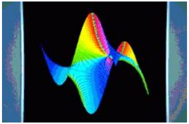
Évaluation numérique de fonctions spéciales et combinatoire analytique avec

NumGfun

Marc Mezzarobba

Projet Algorithms



INRIA Paris Rocquencourt

Séminaire Calin, 15 mars 2011

Dynamic Dictionary of Mathematical Functions - Iceweasel

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http://ddmf.msr-inria.inria.fr/ddmf?service=MainIndex&rendering=jsMath

[01] Dynamic Dictionary of Ma...

Dynamic Dictionary of Mathematical Functions

Welcome to this interactive site on Mathematical Functions, with properties, truncated expansions, numerical evaluations, plots, and more. The functions currently presented are elementary functions and special functions of a single variable. More functions — special functions with parameters, orthogonal polynomials, sequences — will be added with the project advances.

Select a special function from the list

- Help on selecting and configuring the mathematical rendering
- DDMF developers list
- Motivation of the project

Contents

rendering link

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Dynamic Dictionary of Mathematical Functions

Welcome to this interactive site on Mathematical Functions, with properties, truncated expansions, numerical evaluations, plots, and more. The functions currently presented are elementary functions and special functions of a single variable. More functions — special functions with parameters, orthogonal polynomials, sequences — will be added with the project advances.

Select a special function from

- Motivation of the project

Contents

rendering link

jsMath

Benoit, Chyzak, Darrasse, Gerhold, M. & Salvy (2010)

The Special Function $\mathrm{Ai}\,(x)$

1. Differential equation

rendering link

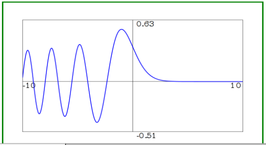
The function $\mathrm{Ai}\,(x)$ satisfies

$$\frac{d^2}{dx^2} y\,(x) - x y\,(x) = 0$$

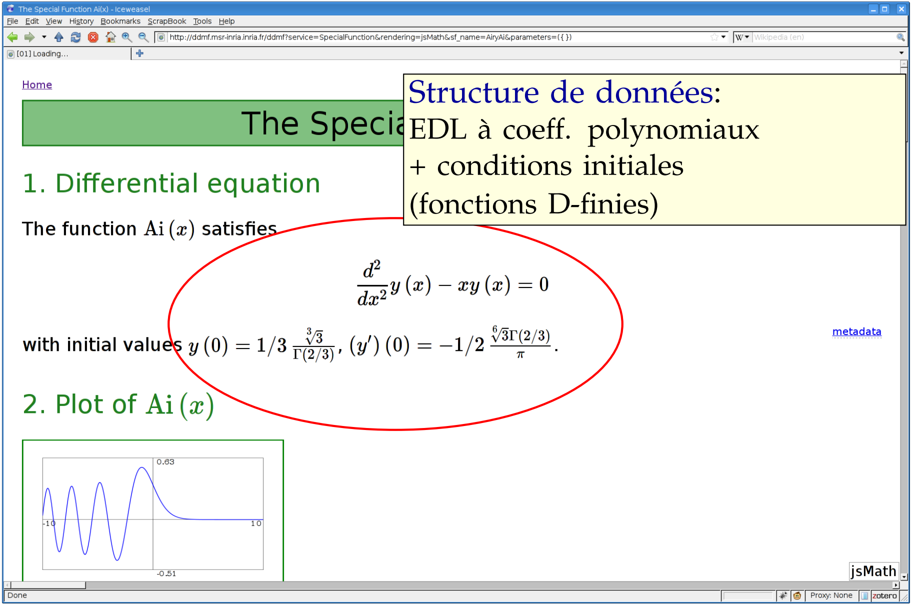
with initial values $y\,(0) = 1/3\,\frac{\sqrt[3]{3}}{\Gamma(2/3)}$, $(y')\,(0) = -1/2\,\frac{\sqrt[6]{3}\Gamma(2/3)}{\pi}$.

metadata

2. Plot of $\mathrm{Ai}\,(x)$



jsMath



Structure de données:
EDL à coeff. polynomiaux
+ conditions initiales
(fonctions D-finies)

The Special Function Ai(x) - Iceweasel

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http://ddmf.msr-inria.inria.fr/ddmf?service=SpecialFunction&rendering=jsMath&sf_name=AiryAi&parameters=({ })

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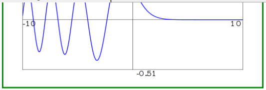

min = -10 max = 10 Submit Query

3. Numerical Evaluation

$$\mathrm{Ai}\,(1/4+1/4\,i)\approx 0.28881085-0.06285935\,i.$$

metadata

(Below, path may be either a point z or a broken-line path $[z_1, z_2, \ldots, z_n]$ along which to perform analytic continuation of the solution of the defining differential equation. Each z_k should be of the form x + y*i.)

path = 1/4+1/4*i precision = 8 Submit Query

4. Taylor expansion of $\mathrm{Ai}\,(x)$ at 0

- Expansion of AiryAi at 0:

$$\mathrm{Ai}\,(x)=\sum_{n=0}^{\infty} 1/3\,\frac{\sqrt[3]{3}x^{3\,n}}{9^n\Gamma\,(n+2/3)\,n!}-1/9\,\frac{3^{2/3}x^{3\,n+1}}{9^n\Gamma\,(n+4/3)\,n!}.$$

