The neighbours of Baxter numbers

Veronica Guerrini

University of Siena, DIISM

31 January 2017, LIPN, Paris

Catalan sequence:
1,2,5,14,42,...(A000108)
Dyck paths, $\mathcal{AV}(132)$,...

↓

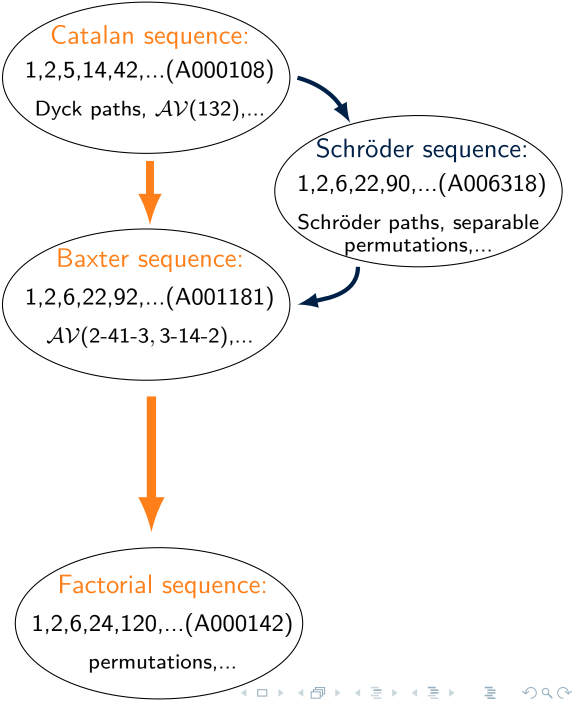
Baxter sequence:
1,2,6,22,92,...(A001181)
$\mathcal{AV}(2\text{-}41\text{-}3, 3\text{-}14\text{-}2)$,...

↓

Factorial sequence:
1,2,6,24,120,...(A000142)
permutations,...

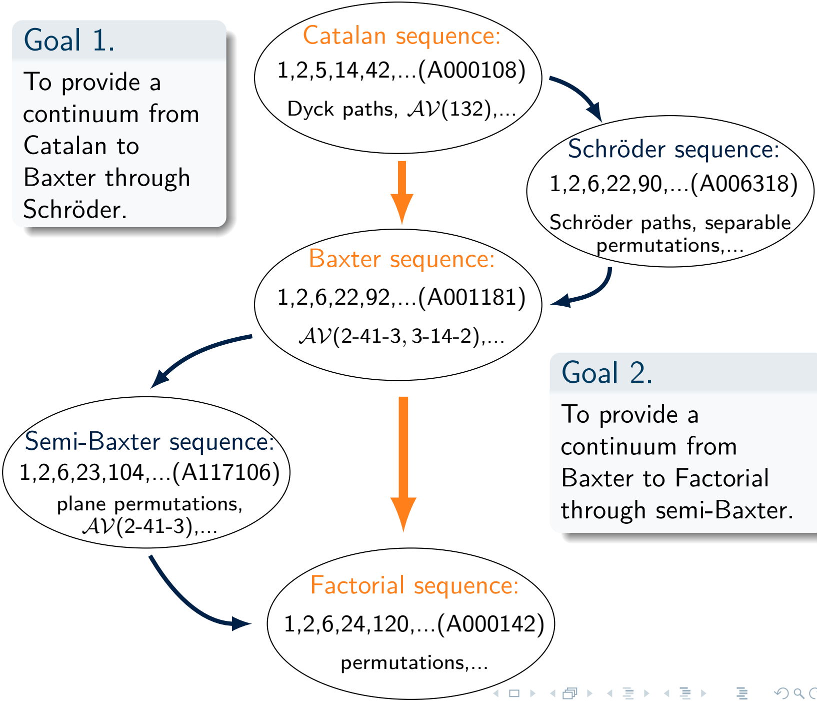
Goal 1.

