A Lattice on Dyck Paths Close to the Tamari Lattice

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Jean-Luc Baril [1] Sergey Kirgizov [1] Mehdi Naima [2]

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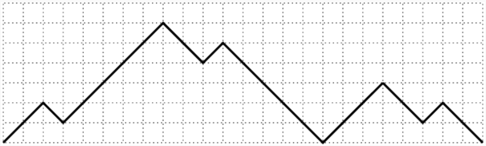
[1]Université de Bourgogne, LIB, Dijon, France

[2]Sorbonne Université, CNRS, LIP6, Paris, France

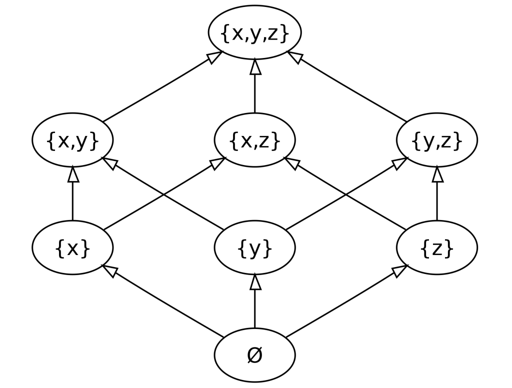


Introduction

Poset

A partial order, is a binary relation $\leq$ on a set P such that for all $a, b, c \in P$

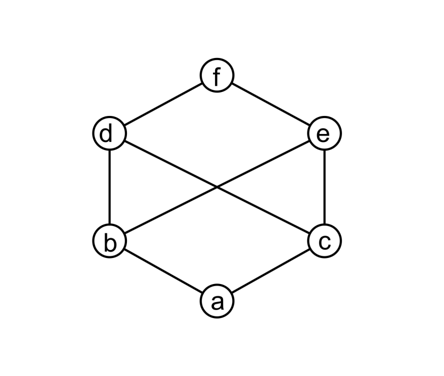
- $a \leq a$ (reflexivity)
- $a \leq b$ and $b \leq a \implies a = b$ (antisymmetry)
- $a \leq b$ and $b \leq c \implies a \leq c$ (transitivity)



set of all subsets of 3 elements ordered by inclusion

Let $(P, \leq)$ be a partially ordered set. Let $x, y, m \in P$, then m is called the **Meet** (greatest lower bound or infimum) $m = x \wedge y$ if :

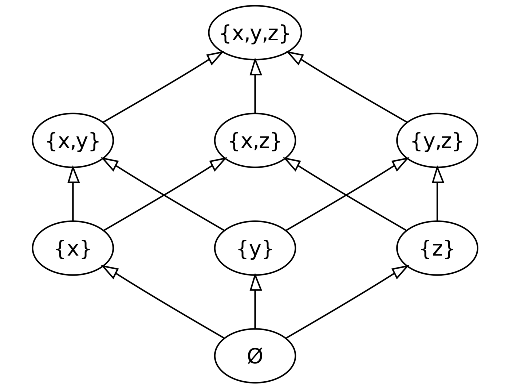
- $m \leq x$ and $m \leq y$
- For any $w \in P$, with $w \leq x$ and $w \leq y$ then $w \leq m$
- Dually a **Join** (Smallest upper bound or supremum) $m = x \vee y$
- If a **meet** (resp. **join**) exists then it is **unique**



Meets do not always exist (for example d, e)

A partially ordered set $(L, \leq)$, is a lattice if $\forall a, b \in L$

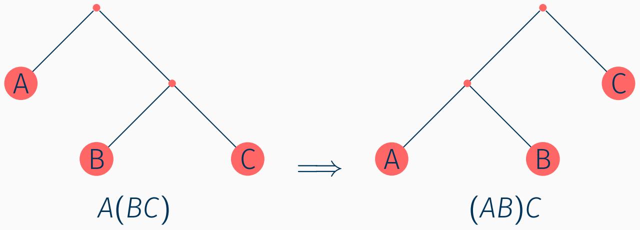
- a, b have a infimum ($a \wedge b$ exists)
- a, b have an supremum ($a \vee b$ exists)