me jsMath

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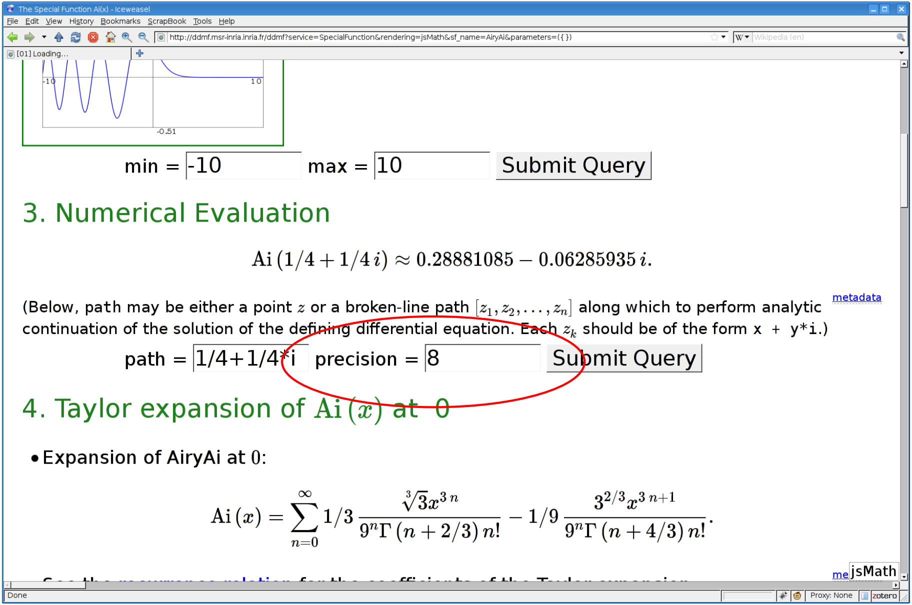
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http://ddmf.msr-inria.inria.fr/ddmf?service=SpecialFunction&rendering=jsMath&sf_name=AiryAi&parameters=({ })
[01] Loading...
-10
10
-0.51
min = -10 max = 10 Submit Query
3. Numerical Evaluation
$$\mathrm{Ai}\,(1/4 + 1/4\,i) \approx 0.28881085 - 0.06285935\,i.$$
metadata
(Below, path may be either a point z or a broken-line path $[z_1, z_2, \ldots, z_n]$ along which to perform analytic continuation of the solution of the defining differential equation. Each z_k should be of the form x + y*i.)
path = 1/4+1/4*i precision = 8 Submit Query
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$$\mathrm{Ai}\,(x) = \sum_{n=0}^{\infty} 1/3\,\frac{\sqrt[3]{3}x^{3\,n}}{9^n\Gamma\,(n+2/3)\,n!} - 1/9\,\frac{3^{2/3}x^{3\,n+1}}{9^n\Gamma\,(n+4/3)\,n!}.$$
jsMath
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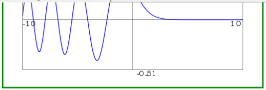

min = -10 max = 10 Submit Query

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$$\mathrm{Ai}\,(1/4+1/4\,i) \approx 0.28881085 - 0.06285935\,i.$$

metadata

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path = 1/4+1/4*i precision = 80 Submit Query

4. Taylor expansion of $\mathrm{Ai}\,(x)$ at 0

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$$\mathrm{Ai}\,(x) = \sum_{n=0}^{\infty} 1/3\,\frac{\sqrt[3]{3}x^{3\,n}}{9^n\Gamma\,(n+2/3)\,n!} - 1/9\,\frac{3^{2/3}x^{3\,n+1}}{9^n\Gamma\,(n+4/3)\,n!}.$$

me jsMath

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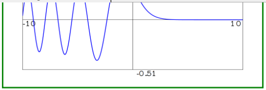

min = -10 max = 10 Submit Query

3. Numerical Evaluation

$$\mathrm{Ai}\,(1/4+1/4\,i) \approx 0.28881085384820872173256483671407046811262524805800436861749378392647$$

(Below, path may be either a point z or a broken-line path $[z_1, z_2, \ldots, z_n]$ along which to perform analytic continuation of the solution of the defining differential equation. Each z_k should be of the form x + y*i.) metadata

path = 1/4+1/4*i precision = 80 Submit Query

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me jsMath

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- précision arbitraire
- résultats garantis
- à partir de l'équa. diff.

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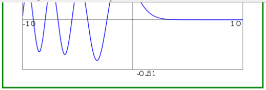

min = -10 max = 10 Su

3. Numerical Evaluation

$$\mathrm{Ai}\,(1/4 + 1/4\,i) \approx 0.28881085384820872173256483671407046811262524805800436861749378392647$$

(Below, path may be either a point z or a broken-line path $[z_1, z_2, \ldots, z_n]$ along which to perform analytic continuation of the solution of the defining differential equation. Each z_k should be of the form x + y*i.) metadata

path = 1/4+1/4*i precision = 80 Submit Query

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me jsMath

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- précision arbitraire
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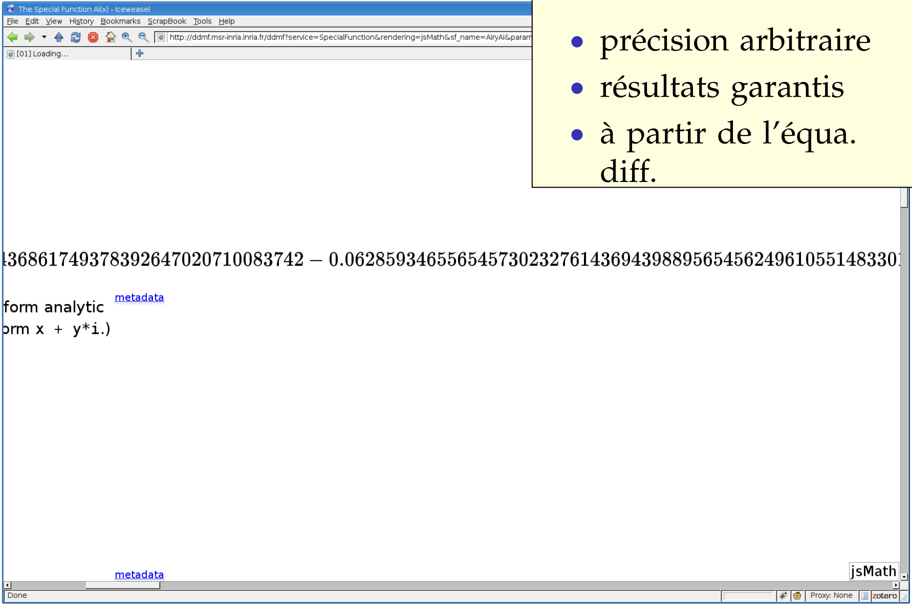

