: 06]

Step 8: put it all together with the help of our friends

[Albenque, Marckert: 08]

$$\left(m_p, \frac{d_{m_p}}{\Lambda_\Delta \sqrt{\frac{(D+1)p}{D}}} \right) \xrightarrow[p\to\infty]{} (\mathcal{T}_{2e}, d_{2e})$$

$$d_H = 2$$

spectral dimension

relies on [Jonsson, Wheater: 97]

diffusion process ⟶ heat kernel ⟶ return probability

$$P(t) \sim t^{-d_S/2} \longrightarrow d_S = -2\frac{d\log P(t)}{d\log t}$$

at short diffusion times

rough argument: lattice with *V* lattice sites

very long times $t \gg V^{\Delta}$

$$\lim_{t\to\infty} P_V(t) = \frac{1}{V}$$

short times $0 \ll t \ll V^{\Delta}$

$$\lim_{V\to\infty} P_V(t) \sim \frac{1}{t^{d_S/2}}$$

interpolating function

$$P_V(t) = \frac{1}{V} + \frac{a}{t^{d_S/2}}\exp\left(-\frac{t}{V^{\Delta}}\right)$$

GF for CE → $P_V(y) = \sum_{t=0}^{\infty} y^t P_V(t)$

$$P_V(y) \approx \frac{1}{V}\frac{1}{1-y} + \frac{a}{(1-y+V^{-\Delta})^{1-d_S/2}}$$

dominant terms as $y \to 1$, $V \to \infty$

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subtract pole contribution

$$\widetilde{P}_V(y) \sim \frac{a}{(1-y+V^{-\Delta})^{1-d_S/2}}$$

non-analytic only in thermodynamic limit

GF for GCE → $\widetilde{Q}(z,y) = \sum_{\mathcal{G}} z^{V_{\mathcal{G}}} \widetilde{P}_{V_{\mathcal{G}}}(y)$

$$\widetilde{Q}(z,y) = \sum_{V=0}^{\infty} z^V G_V \, \widetilde{P}_V(y)$$

$G_V \sim V^{\gamma-2} z_0^{-V}$ as $V \to \infty$

approximating the sum by an integral

$$\widetilde{Q}(z,y) = \left(1 - \frac{z}{z_0}\right)^{\beta} \; \widetilde{\Phi}\left(\frac{1-y}{\left(1-\frac{z}{z_0}\right)^{\Delta}}\right)$$

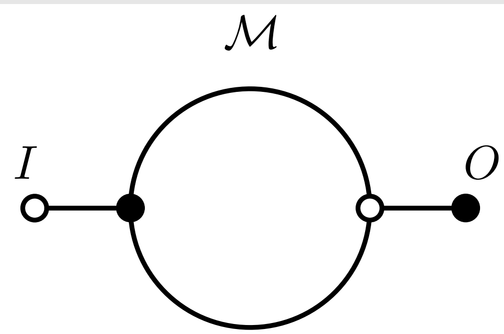
$$\beta = 1 - \gamma + \Delta\left(\frac{d_S}{2} - 1\right)$$

where $\widetilde{\Phi}(v) = \int_1^{\infty} \frac{a e^{-x}}{(v + x^{\Delta})^{1-d_S/2}} \, \mathrm{d}x$

random walks on a melonic graph

return/transit

$$P_{\mathcal{M}}(t) = \begin{pmatrix} P_{\mathcal{M}}^{II}(t) & P_{\mathcal{M}}^{IO}(t) \\ P_{\mathcal{M}}^{OI}(t) & P_{\mathcal{M}}^{OO}(t) \end{pmatrix}$$