set of all subsets is a lattice

Tamari Lattice

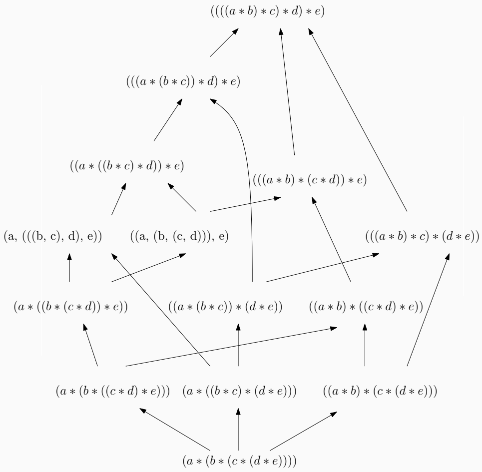
- The Tamari Lattice is a poset introduced by Dov Tamari in 1962
- The Poset has equivalent definitions on bracketed expressions, binary trees, Dyck paths and triangulations
- Many connections with triangulations, combinatorial maps, lambda calculus, ...



If we denote by $\mathcal{T}_n$ the set of bracketed expressions with n atoms.

Definition

The Tamari poset by endowing $\mathcal{T}_n$ with the transitive closure $\preceq$ of the covering relation $A(BC) \longrightarrow (AB)C$ (shifting a parenthesis to the left)



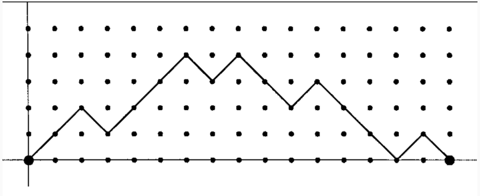
A Dyck path is a lattice path in $\mathbb{N}^2$ starting at the origin, ending on the x-axis and consisting of up steps $U = (1, 1)$ and down steps $D = (1, -1)$.

Catalan numbers

Let $\mathcal{D}_n$ be the set of Dyck paths of semilength n, then :

$$|\mathcal{D}_n| = (2n)!/(n!(n+1)!)$$

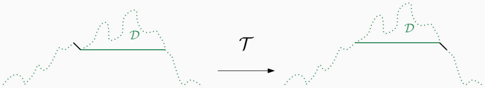
$\mathcal{D} = \bigcup_{n \geq 0} \mathcal{D}_n$

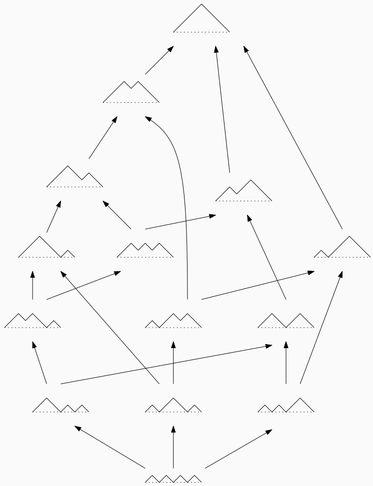


- **first/last return decomposition** of a non-empty Dyck path is unique, $P = URDS$, where $R, S \in \mathcal{D}$
- A Dyck path is **prime** whenever it only touches the x-axis at its beginning and its end

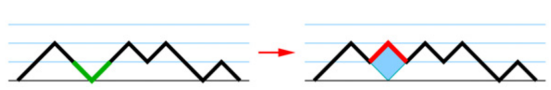
Tamari Lattice

Defined by endowing $\mathcal{D}_n$ with the transitive closure $\preceq$ of the covering relation transforming an occurrence of *DUQD* into an occurrence *UQDD* where $Q \in \mathcal{D}$.

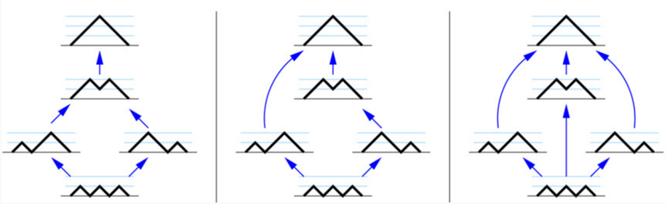




Stanley Lattice : $DU \to UD$



Kreweras Lattice : Swapping descent with Dyck subpath (not necess. prime)