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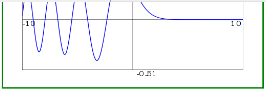

min = -10 max = 10 Su

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$\mathrm{Ai}\,(1/4+1/4\,i) \approx 0.28881085384820872173256483671407046811262524805800436861749378392647$

(Below, path may be either a point z or a broken-line path $[z_1, z_2, \ldots, z_n]$ along which to perform analytic continuation of the solution of the defining differential equation. Each z_k should be of the form `x + y*i`.) metadata

path = 1/4+1/4*i precision = 80 Submit Query

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me jsMath

Done Proxy: None zotero

- précision arbitraire
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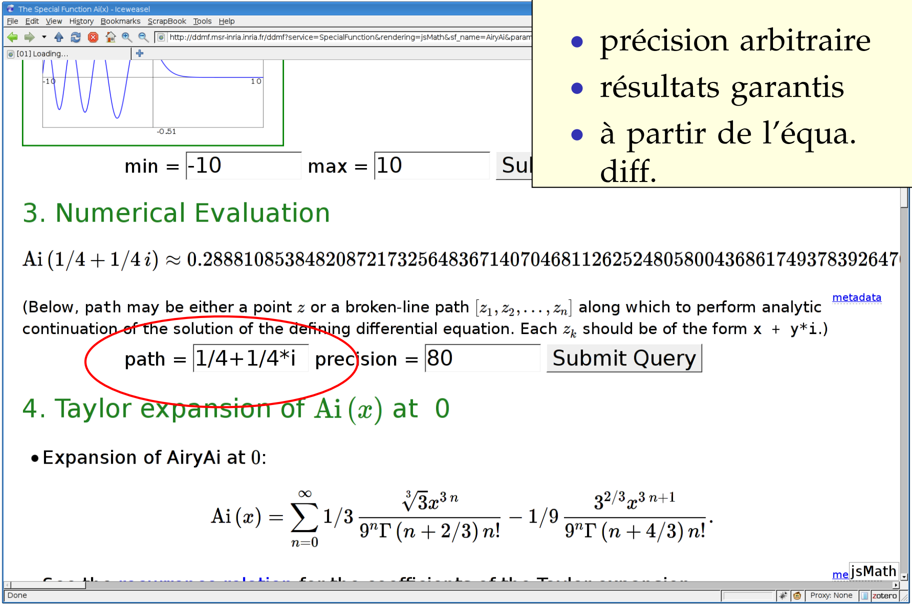
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min = -10 max = 10 Su

3. Numerical Evaluation

$\mathrm{Ai}\,(1/4 + 1/4\,i) \approx 0.28881085384820872173256483671407046811262524805800436861749378392647$

(Below, path may be either a point z or a broken-line path $[z_1, z_2, \ldots, z_n]$ along which to perform analytic continuation of the solution of the defining differential equation. Each z_k should be of the form x + y*i.) metadata

path = 1/4+1/4*i precision = 80 Submit Query

4. Taylor expansion of $\mathrm{Ai}\,(x)$ at 0

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$$\mathrm{Ai}\,(x) = \sum_{n=0}^{\infty} 1/3\,\frac{\sqrt[3]{3}x^{3\,n}}{9^n\Gamma\,(n+2/3)\,n!} - 1/9\,\frac{3^{2/3}x^{3\,n+1}}{9^n\Gamma\,(n+4/3)\,n!}.$$

me jsMath

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- précision arbitraire
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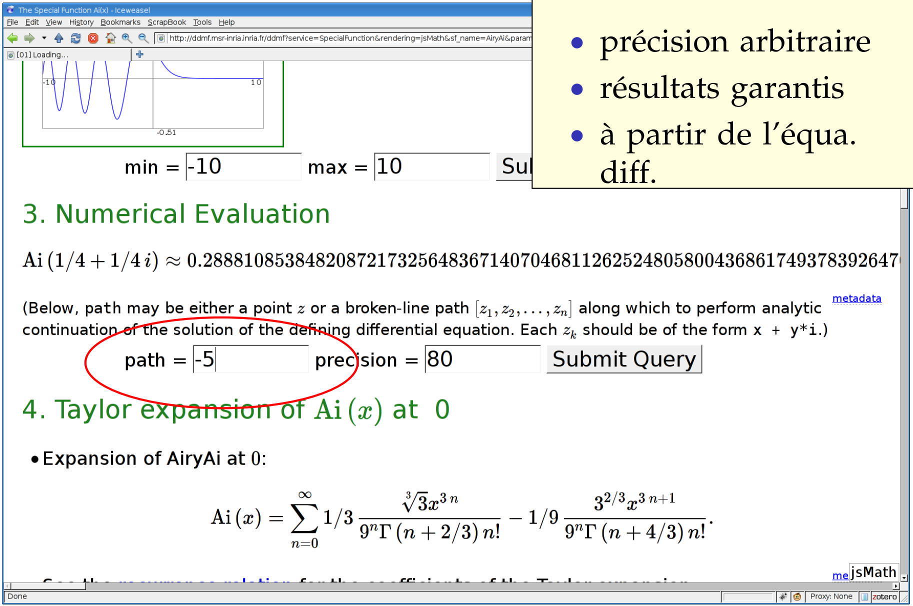
- à partir de l'équa. diff.



