Primitive point packing

(a knapsack problem in the integer lattice)

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joint work with

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Find x in $\mathbb{R}^d$ that maximizes

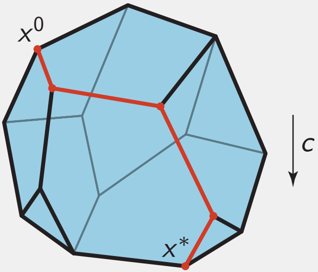
$$z = c \cdot x$$

and satisfies

$$Ax \leq b,$$

where b is in $\mathbb{R}^n$, c in $\mathbb{R}^d$,

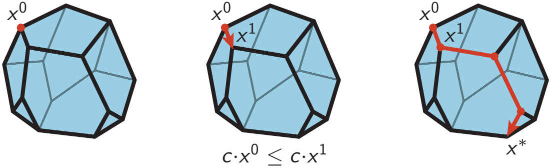
and A is a $n \times d$ matrix.



General question: what is the complexity of linear programming?

Smale's 9th problem: can linear programming be solved with a strongly polynomial algorithm?

Algorithmic, Combinatorial, and Geometric aspects of Linear Optimization



Pivoting algorithms:

$$\frac{21}{20}(n-d) \leq \Delta(d,n) \leq (n-d)^{\log_2 O(d/\log_2(d))}$$

Upper bound: Kalai–Kleitman (1992), ..., Sukegawa (2019).
Lower bound: Santos (2012).

Interior point methods:

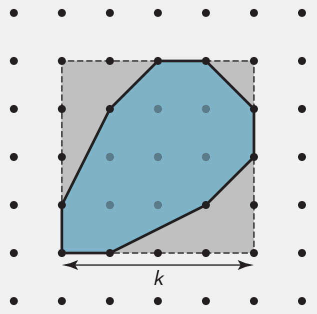
A large class of polynomial interior point methods are not strongly polynomial: Allamigeon–Benchimol–Gaubert–Joswig (2018)

Lattice polytopes

A lattice polytope is a polytope (= a bounded polyhedron) whose vertices belong to $\mathbb{Z}^d$.

Instead of n, we fix an integer k and study the lattice polytopes contained in $[0, k]^d$.

Question: what is the largest possible diameter of a lattice poytope contained in the hypercube $[0, k]^d$? We denote this diameter by $\delta(d, k)$.



Theorem (Naddef, 1989): $\delta(d, 1) = d$.

Theorem (Thiele, 1991, Acketa–Žunić 1995): $\lim_{k \to \infty} \frac{\delta(2, k)}{k^{2/3}} = \frac{6}{(2\pi)^{2/3}}$.

Theorem (Kleinschmid–Onn, 1992): $\delta(d, k) \leq kd$.

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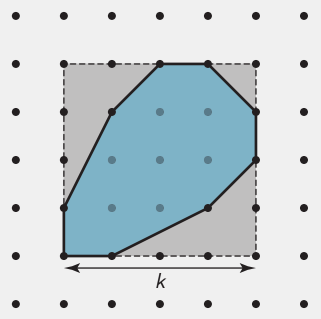


Theorem (Del Pia-Michini, 2016): if $k \geq 2$, then $\delta(d,k) \leq kd - \left\lceil \frac{d}{2} \right\rceil$.

Theorem (Deza-P, 2018): if $k \geq 3$, then $\delta(d,k) \leq kd - \left\lceil \frac{2}{3} d \right\rceil - (k-3)$.

Lattice polytopes

d \ k	1	2	3	4	5	6	7	8	9	10
2	2	3	4	4	5	6	6	7	8	$\cdots$
3	3	4	6	7	9	10				
4	4	6	8							
5	5	7	10							
$\vdots$	$\vdots$	$\vdots$								
d	d	$\lfloor \frac{3}{2} d \rfloor$								

↑

All the known values of $\delta(d, k)$

Naddef, 1989 — $\delta(d, 1) = d$

Thiele, 1991, Acketa-Žunić 1995, Deza-Manoussakis-Onn, 2018

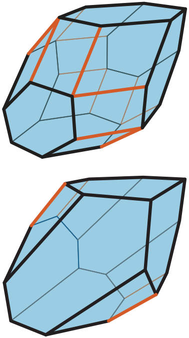
Del Pia-Michini, 2016 — $\delta(d, 2) = \lfloor 3d/2 \rfloor$

Deza-P, 2018 — $\delta(4, 3) = 8$

Chadder-Deza, 2020 — $\delta(3, 4) = 7$, $\delta(3, 5) = 9$

Deza-Deza-Guan-P, 2020 — $\delta(3, 6) = \delta(5, 3) = 10$

P-Rakotonarivo, 2019



↑

Two of the nine (up to symmetry) lattice polytopes of diameter 6 contained in the cube $[0, 3]^3$... among 332 335 207 073.