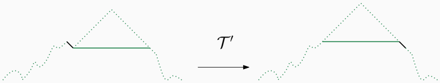


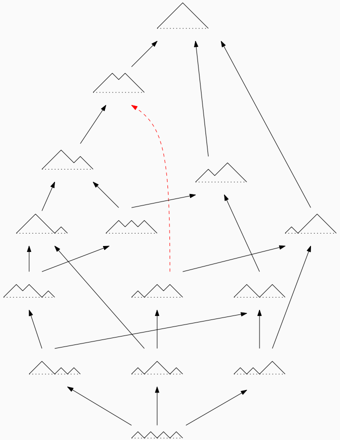
Stanley, Tamari and Kreweras of size 3, Figures from [Bernardi and Bonichon, 2009]

BKN Poset

BKN poset

Defined by endowing $\mathcal{D}_n$ with the transitive closure $\leq$ of the covering relation transforming an occurrence of DU^kD^k into an occurrence U^kD^kD with $k \geq 1$.





The red arrow does not belong to BKN

Unicity of maximum and minimum element

Lemma

For $n \geq 2$, any Dyck path $P \in \mathcal{D}_n$, $P \neq U^n D^n$, contains at least one occurrence of $DU^k D^k$ for some $k \geq 1$.

$\exists$ an occurrence of DU, and the rightmost occurrence of DU always starts an occurrence of $DU\,U^\ell D^\ell D$, $\ell \geq 0$.

Lemma

For $n \geq 2$, any Dyck path $P \in \mathcal{D}_n$, $P \neq (UD)^n$, contains at least one occurrence of $U^k D^k D$ for some $k \geq 1$, and then P contains at least one occurrence of UDD.

By contradiction, assume P does not contain occurrence UDD. Then any peak UD is either at the end of P, or it precedes an up step U, implying that a down step cannot be contiguous to another down step. Thus, $P = (UD)^n$ contradicting $P \neq (UD)^n$.

Propositions :

1. The poset $(\mathcal{D}_n, \leq)$ admits a maximum element and a minimum element.
2. Given $P, Q \in \mathcal{D}_n$ satisfying $P \leq Q$, $P \neq Q$, such that $P = RDS$ and $Q = RUS'$ (R is the maximal common prefix). Let W the Dyck path obtained from P by applying the covering $P \longrightarrow W$ on the leftmost occurrence of DU^kD^k, $k \geq 1$, in DS, then we necessarily have $W \leq Q$.

Theorem

The poset $(\mathcal{D}_n, \leq)$ is a lattice

Existence of a join element. By induction on the semilength of the Dyck paths.

For $n \leq 3$ the poset is isomorphic to the Tamari lattice.

Theorem

The poset $(\mathcal{D}_n, \leq)$ is a lattice

Existence of a join element. By induction on the semilength of the Dyck paths.

Assume $\mathcal{S}_n = (\mathcal{D}_n, \leq)$ is a lattice for $n \leq N$, and show for $N + 1$. Distinguish according to first return decomposition

Theorem

The poset $(\mathcal{D}_n, \leq)$ is a lattice

Existence of a join element. By induction on the semilength of the Dyck paths.

(1) If $P = URDS$ and $Q = UR'DS'$ where $|R| = |R'|$. Apply the recurrence hypothesis for R and R' (resp. S and S'), which means that $R \vee R'$ (resp., $S \vee S'$) exists. Then, the path $U(R \vee R')D(S \vee S')$ is necessarily the least upper bound of P and Q, proving existence of $P \vee Q$.

Theorem

The poset $(\mathcal{D}_n, \leq)$ is a lattice

Existence of a join element. By induction on the semilength of the Dyck paths.

(2) Let us suppose that $P = URDS$ and $Q = UR'DS'$ where $|R'| < |R|$. Let M be an upper bound of P and Q (Prop 1). Since $|R'| < |R|$, M has necessarily a decomposition $M = UM_1DM_2$ where $|M_1| \geq |R|$. In any sequence of coverings

$Q \to \ldots \to M$

from Q to M, there is necessarily a covering that elevates the down-step just after R'

Q =  = Q_1

Theorem

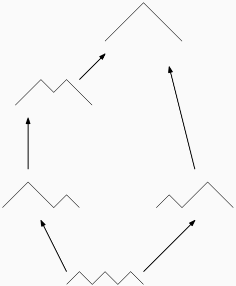
The poset $(\mathcal{D}_n, \leq)$ is a lattice

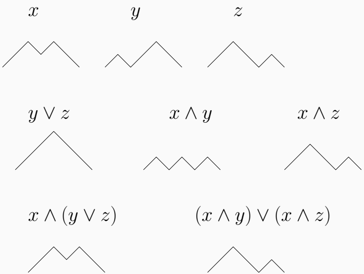
Existence of a join element. By induction on the semilength of the Dyck paths.