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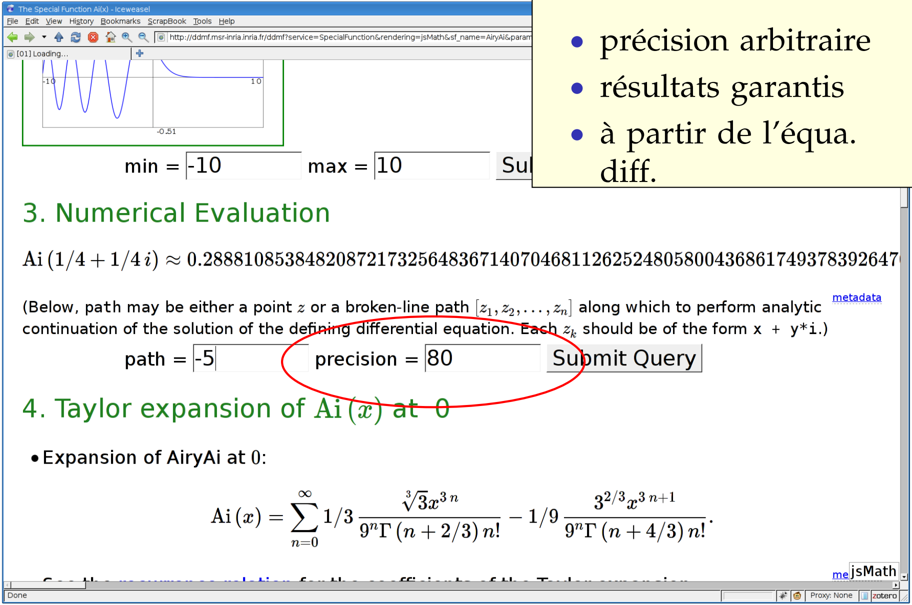

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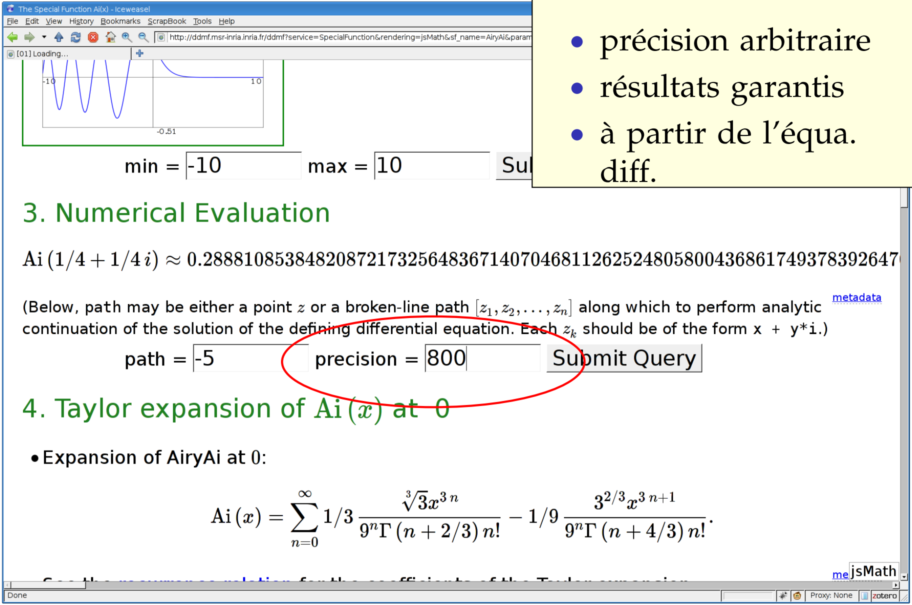

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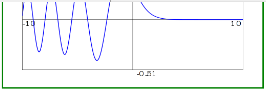

min = -10 max = 10 Su

3. Numerical Evaluation

$$\mathrm{Ai}\left(-5\right) \approx 0.350761009024114319788016327696742221484443250893087208211128178049911192682$$

metadata

(Below, path may be either a point z or a broken-line path $[z_1, z_2, \ldots, z_n]$ along which to perform analytic continuation of the solution of the defining differential equation. Each z_k should be of the form x + y*i.)

path = -5 precision = 800 Submit Query

4. Taylor expansion of $\mathrm{Ai}\left(x\right)$ at 0

- Expansion of AiryAi at 0:

$$\mathrm{Ai}\left(x\right) = \sum_{n=0}^{\infty} 1/3\, \frac{\sqrt[3]{3}x^{3\,n}}{9^n\Gamma\left(n+2/3\right) n!} - 1/9\, \frac{3^{2/3}x^{3\,n+1}}{9^n\Gamma\left(n+4/3\right) n!}.$$

jsMath

Done Proxy: None zotero

This is AsyRec, A Maple package
accompanying Doron Zeilberger's article:

It finds the asymptotics of solutions of (homog.) linear recurrence equations with polynomial coefficients, using the Birkhoff-Trjitzinsky method .

```
> recop := (n+2)^2*N^2-(7*n^2+21*n+16)*N-8*(n+1)^2;
```

$$recop := (n+2)^2 N^2 - \left(7\, n^2 + 21\, n + 16\right) N - 8\,(n+1)^2$$

```
> AsyC(recop, n, N, 5, [2, 10], 1000);
```

0.36755259694786136634,

$$\frac{8^n \left(1 - \frac{1}{3\,n} + \frac{1}{27\,n^2} + \frac{1}{81\,n^3} + \frac{1}{243\,n^4} + \frac{11}{2187\,n^5}\right)}{n}$$

(Wimp & Zeilberger 1985, Zeilberger 2008-2009)

/home/mezzarob/docs/recherche/exposes/2010-07-issac/maple1.mw* - [Server 2] - Maple 13