To provide a continuum from Catalan to Baxter through Schröder.



Goal 1.

To provide a continuum from Catalan to Baxter through Schröder.



Goal 2.

To provide a continuum from Baxter to Factorial through semi-Baxter.

How to establish such continuum?

At the abstract level of generating trees and succession rules so that each inclusion is valid for all the families of objects enumerated by the corresponding sequences.

ECO method. *Enumerating Combinatorial Objects* is a method for the exhaustive generation of a class $\mathcal{C}$ of combinatorial objects equipped with a size $|\cdot| : \mathcal{C} \to \mathbb{N}$.

An ECO-operator is $\vartheta : \mathcal{C}_n \to 2^{\mathcal{C}_{n+1}}$ s.t.
- for any $o, o' \in \mathcal{C}_n$, if $o \neq o'$, then $\vartheta(o) \cap \vartheta(o') = \emptyset$;
- $\bigcup_{o \in \mathcal{C}_n} \vartheta(o) = \mathcal{C}_{n+1}$.

How to establish such continuum?

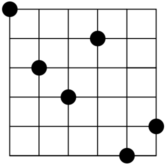
At the abstract level of generating trees and succession rules so that each inclusion is valid for all the families of objects enumerated by the corresponding sequences.

ECO method. *Enumerating Combinatorial Objects* is a method for the exhaustive generation of a class $\mathcal{C}$ of combinatorial objects equipped with a size $|\cdot| : \mathcal{C} \to \mathbb{N}$.

An ECO-operator is $\vartheta : \mathcal{C}_n \to 2^{\mathcal{C}_{n+1}}$ s.t.
- for any $o, o' \in \mathcal{C}_n$, if $o \neq o'$, then $\vartheta(o) \cap \vartheta(o') = \emptyset$;
- $\bigcup_{o \in \mathcal{C}_n} \vartheta(o) = \mathcal{C}_{n+1}$.

A permutation π of length n avoids τ of length $k \leq n$ iff there are no $i_1, \ldots, i_k$ such that $\pi_{i_1} \ldots \pi_{i_k}$ is order isomorphic to τ.

Example. $\pi = 6\,4\,\underline{2}\,1\,\underline{5}\,\underline{3}$ contains $\tau = 1\,3\,2$; $\rho = 6\,4\,3\,5\,1\,2$ avoids τ.



How to establish such continuum?

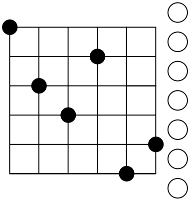
At the abstract level of generating trees and succession rules so that each inclusion is valid for all the families of objects enumerated by the corresponding sequences.

ECO method. *Enumerating Combinatorial Objects* is a method for the exhaustive generation of a class $\mathcal{C}$ of combinatorial objects equipped with a size $|\cdot| : \mathcal{C} \to \mathbb{N}$.

An ECO-operator is $\vartheta : \mathcal{C}_n \to 2^{\mathcal{C}_{n+1}}$ s.t.
- for any $o, o' \in \mathcal{C}_n$, if $o \neq o'$, then $\vartheta(o) \cap \vartheta(o') = \emptyset$;
- $\bigcup_{o \in \mathcal{C}_n} \vartheta(o) = \mathcal{C}_{n+1}$.

A permutation π of length n avoids τ of length $k \leq n$ iff there are no $i_1, \ldots, i_k$ such that $\pi_{i_1} \ldots \pi_{i_k}$ is order isomorphic to τ.

Example. $\pi = 6\,4\,\underline{2}\,1\,\underline{5}\,\underline{3}$ contains $\tau = 1\,3\,2$;
$\rho = 6\,4\,3\,5\,1\,2$ avoids τ.



How to establish such continuum?

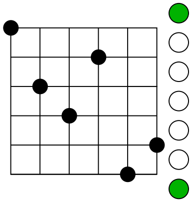
At the abstract level of generating trees and succession rules so that each inclusion is valid for all the families of objects enumerated by the corresponding sequences.