Iterating this process with P and Q_1, construct P', Q' such that $P \leq M, Q \leq M$ $\equiv P' \leq M, Q' \leq M$ where P' and Q' lie **(1)**. Using the hypothesis recurrence $P' \vee Q' = P \vee Q$ exists. The existence of greatest lower bound then follows automatically since the poset is finite with a least and greatest elements. □

Let $(L, \vee, \wedge)$ be a Lattice :

- L is **distributive** if $\forall x, y, z \in L, x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
- The Tamari and BKN lattices are not distributive





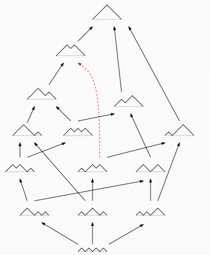
- L is **semidistributive** if it is both **join- and meet-semidistributive** where
  - **meet-semidistributive** if for all elements $e, x, y \in L$ in the lattice we have :

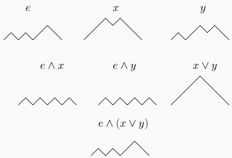
    $$e \wedge x = e \wedge y \implies e \wedge x = e \wedge (x \vee y)$$

  - **join-semidistributive** if for all elements $e, x, y \in L$ in the lattice we have :

    $$e \vee x = e \vee y \implies e \vee x = e \vee (x \wedge y)$$

- Tamari is semidistributive but not BKN





Characteristics of BKN

Number of edges in the poset

Let $A(x,y,z) = \sum_{n\geq 0} a_{n,k,\ell} x^n y^k z^\ell$ be the generating function where $a_{n,k,\ell}$ is the number of Dyck paths of

- semilength n having
- k possible coverings (or equivalently k outgoing edges),
- ℓ incoming edges.

$$A(x,y,z) = \frac{R(x,y,z) - \sqrt{4x\left(xzy - xy - xz + 1\right)\left(xy + xz - x - 1\right) + R(x,y,z)^2}}{2x\left(xzy - xy - xz + 1\right)},$$

where $R(x,y,z) = x^2zy - x^2y - x^2z + x^2 - xy - xz + x + 1$.

Number of edges in the poset

$A(x,y,z) = \frac{R(x,y,z) - \sqrt{4x(xzy-xy-xz+1)(xy+xz-x-1)+R(x,y,z)^2}}{2x(xzy-xy-xz+1)}$, where
$R(x,y,z) = x^2zy - x^2y - x^2z + x^2 - xy - xz + x + 1.$

- Using last return decomposition $P = RUSD$
- 6 different cases according to R and S

$$\begin{aligned} A =& 1 + \underbrace{x}_{R=S=\epsilon} + \underbrace{(A-1)xy}_{\substack{R\neq\epsilon\\ S=\epsilon}} + \underbrace{\frac{x^2z}{1-xz}}_{\substack{R=\epsilon\\ S=U^\alpha D^\alpha}} + \underbrace{\frac{x^2z}{1-xz}(A-1)y}_{R\neq\epsilon, S=U^\alpha D^\alpha} + \underbrace{\frac{x^2z}{1-xz}(A-1)yA}_{S=S'U^\alpha D^\alpha, S'\neq\epsilon} \\ &+ \underbrace{Ax\left(A-1-x-\frac{x^2z}{1-xz}-x(A-1)y-\frac{x^2z}{1-xz}(A-1)y\right)}_{S\neq S'U^\alpha D^\alpha}, \end{aligned} \tag{1}$$