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```
> with(gfun):
> with(NumGfun);
[abs_with_RootOf, analytic_continuation,
    bound_diffeq, bound_diffeq_tail,
    bound_ratpoly, bound_rec, bound_rec_tail,
    diffeqtoproc, dominant_root, evaldiffeq,
    fnth_term, make_waksman_proc,
    needed_terms, transition_matrix]
> evaldiffeq(diff(y(z),z)=y(z), y
    (z), 1, 10000);
2.718281828459045235360287471352662497757
    5724709369995957496696762772407663035
    3547594571382178525166427427466391932
    0030599218174135966290435729003342952
    6059563073813232862794349076323382988
    0753195251019011573834187930702154089
    1499348841675092447614606680822648001
    6847741185374234544243710753907774499
    2069551702761838606261331384583000752
    0449338265602976067371132007
```

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/home/mezzarob/docs/recherche/exposes/2010-07-issac/maple1.mw* - [Server 2] - Maple 13

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$$
\begin{aligned}
&42926741257342244776558417788617173\,7\\
&265462085498294498946787350929581652\\
&632072258992368768457017823038096567\\
&883112289305809140572610865884845873\\
&101658151167533327674887014829167419\\
&701512559782572707406431808601428149\\
&024146780472327597684269633935773542\\
&930186739439716388611764209004068663\\
&398856841681003872389214483176070116\\
&684503887212364367043314091155733280\\
&182977988736590916659612402021778558\\
&854876176161989370794380056663364884\\
&365089144805571039765214696027662583\\
&59905198704230017946553678 9\ \_C_0
\end{aligned}
$$

```
> bound_diffeq_tail({(1+z^2)*diff(y(z),z)-1, y(0) = 0}, y(z), n);
```

$$
-\frac{500000009001}{1000000000000}\,\frac{(n\,|z| - n - 1)\,|z|^n}{(|z| - 1)^2}
$$

\>

Ready Memory: 12.87M Time: 0.45s Math Mode http://algo.inria.fr/libraries/ (LGPL)

B. Salvy and P. Zimmermann. Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. 1994.

© Bruce Marlin

```
> diffeq := random_diffeq(3, 2);
```

```
> diffeq := random_diffeq(3, 2);
```

$$diffeq := \left\{ \left(\frac{13}{30} + \frac{8}{15} z + \frac{7}{30} z^2 \right) y(z) + \left(-\frac{9}{20} + \frac{29}{30} z - \frac{1}{12} z^2 \right) \left(\frac{d}{dz} y(z) \right) + \left(-\frac{43}{60} + \frac{49}{60} z + \frac{11}{30} z^2 \right) \left(\frac{d^2}{dz^2} y(z) \right) + \left(-\frac{7}{12} + \frac{17}{30} z - \frac{3}{5} z^2 \right) \left(\frac{d^3}{dz^3} y(z) \right), y(0) = 0, D(y)(0) = \frac{7}{30}, D^{(2)}(y)(0) = -\frac{43}{60} \right\}$$

```
> diffeq := random_diffeq(3, 2);
```

$$diffeq := \left\{ \left(\frac{13}{30} + \frac{8}{15} z + \frac{7}{30} z^2\right) y(z) + \left(-\frac{9}{20} + \frac{29}{30} z - \frac{1}{12} z^2\right) \left(\frac{d}{dz} y(z)\right) + \left(-\frac{43}{60} + \frac{49}{60} z + \frac{11}{30} z^2\right) \left(\frac{d^2}{dz^2} y(z)\right) + \left(-\frac{7}{12} + \frac{17}{30} z - \frac{3}{5} z^2\right) \left(\frac{d^3}{dz^3} y(z)\right), y(0) = 0, D(y)(0) = \frac{7}{30}, D^{(2)}(y)(0) = -\frac{43}{60} \right\}$$

```
> evaldiffeq(diffeq, y(z), (1+I)/5, 40);
```

```
> diffeq := random_diffeq(3, 2);
```

$$diffeq := \left\{ \left(\frac{13}{30} + \frac{8}{15} z + \frac{7}{30} z^2 \right) y(z) + \left(-\frac{9}{20} + \frac{29}{30} z - \frac{1}{12} z^2 \right) \left(\frac{d}{dz} y(z) \right) + \left(-\frac{43}{60} + \frac{49}{60} z + \frac{11}{30} z^2 \right) \left(\frac{d^2}{dz^2} y(z) \right) + \left(-\frac{7}{12} + \frac{17}{30} z - \frac{3}{5} z^2 \right) \left(\frac{d^3}{dz^3} y(z) \right), y(0) = 0, D(y)(0) = \frac{7}{30}, D^{(2)}(y)(0) = -\frac{43}{60} \right\}$$

```
> evaldiffeq(diffeq, y(z), (1+I)/5, 40);
```

$$0.04485557487767843131893308147593115486 63 + 0.01990489830212805305047897725810997882 82\, I$$

