Collegium Urbis Nov Eborac $\rightarrow$ *Labo d'Informatique Paris Nord*

Algebraic area enumeration for lattice paths

Work (mostly) with Stéphane Ouvry and (lately) Li Gan

arXiv:1908.00990, 2103.15827, 2105.14042, 2107.10851, 2110.06235, 2110.09394

February 8, 2022

What's this talk really about?

- Exploring the combinatorics of various types of random walks
- Using approach that trades combinatorial ingenuity for algebraic dexterity
- Using physics perspective and results to derive combinatorial quantities
- Receiving feedback from combinatorics community

Basic setup

The basic playground: random walks on various lattices

- Sort them by length and area
- Package them in Generating (aka Partition) Functions

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Basic steps and techniques

- Map process to a Matrix (Hamiltonian)
- Derive matrix-based expressions for generating functions
- Evaluate them using physics techniques (and bias):
  - -Exclusion statistics
  - -Bosonization
  - -Cluster coefficients

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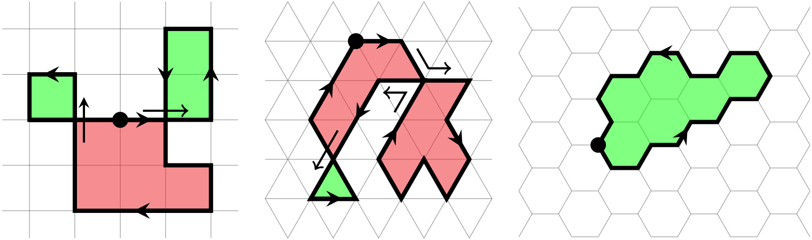
- See if either Mathematicians or Physicists care

Mathematicians	Physicists
Combinatorics	Diffusion, Adsorption
q-Hypergeometric functions	Hofstadter butterflies
	Phase transitions

Types of paths

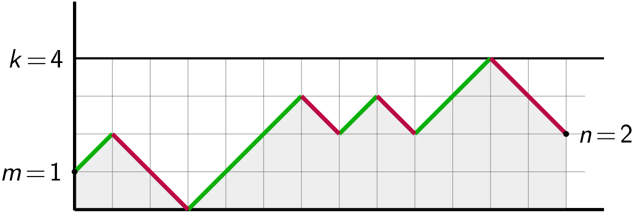
- Random walks on planar lattices



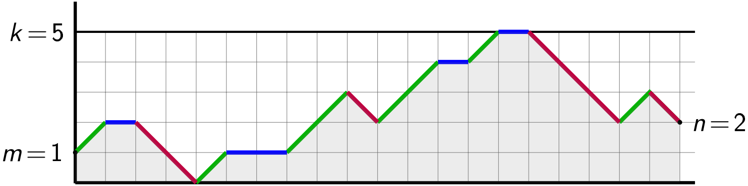
Square | Triangular | Honeycomb

Algebraic area can be positive or negative

- Forward-moving paths on lattices



Dyck meander



Motzkin meander

Basic device: the "quantum torus"

- Associate steps in each direction with operators and consider the Hamiltonian $$H = \underset{\rightarrow}{u} + \underset{\leftarrow}{u^\dagger} + \underset{\uparrow}{v} + \underset{\downarrow}{v^\dagger}$$

  [Related to symbolic methods, (Flajolet & Sedgewick)]
- If $\{u, u^\dagger, v, v^\dagger\}$ is a free algebra each monomial in H^ℓ represents a unique walk
- H^ℓ generates all walks of length ℓ from a fixed origin

Case study on the plane: square lattice closed walks

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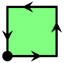
- If $\{u, u^\dagger, v, v^\dagger\}$ is a free algebra each monomial in H^ℓ represents a unique walk
- H^ℓ generates all walks of length ℓ from a fixed origin
- If, instead, we impose

$$vv^\dagger = v^\dagger v = uu^\dagger = u^\dagger u = 1 \;, \quad v\,u = \mathrm{Q}\,u\,v$$

$$\text{then} \quad H^\ell = \sum_A C_{\ell;m,n}(A)\,\mathrm{Q}^A v^n u^m$$

$C_{\ell;m,n}(A)$: # walks from $(0,0)$ to (m,n) of length ℓ and area A between the walk and horizontal axis