Comparison with Tamari Lattice

G.F $E(x)$ of the total number of possible coverings over all Dyck paths of semilength n (or equivalently the number of edges in the Hasse diagram) is

$$E(x) = \frac{-1 + 4x + (1-2x)\sqrt{1-4x}}{2(1-4x)(1-x)}.$$

From $A(x,y,z)$ simply compute $\partial_y(A(x,y,1))|_{y=1}$.
$[x^n]E(x) = \sum_{k=0}^{n-2} \binom{2k+2}{k}$ (A057552 in [Sloane et al., 2003])
coverings Tamari Lattice : $\frac{(n-1)}{2} C_n$

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$\#$ coverings Tamari Lattice : $\frac{(n-1)}{2} C_n$

The ratio between the numbers of coverings in $\mathcal{T}_n$ and $\mathcal{S}_n$ tends towards $3/2$.

- An interval is an ordered pair of elements (P, Q) with $P \leq Q$
- Inspired by [Bousquet-Mélou and Chapoton, 2023]
- Let $I(x, y) = \sum_{n,k\geq 1} a_{n,k} x^n y^k$, where $a_{n,k}$ number of intervals in $\mathcal{S}_n$ with upper path ends with k down-steps exactly
- Let $J(x, y) = \sum_{n,k\geq 1} b_{n,k} x^n y^k$, where $b_{n,k}$ number of intervals (P, Q) in $\mathcal{S}_n$ such that the upper path Q is **prime** and ends with k down-steps exactly

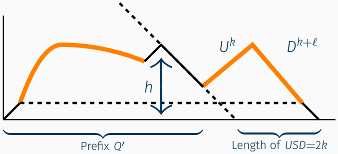
$$I(x,y) = \underbrace{J(x,y)}_{\substack{\text{Interval is} \\ \text{either prime}}} + \underbrace{I(x,1)\cdot J(x,y)}_{\substack{Q=RUSD, P=P_1P_2 \\ I_1:=(P_1,R) \text{ and } I_2:=(P_2,USD)}} \tag{2}$$

The following also holds :

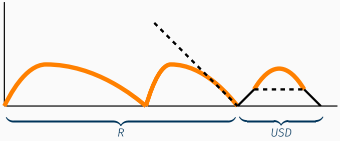
$$J(x,y) = \underbrace{xy}_{\text{P=UD and Q=UD}} + \underbrace{xyI(x,y)}_{\substack{P \text{ is prime, } P = UP'D \\ \text{and necess. } Q = UQ'D}} + \underbrace{\frac{J(x,y) - J(x,1)}{y-1} \cdot C(xy)xy^2}_{\substack{P \text{ is not prime, } P = RUSD \\ \text{const. } h \text{ intervals}}}, \quad (3)$$

where $C(x)$ is the g.f. for Catalan numbers, i.e., $C(x) = 1 + xC(x)^2$.

$Q = Q'U^kD^{k+\ell} =$



$P = RUSD =$



With little rearrangments

$$\begin{cases} I(x,y) &= \frac{J(x,y)}{1-J(x,1)}, \\ J(x,y) &= xy + xy\frac{J(x,y)}{1-J(x,1)} + \frac{J(x,y)-J(x,1)}{y-1} \cdot C(xy)xy^2. \end{cases}$$

In order to compute $J(x,1)$ use the kernel method [Banderier et al., 2002] on

$$J(x,y) \cdot \left(1 - \frac{xy}{1-J(x,1)} - \frac{C(xy)xy^2}{y-1}\right) = xy - \frac{J(x,1)}{y-1} \cdot C(xy)xy^2.$$

Cancel the factor of $J(x,y)$ by finding y as a function y_0 of $J(x,1)$ and x to find :

$$\begin{cases} 1 - \frac{xy_0}{1-J(x,1)} - \frac{C(xy_0)xy_0^2}{y_0-1} &= 0, \\ xy_0 - \frac{J(x,1)}{y_0-1} \cdot C(xy_0)xy_0^2 &= 0. \end{cases}$$

Then $y_0 = \frac{1+4x-\sqrt{1-8x}}{8x}$.