```
> evaldiffeq(diffeq, y(z), 1/5, 1000000);
```

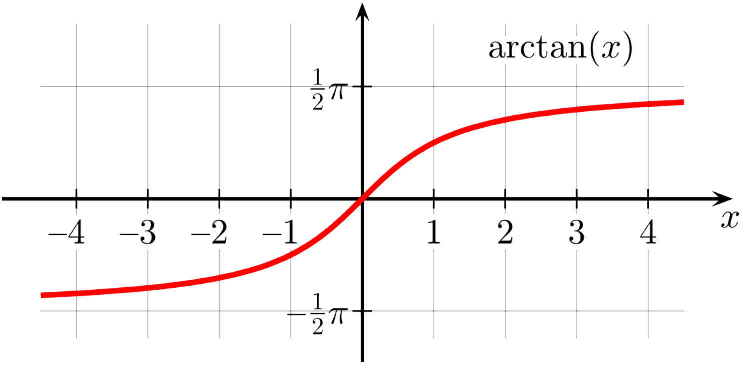
```
> evaldiffeq(diffeq, y(z), 1/5, 1000000);
```

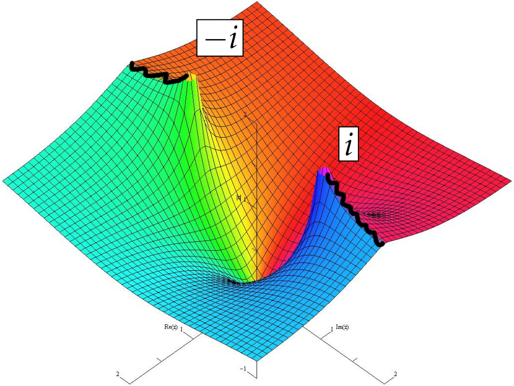
(29 min plus tard...)

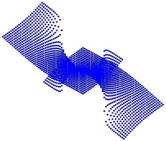
```
> evaldiffeq(diffeq, y(z), 1/5, 1000000);
0.033253281257567506772459381920024394391065961347292863\
    13611785593075654371610784719859620906805710762776061\
    65993844793918297941976188620650536691082179149605904\
    31080482988558239935175505111768194891591740446771304\
    74730251896359727561534310095807343639273056518962333\
    97217595138842309884016425632431029577130431472108646\
    95485154767624024297343851584414126056237771911489680\
. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
    97933258259972366466573219602501650218139747781157348\
    78322628655747195818205282428148240800376913561455564\
    29598794491231828039584256430669932365880956101719727\
    33806130243940574539991121877851105270752378138422728\
    76176859592508040781771637205060431902227437673286901\
    71292574098466950906705927590030494460150099288210121\
    868701569
```

Évaluation numérique

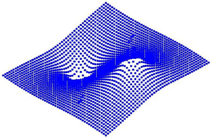
générale
automatique
garantie
asymptotiquement rapide


arctan(x)
½π
−½π
−4
−3
−2
−1
1
2
3
4
x

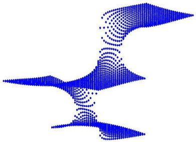




(Re)

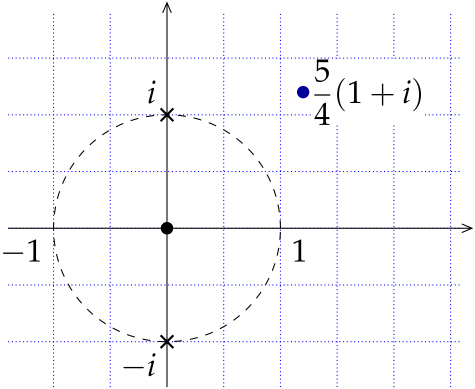


(Im)

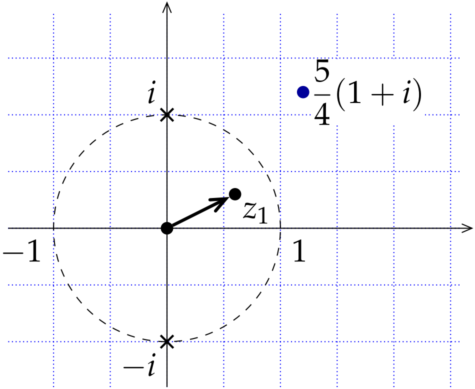


(arg)

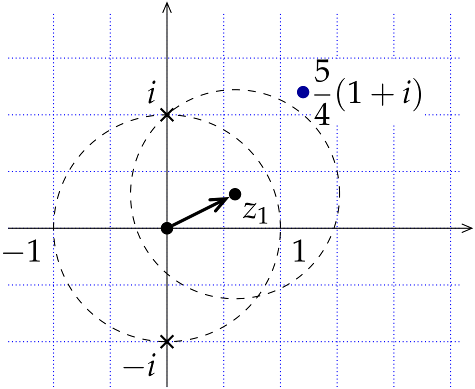
$$(1+z^2)\, y''(z) + 2z\, y'(z) = 0,$$
$$y(0) = 0, \quad y'(0) = 1$$



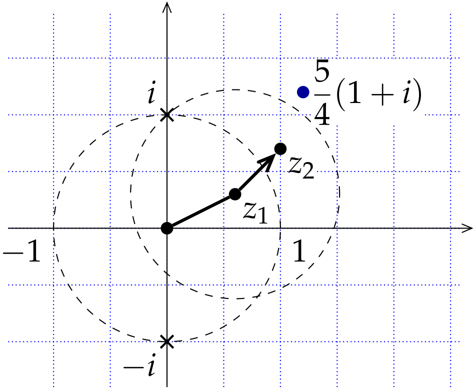
$\frac{5}{4}(1+i)$
i
-1
1
$-i$



$$\begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix} = \begin{bmatrix} 1 & 0,5705170238\cdots + 0,2200896807\ldots i \\ 0 & 0,7288378766\cdots - 0,2065997130\ldots i \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

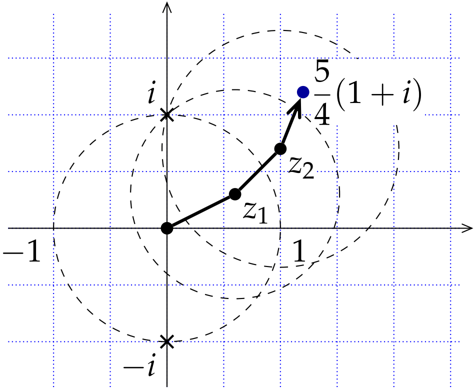


$$\begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix} = \begin{bmatrix} 1 & 0,5705170238\cdots + 0,2200896807\ldots i \\ 0 & 0,7288378766\cdots - 0,2065997130\ldots i \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$



$$\begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix} = \begin{bmatrix} 1 & 0,5705170238\cdots + 0,2200896807\ldots i \\ 0 & 0,7288378766\cdots - 0,2065997130\ldots i \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