 $= v^\dagger u^\dagger v u = \mathrm{Q}$ elementary plaquette

The total number of closed walks of area A is $C_{\ell;0,0}(A)$ and the area generating functions is

$$G_\ell(\mathrm{Q}) = \sum_A C_{\ell;0,0}\, Q^A$$

- Need to extract term $v^0 u^0$ in H^ℓ
- Define the "trace" mapping

$$\mathrm{Tr}\,(v^n u^m) = \delta_{n,0}\, \delta_{m,0}$$

Then $\quad G_\ell(\mathrm{Q}) = \mathrm{Tr}\, H^\ell$

- Main task: calculate the formal trace $\mathrm{Tr}\, H^\ell$ effectively

Matrices to the rescue

- For $\mathrm{Q} = e^{i\phi}$ u, v can be realized as unitary operators
- If $\mathrm{Q} = e^{i2\pi p/q}$ (p, q coprime) the algebra $vu = \mathrm{Q}uv$ has a single q-dimensional unitary irreducible representation

$$u = e^{ik_x}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathrm{Q} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathrm{Q}^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathrm{Q}^{q-2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \mathrm{Q}^{q-1} \end{pmatrix} \quad (\text{"clock" matrix})$$

$$v = e^{ik_y}\begin{pmatrix} 0 & 1 & 0 & 0 \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 \ \cdots & 0 & 0 \end{pmatrix} \quad (\text{"shift" matrix})$$

with Casimirs: $u^q = e^{iqk_x}$, $v^q = e^{iqk_y}$

Fysics Fun Facts:

- Hamiltonian of an electron hopping on the square lattice in a constant normal magnetic field: Hofstadter model (1976)
- $p/q = \phi/\phi_0$ is the flux per lattice cell measured in units of the elementary flux quantum $\phi_0 = \hbar/e$
- u, v are noncommuting "magnetic translation operators"

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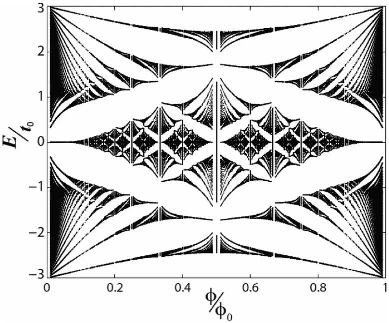
- u, v are noncommuting "magnetic translation operators"
- The energy spectrum of the particle (eigenvalues of H) as a function of the flux becomes fractal: "Hofstadter butterfly"



Back to our purpose: Tr H^ℓ

The ordinary (matrix) trace of u, v is

$$\mathrm{tr}\,(v^n u^m) = q \sum_{s,t=-\infty}^{\infty} e^{isqk_x + itqk_y} \delta_{m,sq}\, \delta_{n,tq}$$

$$\text{and for } |m|, |n| < q, \quad \mathrm{tr}\,(v^n u^m) = \delta_{m,0}\, \delta_{n,0}$$

$$\text{So for } \ell < q: \qquad \mathrm{Tr}\, H^\ell = \frac{1}{q} \mathrm{tr}\, H^\ell$$

(For $\ell \geq q$: $\mathrm{tr}\, H^\ell$ includes paths closing on a q-periodic lattice)

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Define the secular determinant

$$\det(1 - zH) = 1 - z\,\mathrm{tr}\, H + \frac{z^2}{2}\left[(\mathrm{tr}\, H)^2 - \mathrm{tr}\, H^2\right] + \cdots$$

so $$\ln\det(1 - zH) = \mathrm{tr}\, \ln(1 - zH) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}\, H^n$$