- The generating function $J(x,y)$ can be found explicitly
- From $J(x,y)$ we exhibit
  (prime intervals) $J(x,1) = \frac{1-\sqrt{1-8x}}{4} = x + 2x^2 + 8x^3 + 40x^4 + 224x^5 + \dots$
  (A052701) $(2^{n-1}c_{n-1})$
- We then obtain : $I(x,y) = J(x,y) \cdot \frac{3-\sqrt{1-8x}}{2(x+1)}$
- **(intervals)** $\mathbf{I(x,1) = \frac{1-2x-\sqrt{1-8x}}{2(x+1)} = x + 3x^2 + 13x^3 + 67x^4 + 381x^5 + \dots}$
  (A064062) $\left(\frac{1}{n}\sum_{m=0}^{n-1}(n-m)\binom{n+m-1}{m}2^m\right) \overset{n\to\infty}{=} \frac{2^{3n}n^{-3/2}}{36\sqrt{\pi}}$

Both sequences count **outerplanar maps** and **bi-colored Dyck Paths** [Geffner and Noy Serrano, 2017]

Asymptotic exponential growth of intervals in $\mathcal{T}_n$ and $\mathcal{S}_n$ is $\left(\frac{32}{27}\right)^n$

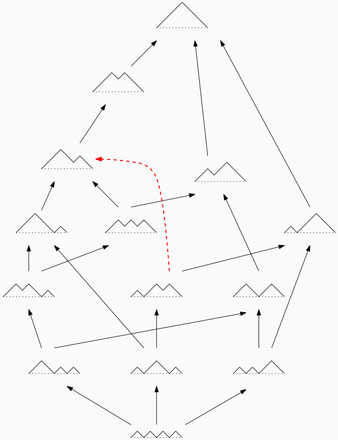
Generalization

BKN poset

Defined by endowing $\mathcal{D}_n$ with the transitive closure $\leq$ of the covering relation transforming an occurrence of DU^kD into an occurrence U^kDD with $k \geq 1$.

Reminder BKN :

DU^kD^k into an occurrence U^kD^kD with $k \geq 1$.



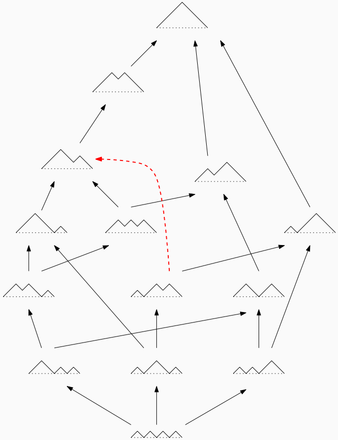
The red arrow does not belong to BKN

- Joint work BKN and Bousquet-Mélou
- The resulting poset is a lattice

	meet-semidistributive	join-semidistributive
Tamari	yes	yes
BKN	no	no
BBKN	yes	no

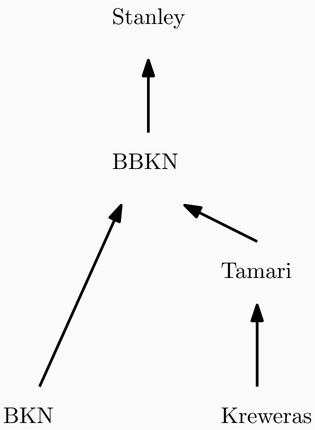
- As n tends to infinity, the number of intervals

$$\kappa\,\mu^n n^{-7/2}, \mu = \frac{11+5\sqrt{5}}{2}, \qquad \kappa = \frac{3}{8}\sqrt{\frac{275+123\sqrt{5}}{10\,\pi}}$$



The red arrow does not belong to BKN

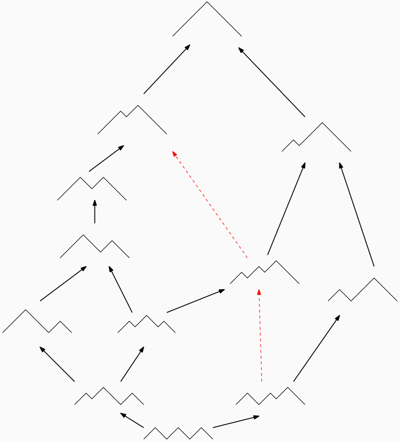
	Asymptotic form	
Kreweras	$c_1\,\mu_4^n n^{-3/2}$	$\mu_1 = 6.75$
BKN	$c_2\,\mu_1^n n^{-3/2}$	$\mu_2 = 8$
Tamari	$c_3\,\mu_2^n n^{-5/2}$	$\mu_3 = \frac{256}{27} \approx 9.48148$
BBKN	$c_4\,\mu_3^n n^{-7/2}$	$\mu_4 = \frac{11+5\sqrt{5}}{2} \approx 11.09$
Stanley	$c_5\,\mu_4^n n^{-10/2}$	$\mu_5 = 16$