Lattice polytopes

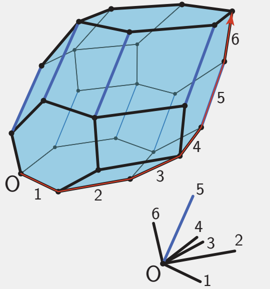
A zonotope is a Minkowski sum of line segments. Denote by $\delta_z(d,k)$ the largest possible diameter of a lattice zonotope contained in $[0,k]^d$.

Theorem (Deza–Manoussakis–Onn, 2018):

$$\delta_z(d,k) \geq \left\lfloor \frac{(k+1)d}{2} \right\rfloor \text{ when } k < 2d.$$

Conjecture (Deza–Manoussakis–Onn, 2018):

$$\delta(d,k) = \delta_z(d,k).$$



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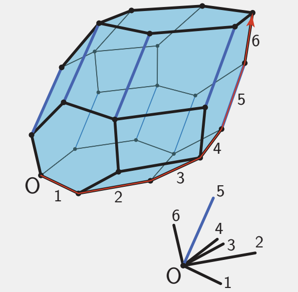
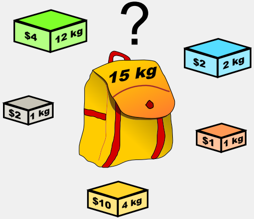


Primitive point packing

(a knapsack problem in the integer lattice)

The diameter $\delta(Z)$ of a zonotope is the number of its generators.

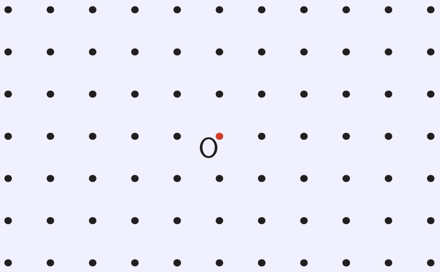
The Zonotope Z is contained in $[0, \kappa(Z)]^d$ where $\kappa(Z)$ is the largest coordinate of the sum of its generators (thought of as vectors).



Objects $\rightarrow$ vectors from $\mathbb{Z}^d$.

Capacity $\rightarrow$ the largest coordinate κ of the sum of the selected vectors.

How many such vectors can we select under the requirement that $\kappa \leq k$?



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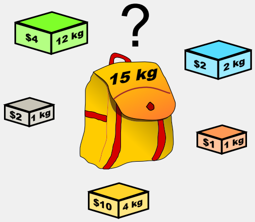
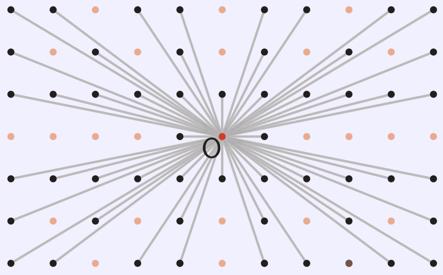


Objects → primitive vectors whose first non-zero coordinate is positive.
Capacity → the largest coordinate κ of the sum of the absolute value of the selected vectors.

How many primitive vectors can we select, such that $\kappa \leq k$?



Asymptotic estimates

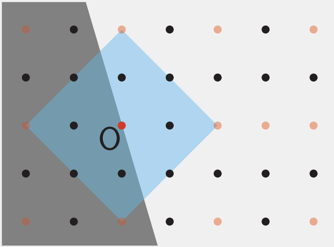
Theorem (Thiele, 1991, Acketa-Žunić 1995): $\lim_{k\to\infty} \frac{\delta(2,k)}{k^{2/3}} = \frac{6}{(2\pi)^{2/3}}$.

But, when $d > 2$ and k grows large,

$?? \leq \delta(d,k) \leq k(d-1)$ (minus a term that does not depend on k).

Theorem (Deza-P-Sukegawa, 2020): For any fixed d,

$$\lim_{k\to\infty} \frac{\delta_z(d,k)}{k^{\frac{d}{d+1}}} = \left(\frac{2^{d-1}(d+1)^d}{d!\zeta(d)}\right)^{\frac{1}{d+1}}$$

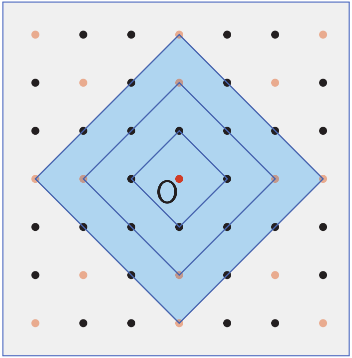


Theorem (Deza-P-Sukegawa, 2020): If $\mathcal{X}_p$ is the set of the primitive points (whose first non-zero coordinate is positive) contained in the ball $B(d,p)$ for the 1-norm centered at O and of radius p, then

$$|\mathcal{X}_p| = \delta_z(d, \kappa_p).$$

Moreover, $\mathcal{X}_p$ is the unique such set!