Finishing touch

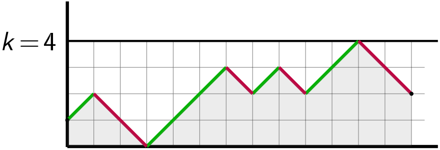
- Automorphism $u \to -uv$ preserves algebra (more on that later)
- Hamiltonian becomes 2-paradiagonal (0 diagonal)
- Putting Casimirs $u^q = v^q = 1$ the secular matrix becomes

$$1 - zH = \begin{pmatrix} 1 & (Q-1)z & 0 & \cdots & 0 & 0 \\ (\frac{1}{Q}-1)z & 1 & (Q^2-1)z & \cdots & 0 & 0 \\ 0 & (\frac{1}{Q^2}-1)z & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & (Q^{q-1}-1)z \\ 0 & 0 & 0 & \cdots & (\frac{1}{Q^{q-1}}-1)z & 1 \end{pmatrix}$$

- $1 - zH$ becomes special 3-diagonal

Save it in the fridge and move on to another type of walks

Case study of forward-moving walks: Dyck paths



$$H = UV + V^{\dagger}U$$

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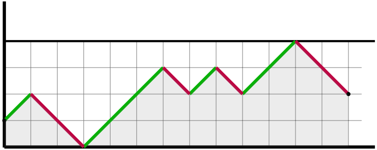

$$H = UV + V^\dagger U$$

- No lattice points below 0 or $k \Rightarrow V^{k+1} = {V^\dagger}^{k+1} = 0$
- V^{-1} does not exist;

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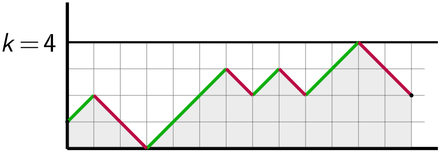

$$H = UV + V^\dagger U$$

- No lattice points below 0 or $k \Rightarrow V^{k+1} = {V^\dagger}^{k+1} = 0$
- V^{-1} does not exist; $\mathrm{Q} \to q$ (real) and $(k+1)\times(k+1)$ matrices

$$U = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & q & 0 & \cdots & 0 & 0 \\ 0 & 0 & q^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & q^{k-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & q^k \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 & 0 & 0\cdots & 0 & 0 \\ 0 & 0 & 1 & 0\cdots & 0 & 0 \\ 0 & 0 & 0 & 1\cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0\cdots & 0 & 1 \\ 0 & 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix}$$

$$VU = q\,UV, \quad VV^\dagger = 1 - |k\rangle\langle k|\,, \quad V^\dagger V = 1 - |0\rangle\langle 0|$$

$$H = q^{1/2} \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & q & 0 & \cdots & 0 & 0 \\ 0 & q & 0 & q^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & q^{k-1} \\ 0 & 0 & 0 & 0 & \cdots & q^{k-1} & 0 \end{pmatrix} \quad \text{(adds sierra)}$$

- "States" in $(k+1)$-dim space correspond to vertical position
- $\langle m| H |n\rangle = H_{mn}$: step from height m to n weighted by area
- Hamiltonian is the transition matrix of the path

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Define full length and area generating function

$$\begin{aligned} G_{k,mn}(z,q) &= \sum_{\text{paths}} N(\ell, A; m, n) z^\ell q^A \\ &= \sum_\ell z^\ell \langle m| H^\ell |n\rangle = \langle m| (1 - zH)^{-1} |n\rangle \end{aligned}$$

(no secular determinant yet, but...)

...Voilà! $\langle m| (1 - zH)^{-1} |n\rangle = \dfrac{\det (1 - zH)_{(mn)}}{\det (1 - zH)}$ (mn-minor)

First real use of matrices: because of special structure of H, minors can be related to full secular determinant

Define $\mathcal{Z}_k(z, q) = \det(1 - zH)$ [for $(k+1)$-dim H]. Then

$$G_{k,mn}(z, q) = z^{m-n} q^{\frac{m^2-n^2}{2}} \frac{\mathcal{Z}_{m-1}(z, q)\mathcal{Z}_{k-n-1}(zq^{n+1}, q)}{\mathcal{Z}_k(z, q)}$$

$$\ldots\textit{Voilà!}\quad \langle m|\,(1-zH)^{-1}\,|n\rangle = \frac{\det\,(1-zH)_{(mn)}}{\det\,(1-zH)} \quad (mn\text{-minor})$$

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Result for unrestricted $(k=\infty)$ excursions $(m=n=0)$ as a ratio $f(zq,q)/f(z,q)$ was known (Bousquet-Mélou, Bacher)