Inclusion of lattices

Open questions

- Fix $m \geq 1$, an m-Dyck path is a path in $\mathbb{N}^2$ starting at $(0,0)$ ending on the x-axis and consisting of $U = (m, m)$ and $D = (1, -1)$.
- m-BKN poset is defined by endowing $\mathcal{D}_n^m$ with the transitive closure $\leq$ of the covering transforming an occurrence of DU^kD^{mk} into an occurrence $U^kD^{mk}D$ with $k \geq 1$.
- m-BKN seems to always give lattices
- Can we extend our approach to count intervals in m-BKN ?
- $I_n^2 = 0, 1, 6, 55, 600, 7192, 91470, \ldots$
- $I_n^3 = 0, 1, 10, 152, 2723, 53307, 1104003, \ldots$



The red arrows belong to 2-Tamari but not to 2-BKN

Sequent Calculus

- In [Zeilberger, 2019] showed a sequent calculus capturing the Tamari order (semi-associative law)
- Can we find a calculus capturing the BKN order ?
- Currently working on proofs

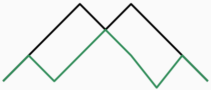
$$\frac{}{A \Rightarrow A}\ \text{id}$$

$$\frac{A, B, \Delta \Rightarrow C}{A * B, \Delta \Rightarrow C}\ \text{L}$$

$$\frac{\Delta \Rightarrow A \qquad \Gamma \Rightarrow B}{\Delta, \Gamma \Rightarrow A * B}\ \text{R}$$

- A, B, C are formulas, Δ, Γ are lists of formulas
- **(atoms)** lowercase latin letters
- **(Formulas)** $\mathcal{F} := a, b, \ldots \mid (\mathcal{F} * \mathcal{F})$

$$\dfrac{\dfrac{}{a \Rightarrow a}\,\text{id} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{b \Rightarrow b}\,\text{id} \quad \dfrac{}{c \Rightarrow c}\,\text{id}}{b, c \Rightarrow (b * c)}\,\text{R1} \qquad \dfrac{\dfrac{}{d \Rightarrow d}\,\text{id} \quad \dfrac{}{e \Rightarrow e}\,\text{id}}{d, e \Rightarrow (d * e)}\,\text{R1}}{b, c, d, e \Rightarrow ((b * c) * (d * e))}\,\text{R2}}{(b * c), d, e \Rightarrow ((b * c) * (d * e))}\,\text{L}}{\dfrac{\dfrac{\dfrac{a, (b * c), d, e \Rightarrow (a * ((b * c) * (d * e)))}{(a * (b * c)), d, e \Rightarrow (a * ((b * c) * (d * e)))}\,\text{L}}{((a * (b * c)) * d), e \Rightarrow (a * ((b * c) * (d * e)))}\,\text{L}}{(((a * (b * c)) * d) * e) \Rightarrow (a * ((b * c) * (d * e)))}\,\text{L}}\,\text{R2}$$



$$\dfrac{}{A \Rightarrow A}\,\text{id}$$

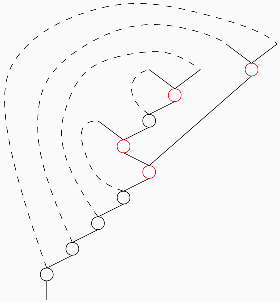
$$\dfrac{A, B, \Delta \Rightarrow C}{A * B, \Delta \Rightarrow C}\,\text{L}$$

$$\dfrac{\Delta \Rightarrow A \qquad C \Rightarrow B}{\Delta, C \Rightarrow A * B}\,\text{R1}$$

$$\dfrac{\mathfrak{T} \Rightarrow A \qquad \Delta \Rightarrow B}{\mathfrak{T}, \Delta \Rightarrow A * B}\,\text{R2}$$

A, B, C are formulas, Δ a list of formulas and $\mathfrak{T}$ a list of atoms.