$$\begin{bmatrix} y(z_2) \\ y'(z_2) \end{bmatrix} = \begin{bmatrix} 1 & 0,3656231471\cdots + 0,3290407483\ldots i \\ 0 & 0,7515011402\cdots - 0,0792619810\ldots i \end{bmatrix} \begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix}$$



$$\begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix} = \begin{bmatrix} 1 & 0,5705170238\cdots + 0,2200896807\ldots i \\ 0 & 0,7288378766\cdots - 0,2065997130\ldots i \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

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- Schroeppel (1972) – Points particuliers
- Brent (1976) – Fonctions particulières, points quelconques
- Chudnovsky & Chudnovsky (1986-1988) – Méthode générale, esquisse points singuliers réguliers
- van der Hoeven (1999, 2001) – Algorithme complet avec bornes

Théorème (Chudnovsky[2])

Soit y solution d'une équation différentielle linéaire à coefficients polynomiaux. Soit z un point de la surface de Riemann de y.

On peut calculer $y(z)$ à 2^{-n} près en

$$O\left(M\left(n \cdot (\log n)^3\right)\right)$$

opérations binaires.

Théorème (Chudnovsky^2, van der Hoeven)

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opérations binaires.

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Soit y solution d'une équation différentielle linéaire à coefficients polynomiaux. Soit z un point de la surface de Riemann de y.

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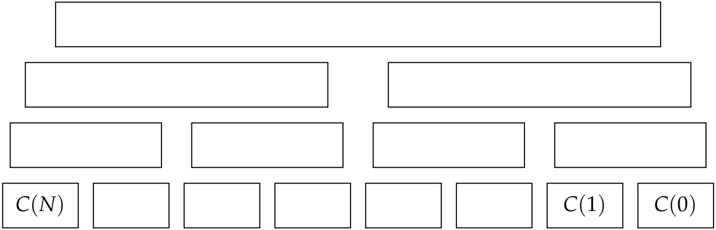
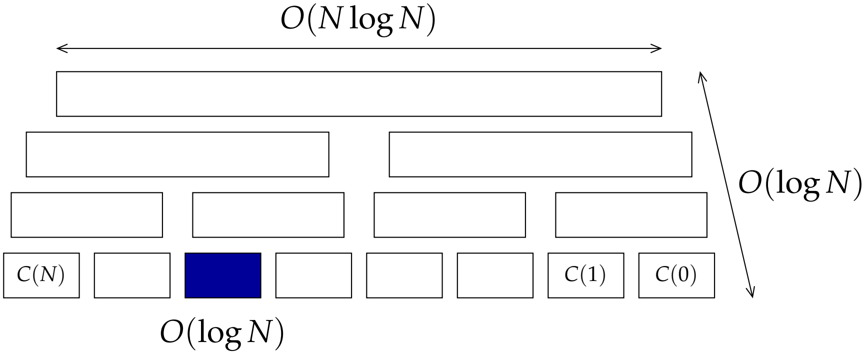

C(N)
C(1)
C(0)



O(N log N)
O(log N)
C(N)
C(1)
C(0)
O(log N)

Paramètres

$\kappa, \alpha, \ldots \in \mathbb{Q}$ ou $\bar{\mathbb{Q}}$ t.q.

$$|y_n| \leqslant n!^{\kappa} \cdot \alpha^n \cdot \varphi(n)$$

Outils : méthode des séries majorantes + analyse asymptotique élémentaire (M. & Salvy 2010)

Bornes symboliques

- Lisibles (presque !)
- Asymptotiquement fines

Bornes numériques

- Approx. sûres des paramètres
- Plus rapide (pas d'algébriques)

Idée: Replacer y par une fonction simple qui la “domine”

$$z^2\, y''(z) + z\, y'(z) + (z^2 - \nu^2)\, y(z)$$

0 point singulier

régulier

pour toute solution y,
$\exists N$ tq $y(z) = O(1/|z|^N)$
quand $z \to 0$

ex. : $y(z) = z^{\sqrt{2}}$, $y(z) = \dfrac{\log z}{z}$

irrégulier

croissance non-poly.
en $1/|z|$ possible
quand $z \to 0$

ex. : $y(z) = e^{1/z}$

Théorème (Fuchs, 1866)