New result*/insight:

- Derivation for bounded meanders $(k,m,n$ nontrivial)
- Identification of components as determinants
- Several (old and new) dualities and recursion relations

algebraic manipulations instead of combinatorial relations

Physics-solving the secular determinant

Consider: $$\mathcal{Z} = \det \begin{pmatrix} 1 & z\alpha_1 & 0 & \cdots & 0 & 0 \\ z\beta_1 & 1 & z\alpha_2 & \cdots & 0 & 0 \\ 0 & z\beta_2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & z\alpha_k \\ 0 & 0 & 0 & \cdots & z\beta_k & 1 \end{pmatrix}$$

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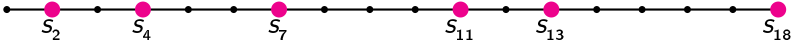
Basic observation: $\mathcal{Z}$ is the grand canonical partition function of exclusion statistics-2 particles on energy levels $1, 2, \ldots, k$ with spectral factors $s_j = e^{-\beta\epsilon_j} = \alpha_j\beta_j$ and fugacity $y = -z^2$



$N = 6$ particles in $k = 18$ states

$$\mathcal{Z} = \sum_{N=0}^{\lfloor (k+1)/2 \rfloor} y^N Z_{k,N}^{(2)} = \sum_{N=0}^{\lfloor (k+1)/2 \rfloor} y^N \sum_{j_i \le j_{i+1} - 2} s_{j_1} s_{j_2} \cdots s_{j_N}$$

Square lattice walks: secular matrix gives spectral factors

$$s_j = (\mathrm{Q}^j - 1)(\mathrm{Q}^{-j} - 1) = 4\sin^2(\pi jp/q)$$

Mapping $\quad \ln \det(1 - zH) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}\, H^n$

to $\quad$ (Grand potential) $\ln \mathcal{Z} = \sum_{n=1}^{\infty} y^n b_n$ (cluster coefficients)

and calculating cluster coefficients for exclusion-2 particles, we get

$$\mathrm{tr}\, H^{2n} = \sum_{\substack{l_1, l_2, \dots, l_i \\ \text{composition of } n}} c_2(l_1, l_2, \dots, l_i) \sum_{j=1}^{k-i+1} s_j^{l_1} s_{j+1}^{l_2} \cdots s_{j+i-1}^{l_i}$$

$$c_2(l_1, l_2, \dots, l_i) = \frac{2n}{l_1} \binom{l_1 + l_2 - 1}{l_2} \binom{l_2 + l_3 - 1}{l_3} \cdots \binom{l_{i-1} + l_i}{l_i}$$

Compositions: ordered partitions (e.g., $3 = 2+1 = 1+2 = 1+1+1$), 2^{n-1} in all

Finally, expanding the trigo sum and extracting its Q^A part, we obtain the number of walks of length $\ell = 2n$ and area A

$$C_{2n}(A) = \sum_{\substack{l_1, l_2, \ldots, l_j \\ \text{composition of n}}} \sum_{k_3=0}^{2l_3} \sum_{k_4=0}^{2l_4} \cdots \sum_{k_j=0}^{2l_j}$$

$$\prod_{i=3}^{j} \binom{2l_i}{k_i} \binom{2l_1}{l_1 + A + \sum_{i=3}^{j}(i-2)(k_i - l_i)} \binom{2l_2}{l_2 - A - \sum_{i=3}^{j}(i-1)(k_i - l_i)}$$

(Ouvry & Wu, based on earlier work by Kreft)

- Explicit, but quite complicated
- Computational complexity increases with $\ell = 2n$
- Perhaps other expressions available?
- $\operatorname{tr} H^{2n}$ for special values of p, q leads to sequences of Apéry-like numbers