A formula for $\delta_z(d, k)$

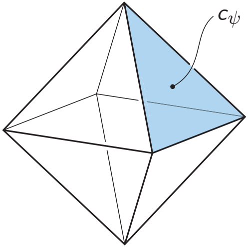


Denote by $S(d, i)$ the boundary of $B(d, i)$.

$$|\mathcal{X}_p| = \frac{1}{2}\#\text{PP}^\star \text{ in } B(d, p)$$
$$= \frac{1}{2}\sum_{i=1}^{p} \#\text{PP in } S(d, i)$$

$$\kappa_p = \frac{1}{2d}\sum_{i=1}^{p} i\#\text{PP in } S(d, i)$$

* PP stands for "primitive points"



$$\#\text{PP in } S(d, i) = \sum_{j=1}^{d} 2^j \binom{d}{j} c_\psi(i, j)$$

$2^j\binom{d}{j}$ is the number of j-dimensional faces of a d-dimensional cross-polytope and $c_\psi(i, j)$ the number of compositions of i into j relatively prime integers.

A formula for $\delta_z(d,k)$

Proposition (Deza-P, 2020): $c_\psi(p,d) = \frac{1}{(d-1)!}\sum_{i=1}^{d} s(d,i)J_{i-1}(p)$.

In this expression, $s(d,i)$ are the Stirling numbers of the first kind and $J_i(p)$ is Jordan's totient function. Both be computed efficiently:

$$J_i(p) = p^i \prod_{q|p}\left(1-\frac{1}{q^i}\right), \text{ where } q \text{ ranges over prime numbers.}$$

$$s(d+1,i) = -ds(d,i)+s(d,i-1) \text{ with } \begin{cases} s(d,d)=1 \text{ for all } d, \\ s(d,0)=0 \text{ when } d>0. \end{cases}$$

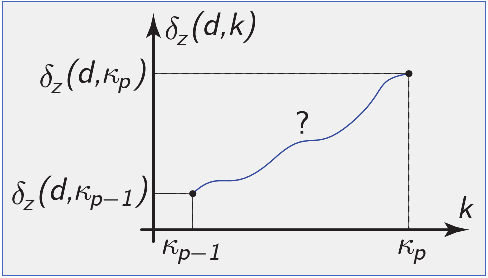
Theorem (Deza-P, 2020):

$$\delta_z(d,k) = \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{d}\frac{2^j}{(j-1)!}\binom{d}{j}\sum_{m=1}^{j}s(j,m)J_{m-1}(i)$$

when, for some positive integer p,

$$k = \kappa_p = \frac{1}{2d}\sum_{i=1}^{p}\sum_{j=1}^{d}\frac{i2^j}{(j-1)!}\binom{d}{j}\sum_{m=1}^{j}s(j,m)J_{m-1}(i)$$

A formula for $\delta_z(d,k)$



For fixed d, our point packing problem is solved (uniquely)

- at κ_{p-1} by the set of the primitive points of 1-norm $p-1$,*
- at κ_p by the set of the primitive points of 1-norm p.*

* whose first non-zero coordinate is positive.

What happens between κ_{p-1} and κ_p?

(1) Can we only add primitive points of 1-norm p?

(2) If yes, each additional point increases κ by p/d on average.

(3) In this case, is it a (discrete) straight line of slope p/d?

(4) Is there sometimes unicity between κ_{p-1} and κ_p?

(1) $\rightarrow$ not always, (3) $\rightarrow$ almost, but on two parallel lines, (4) $\rightarrow$ never.

A formula for $\delta_z(d, k)$

Consider the map $k \mapsto \lambda(d, k)$ such that, when $\kappa_{p-1} < k < \kappa_p$,

$$\frac{\lambda(d, k) - \delta_z(d, \kappa_{p-1})}{k - \kappa_{p-1}} = p/d.$$

Theorem (Deza-P, 2020): For any fixed d, the maps $k \mapsto \delta_z(d, k)$ and $k \mapsto \lfloor \lambda(d, k) \rfloor$ coincide, except on a subset $\mathbb{E}$ of $\mathbb{N} \backslash \{0\}$ such that

$$\lim_{k \to \infty} \frac{|\mathbb{E} \cap [1, k]|}{k^{1/(d-1)}} = 0.$$

Moreover, $k \mapsto \delta_z(d, k)$ coincides on $\mathbb{E}$ with $k \mapsto \lfloor \lambda(d, k) \rfloor - 1$.

The exceptions only occur for values of k such that $\kappa_{p-1} < k < \kappa_p$ where d is a proper divisor of p (at most twice in that range when $d > 2$.)

In fact, $\lim_{k \to \infty} \frac{\mathbb{E} \cap [1, k]}{k^{1/(d+1)}} = cte$ $(d > 2)$ and $\lim_{k \to \infty} \frac{\mathbb{E} \cap [1, k]}{k^{2/3}} = cte'$ $(d = 2)$