strategy: successive decomposition

1st-return/1st-transit $P_{\mathcal{M}}^{1}(t)$

w_q word of length q on the alphabet $\{I,O\}$

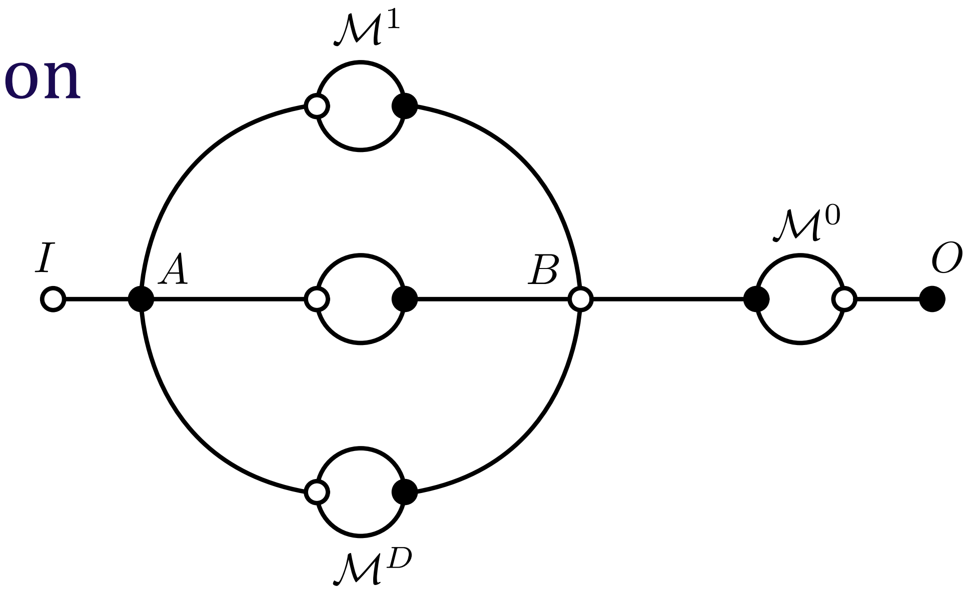
$$P_{\mathcal{M}}^{XY}(t) = \delta^{XY}\delta_{t,0} + P_{\mathcal{M}}^{1;XY}(t)$$
$$+ \sum_{q=1}^{\infty}\sum_{w_q}\sum_{t_0+\cdots+t_q=t} P_{\mathcal{M}}^{1;Xw_q(1)}(t_0)P_{\mathcal{M}}^{1;w_q(1)w_q(2)}(t_1)\ldots P_{\mathcal{M}}^{1;w_q(q-1)w_q(q)}(t_{q-1})P_{\mathcal{M}}^{1;w_q(q)Y}(t_q)\ .$$

GF

$$P_{\mathcal{M}}^{XY}(y) = \sum_{t=0}^{\infty} y^t P_{\mathcal{M}}^{XY}(t) \qquad P_{\mathcal{M}}^{1;XY}(y) = \sum_{t=0}^{\infty} y^t P_{\mathcal{M}}^{1;XY}(t)$$

$$P_{\mathcal{M}}(y) = \frac{1}{1 - P_{\mathcal{M}}^{1}(y)}$$

melonic factorization



w_q word of length *q* on the alphabet *{A,B}*

$$P^{1,II}_{\mathcal{M}}(t) = \frac{1}{D+1}\delta_{t,2} + \frac{1}{D+1}P^{1;AA}(t-2)$$

$$+\frac{1}{D+1}\sum_{q=1}^{\infty}\sum_{w_q}\sum_{t_0+\cdots+t_q=t-2}\Big(P^{1;Aw_q(1)}(t_0)P^{1;w_q(1)w_q(2)}(t_1)\dots P^{1;w_q(q-1)w_q(q)}(t_{q-1})P^{1;w_q(q)A}(t_q)\Big)\ .$$