Dyck paths: Secular determinant gives spectral factors $s_j = q^{2j}$

- $s_j = e^{-\beta\epsilon_j}$: equidistant spectrum (harmonic oscillator)
- Full machinery of SM/QM and bosonization at play

$$\mathcal{Z}_k = \sum_{N=0}^{\lfloor (k+1)/2 \rfloor} y^N Z_{k,N}^{(2)}$$

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\begin{aligned}
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\text{(bosonization)} &= \sum_{N=0}^{\lfloor (k+1)/2 \rfloor} (-z^2)^N q^{N(N-1)} Z_{k-2(N-1),N}^{B}
\end{aligned}
$$

$$
\begin{aligned}
\text{where} \quad Z_{k,N}^{B} &= \prod_{j=1}^{k-1} \frac{1-q^{j+N}}{1-q^j} = \prod_{j=1}^{N} \frac{1-q^{j+k-1}}{1-q^j} \\
&= \frac{[k+N-1]!_q}{[N]!_q\,[k-1]!_q} = \binom{k+N-1}{N}_q
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\text{so} \quad \mathcal{Z}_k(z,q) = \sum_{N=0}^{\lfloor (k+1)/2 \rfloor} (-z^2)^N q^{N(N-1)} \binom{k-N+1}{N}_q
$$

$\mathcal{Z}_k(z,q)$ is a q-deformed Fibonacci polynomial in z^2

Summarizing, full length and area Dyck paths generating function

$$G_{k,mn}(z,q) = z^{m-n} q^{\frac{m^2-n^2}{2}} \frac{\mathcal{Z}_{m-1}(z,q)\mathcal{Z}_{k-n-1}(zq^{n+1},q)}{\mathcal{Z}_k(z,q)}$$

$$\mathcal{Z}_k(z,q) = \sum_{N=0}^{\lfloor (k+1)/2 \rfloor} q^{N(N-1)} \prod_{j=1}^{N} (-z^2) \frac{1-q^{j+k-2N+1}}{1-q^j}$$

We can also derive

$$\ln G_{k,mn}(z,q) = (n-m)\ln z + \frac{(n^2-m^2)}{2}\ln q + \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell} p_{k,mn;\ell}(q)$$

with $$p_{k,mn;\ell} = \sum_{\substack{l_1,l_2,\dots,l_j;\, j\le k \\ \text{composition of } n}} c_2(l_1,l_2,\dots,l_j)\, q^{\sum_{i=1}^{j}(i-1)l_i} \sum_{r=\max(m-j,0)}^{\min(k-j,n)} q^{r\ell}$$

"Close" to $\langle m|\, H^\ell \,|n\rangle$ (cf. $\ln \mathcal{Z} = \ln(1-zH) = -\sum_\ell \mathrm{tr}\, H^\ell/\ell$) but...

Other walks, other statistics: a taste

- What about other types of (planar or forward-moving) walks?
- Why only exclusion-2? What about other statistics?

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Consider the off-diagonal matrix H_g with matrix elements

$$(H_g)_{ij} = \alpha_i \, \delta_{i+1,j} + \beta_i \, \delta_{i,j+g-1} \ , \qquad 0 \le i,j \le k$$

One side-diagonal above the diagonal, the other $g-1$ steps below it

Basic fact: $$\mathcal{Z}_g(y_g) = \det(1 - zH_g)$$

- $\mathcal{Z}(y_g)$ grand partition function of exclusion-g particles
- Fugacity $y_g = -z^g$
- Spectral factors $s_i = \beta_i \, \alpha_i \alpha_{i+1} \cdots \alpha_{i+g-2}$
- $g = 2$ reduces to previous cases (planar & Dyck walks)

(A wealth of) Examples

Planar walks:

$$H_g = A(u)v + v^{1-g}B(u) \;\Rightarrow\; \alpha_i = A(\mathrm{Q}^i),\; \beta_i = B(\mathrm{Q}^i)$$

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Many walks can be brought to this form via a

- square lattice modular transformation, or
- equivalently, u, v algebra automorphism

$$u \to e^{i\phi} u^{m_1} v^{m_2} \;,\quad v \to e^{i\theta} u^{n_1} v^{n_2} \;,\quad \mathrm{Q} \to \mathrm{Q}^{m_1 n_2 - m_2 n_1}$$

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Square lattice walk: $H = u + v + u^{-1} + v^{-1}$

$$u \to -uv : \quad H \;\to\; (1-u)v + v^{-1}(1-u^{-1}) \quad (g=2)$$

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$$u \to e^{i\phi} u^{m_1} v^{m_2} \;,\quad v \to e^{i\theta} u^{n_1} v^{n_2} \;,\quad \mathrm{Q} \to \mathrm{Q}^{m_1 n_2 - m_2 n_1}$$