Si 0 est un point singulier régulier d'une équation différentielle linéaire à coefficients analytiques, celle-ci admet pour un certain voisinage D de 0 une base de solutions de la forme

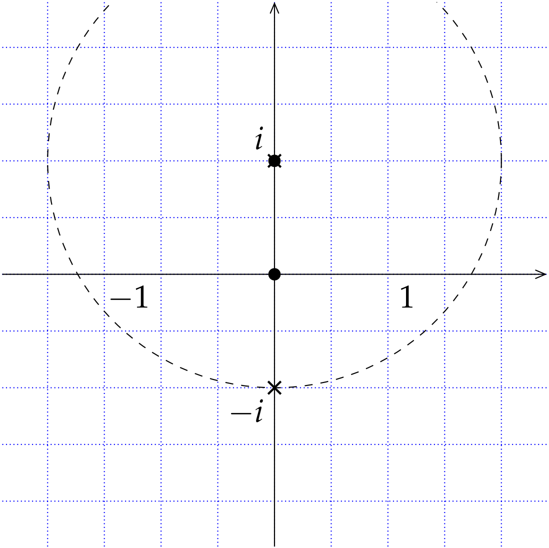
$$z^{\lambda}\big(y_0(z) + y_1(z)\log z + \cdots + y_t(z)\log^t z\big), \quad z \in D \setminus \{0\}$$

où $\lambda \in \mathbb{C}$ et les y_i sont analytiques sur D.

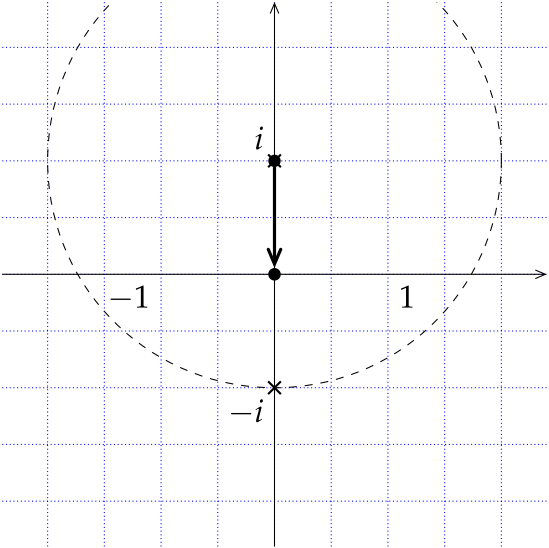
$$L\left(z, z\frac{\mathrm{d}}{\mathrm{d}z}\right) \cdot y(z) = 0$$

$$y(z) = \sum_{n \in \mathbb{Z}} y_n \, z^n \qquad\qquad L(S_n^{-1}, n) \cdot (y_n) = 0$$

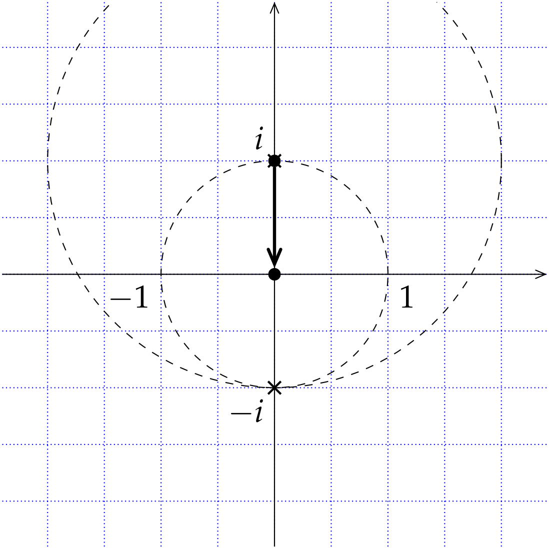
$$y(z) = \sum_{n \in \lambda + \mathbb{Z}} \sum_{k \geqslant 0}^{\text{(finie)}} y_n \, \frac{\log^k z}{k!} \, z^n \qquad\qquad L(S_n^{-1}, n + S_k) \cdot (y_{n,k}) = 0$$



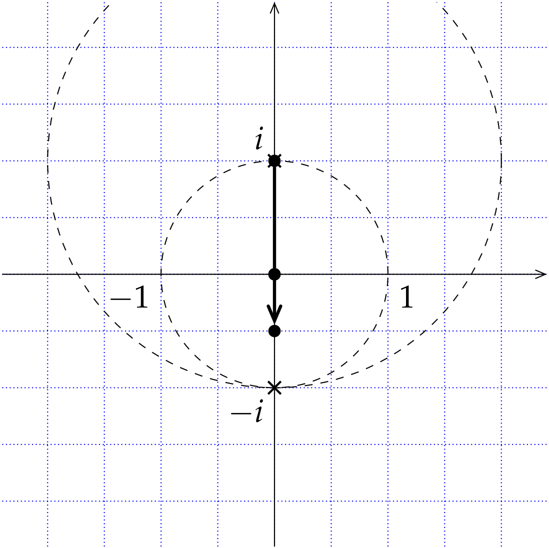
i
-1
1
$-i$



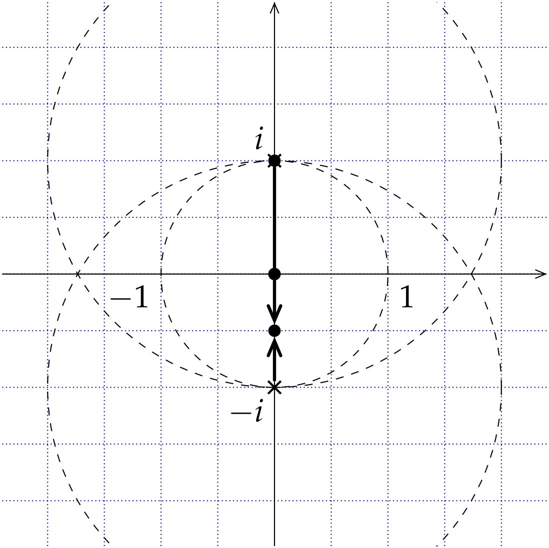
i
−1
1
−i



i
−1
1
−i



i
−1
1
−i



i
−1
1
−i

- Wimp et Zeilberger (1985), Zeilberger (2008) – Méthode de Birkhoff-Trjitzinsky (heuristique)
- Flajolet et Puech (1986) – Prolongement analytique numérique pour l’asymptotique
- Banderier, Chern et Hwang (WIP ?) – Calcul de constantes de connection par resommation

Analyse de singularité (Flajolet, Odlyzko)

asymptotique de $y(z) = \sum_n y_n z^n$ en ses singularités

↓ transfert mécanique

asymptotique de (y_n) à l'infini

This is AsyRec, A Maple package
accompanying Doron Zeilberger's article:

It finds the asymptotics of solutions of (homog.) linear recurrence equations with polynomial coefficients, using the Birkhoff-Trjitzinsky method .

```
> recop := (n+2)^2*N^2-(7*n^2+21*n+16)*N-8*(n+1)^2;
```

$$recop := (n+2)^2 N^2 - \left(7 n^2 + 21 n + 16\right) N - 8 (n+1)^2$$

```
> AsyC(recop, n, N, 5, [2, 10], 1000);
```

0.3675525969478613663 4,

$$\frac{8^n \left(1 - \frac{1}{3 n} + \frac{1}{27 n^2} + \frac{1}{81 n^3} + \frac{1}{243 n^4} + \frac{11}{2187 n^5}\right)}{n}$$

(Wimp & Zeilberger 1985, Zeilberger 2008-2009)

NumGfun en bref

- Prolongement analytique numérique multiprécision
  général – garanti – automatique – rapide
- Bornes fines
  suites – séries majorantes – restes de séries

Code disponible

`http://algo.inria.fr/libraries/` (GNU LGPL)

Perspectives

- Points singuliers réguliers avec garanties
- Asymptotique automatique
- Aller (plus) vite
- Moins de dépendance à Maple

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Merci !

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