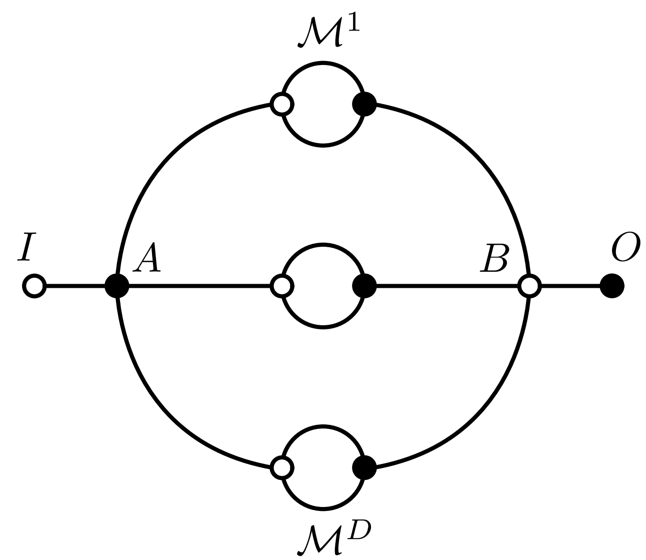
$$P^1_{\mathcal{M}} = E^{22}P^1_{\mathcal{M}^0}E^{22} + \Big(E^{12}y + E^{22}P^1_{\mathcal{M}^0}E^{11}\Big) \times \frac{1}{D+1-\sum_{i=1}^{D}P^1_{\mathcal{M}^i} - E^{11}P^1_{\mathcal{M}^0}E^{11}}\Big(yE^{21}+E^{11}P^1_{\mathcal{M}^0}E^{22}\Big)$$

$$P^1_{\mathcal{M}_{(0)}} = \begin{pmatrix} 0 & y \\ y & 0\end{pmatrix} \equiv y\,\sigma\ .$$

$$E^{ab}_{\alpha\beta} = \delta^a_\alpha\,\delta^b_\beta \qquad \text{where} \qquad a,b,\alpha,\beta\in\{1,2\}\ .$$

simple melonic graphs

$$P^1_{\mathcal{M}}(y) = \frac{y^2}{D+1-\sum_{i=1}^{D} P^1_{\mathcal{M}^i}(y)}$$



lemma: $P^1_{\mathcal{M}}(y) = a_{\mathcal{M}} + b_{\mathcal{M}}\,\sigma$ $\qquad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

simultaneously diagonalize

eigenvalue equation

$$\lambda^{1;\pm}_{\mathcal{M}}(y) = \frac{y^2}{D+1-\sum_{i=1}^{D} \lambda^{1;\pm}_{\mathcal{M}^i}(y)}$$

$$\lambda^{1;\pm}_{\mathcal{M}_{(0)}}(y) = \pm y$$

GF $\qquad Q^{\pm}(z,y) = \sum_{\mathcal{M}} z^{p_{\mathcal{M}}} \frac{1}{1-\lambda^{1;\pm}_{\mathcal{M}(y)}}$

pole at $y = \pm 1$

non-pole contribution

$$\widetilde{Q}^{\pm}(z,y) = -\frac{\mathrm{d}}{\mathrm{d}y}(1 \mp y) Q^{\pm}(z,y)$$

$$\widetilde{Q}^{\pm}(z,y) = \sum_{n=0}^{\infty} (1 \mp y)^n \, \widetilde{Q}^{\pm}_n(z)$$

$$\widetilde{Q}^{\pm}_0(z) = -\frac{1}{8} \log(1 - z/z_0)$$

$$\sum_{n=1}^{\infty} (1 \mp y)^n \, \widetilde{Q}^{\pm}_n(z) = F\left(\frac{1 \mp y}{(1 - z/z_0)^{3/2}}\right)$$

analyse higher derivative

$$\frac{\partial \widetilde{Q}^{\pm}}{\partial z}(z,y) = \frac{1}{1 - z/z_0} \widetilde{\Phi}\left(\frac{1 \mp y}{(1 - z/z_0)^{3/2}}\right)$$

undiagonalize AND non-simple melons

extract spectral dimension

$$d_S = \frac{4}{3}$$

Outlook

extend analysis to double scaling limit