Square lattice walk: $H = u + v + u^{-1} + v^{-1}$

$$u \to -uv : \quad H \;\to\; (1-u)v + v^{-1}(1-u^{-1}) \quad (g=2)$$

Chiral triangular walk: $H = u + v + \mathrm{Q}^{\frac{1+a}{2}} u^{-1} v^{-1} \;,\; a \in \mathrm{R}$

$$u \to -iuv,\; v \to iu^{-1}v : \quad H \to i(-u + u^{-1})\, v + \mathrm{Q}^a v^{-2} \quad (g=3)$$

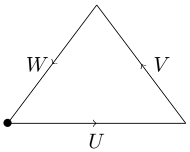
$$\alpha_j = 2\sin\frac{2\pi pj}{q}\ , \quad \beta_j = e^{i2\pi ap/q}$$

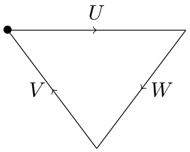
$$s_j = \beta_j\alpha_j\alpha_{j+1} = 4e^{i2\pi ap/q}\sin\frac{2\pi pj}{q}\sin\frac{2\pi p(j+1)}{q}$$

Interpretation: chiral hopping on triangular lattice

- u, v and $w = \mathrm{Q}^{1+a}\,u^{-1}v^{-1}$ represent jumps at 120° angles

- $wvu = vuw = uwv = \mathrm{Q}^{1+a}$
  $\rightarrow$ *up-cell* of area $1+a$

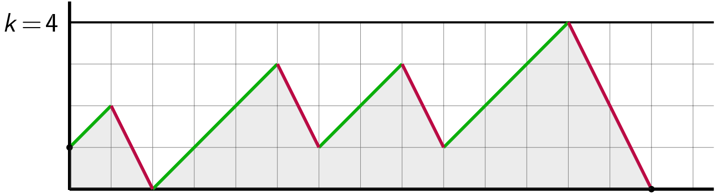




- $vwu = wuv = uvw = (\mathrm{Q}^{1-a})^{-1}$
  $\rightarrow$ down-cell of area $1-a$

Forward-moving walks:

$$H = UV + q^{\frac{g-2}{2}} V^{\dagger g-1} U$$



$g=3$, $(1,2)$ path

- Represents a "one step up, $g-1$ steps down" process [$(1, g-1)$ Lukasiewicz(?) paths]
- Corresponds to an exclusion-g grand partition function
- $g=2$: Dyck paths, as before

...and so on.

Exclusion-g connection allows for explicit solutions

Planar paths: $\mathrm{tr}\, H^{\ell}$ given in terms of cluster coefficients

$$\mathrm{tr}\, H^{gn} = \sum_{\substack{l_1, l_2, \ldots, l_i \\ g\text{-composition of } n}} c_g(l_1, l_2, \ldots, l_i) \sum_{j=1}^{k-i+1} s_j^{l_1} s_{j+1}^{l_2} \cdots s_{j+i-1}^{l_i}$$

with $$c_g(l_1, l_2, \ldots, l_j) = \frac{\prod_{i=1}^{j-g+1} (l_i + \cdots + l_{i+g-1} - 1)!}{\prod_{i=1}^{j-g} (l_{i+1} + \cdots + l_{i+g-1} - 1)!} \prod_{i=1}^{j} \frac{1}{l_i!}$$

g-compositions: compositions with up to $g-2$ zeros inserted between parts (e.g., $g=3$: $2 = 1+1 = 1+0+1$); g^{n-1} in all

- Leads to complicated but explicit trigo sums for number of paths of given area & length $C(\ell, A)$

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Forward-moving paths: bosonization still works, yields explicit expressions for $G_{m,n}(z,q)$ and expansion for $\ln G_{m,n}(z,q)$

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Several other models have been studied (honeycomb lattice, Motzkin paths) or can be studied [Kagomé lattice, (1,p)-paths]

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Suggestions welcome – Thank You!