- A term with no free variables is **closed**
- A term is **indecomposable** if it has no closed proper subterms
- An abstraction $\lambda x.M$ is linear if the x has exactly one free occurrence in M. By extension, a term is **linear** if every abstraction subterm is linear
- A linear term M is **planar** if its binding diagram is planar
- A term is β-**normal** if it can not be reduced further by β-reductions



$\lambda e.(\lambda d.(\lambda c.(\lambda a.(((a)@(\lambda b.((b)@(c))))@((d)@(e))))))$

Fragment of Lambda Calculus

- In [Zeilberger, 2019] showed that **Tamari intervals** are in bijection with **Closed indecomposable β-normal linear planar lambda terms**
- BKN Lattice being a restriction of the Tamari Lattice
- Can we characterize the properties of the fragment of Lambda Calculus induced by BKN ?

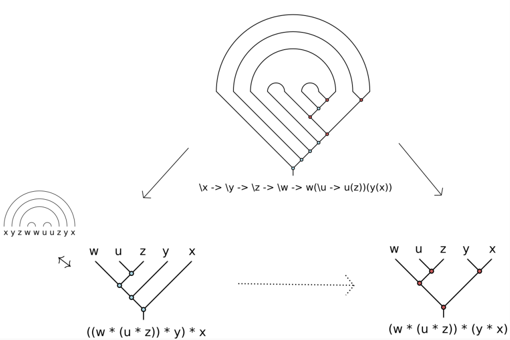


Figure from [Zeilberger, 2019]

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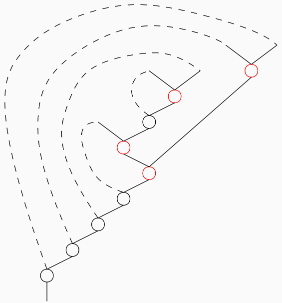
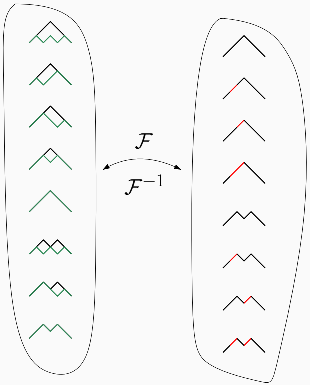


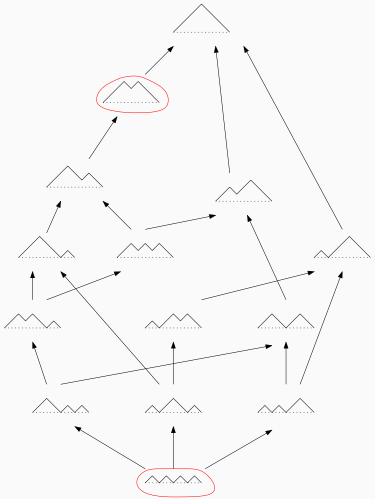
$\lambda e.(\lambda d.(\lambda c.(\lambda a.(((a)@(\lambda b.((b)@(c))))@((d)@(e))))))$

First term belonging to Tamari but not to BKN

- Sequence of prime intervals : $x + 2\,x^2 + 8\,x^3 + 40\,x^4 + 224\,x^5 + 1344\,x^6 + \ldots$ (A052701) $(2^{n-1}c_{n-1})$
- Also corresponds to Number of Dyck paths of semilength n in which the step $U = (1, 1)$ not on ground level comes in 2 colors
- Can we find a bijection between these classes ?



- The *diameter* is the maximum distance between any two vertices
- The diameter of BKN gives an upper bound on the diameter of the Tamari Lattice
- For $n \geq 3$, we conjecture that the diameter of $\mathcal{S}_n$ is $2n-4$, and that this value corresponds to the distance between $(UD)^n$ and $UU(UD)^{n-2}DD$.



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$$J(x,y) = \frac{xy\left(-1+J(x,1)\right)\left(J(x,1)\,C\left(xy\right)y - y + 1\right)}{J(x,1)\,C\left(xy\right)xy^2 - C\left(xy\right)xy^2 - xy^2 - J(x,1)\,y + xy + J(x,1) + y - 1}$$