ECO method. *Enumerating Combinatorial Objects* is a method for the exhaustive generation of a class $\mathcal{C}$ of combinatorial objects equipped with a size $|\cdot|:\mathcal{C}\to\mathbb{N}$.

An ECO-operator is $\vartheta:\mathcal{C}_n\to 2^{\mathcal{C}_{n+1}}$ s.t.
- for any $o,o'\in\mathcal{C}_n$, if $o\neq o'$, then $\vartheta(o)\cap\vartheta(o')=\emptyset$;
- $\bigcup_{o\in\mathcal{C}_n}\vartheta(o)=\mathcal{C}_{n+1}$.

A permutation π of length n avoids τ of length $k\leq n$ iff there are no $i_1,\ldots,i_k$ such that $\pi_{i_1}\ldots\pi_{i_k}$ is order isomorphic to τ.

Example. $\pi=6\,4\,\underline{2}\,1\,\underline{5}\,\underline{3}$ contains $\tau=1\,3\,2$; $\rho=6\,4\,3\,5\,1\,2$ avoids τ.



How to establish such continuum?

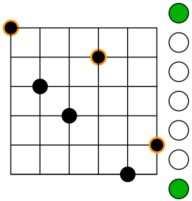
At the abstract level of generating trees and succession rules so that each inclusion is valid for all the families of objects enumerated by the corresponding sequences.

ECO method. *Enumerating Combinatorial Objects* is a method for the exhaustive generation of a class $\mathcal{C}$ of combinatorial objects equipped with a size $|\cdot| : \mathcal{C} \to \mathbb{N}$.

An ECO-operator is $\vartheta : \mathcal{C}_n \to 2^{\mathcal{C}_{n+1}}$ s.t.
- for any $o, o' \in \mathcal{C}_n$, if $o \neq o'$, then $\vartheta(o) \cap \vartheta(o') = \emptyset$;
- $\bigcup_{o \in \mathcal{C}_n} \vartheta(o) = \mathcal{C}_{n+1}$.

A permutation π of length n avoids τ of length $k \leq n$ iff there are no $i_1, \ldots, i_k$ such that $\pi_{i_1} \ldots \pi_{i_k}$ is order isomorphic to τ.

Example. $\pi = 6\,4\,\underline{2}\,1\,\underline{5}\,\underline{3}$ contains $\tau = 1\,3\,2$; $\rho = 6\,4\,3\,5\,1\,2$ avoids τ.



How to establish such continuum?

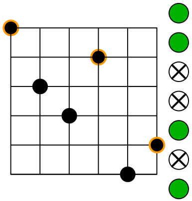
At the abstract level of generating trees and succession rules so that each inclusion is valid for all the families of objects enumerated by the corresponding sequences.

ECO method. *Enumerating Combinatorial Objects* is a method for the exhaustive generation of a class $\mathcal{C}$ of combinatorial objects equipped with a size $|\cdot| : \mathcal{C} \to \mathbb{N}$.

An ECO-operator is $\vartheta : \mathcal{C}_n \to 2^{\mathcal{C}_{n+1}}$ s.t.
- for any $o, o' \in \mathcal{C}_n$, if $o \neq o'$, then $\vartheta(o) \cap \vartheta(o') = \emptyset$;
- $\bigcup_{o \in \mathcal{C}_n} \vartheta(o) = \mathcal{C}_{n+1}$.

A permutation π of length n avoids τ of length $k \leq n$ iff there are no $i_1, \ldots, i_k$ such that $\pi_{i_1} \ldots \pi_{i_k}$ is order isomorphic to τ.

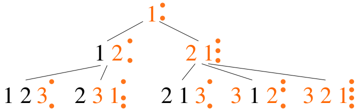
Example. $\pi = 6\,4\,\underline{2}\,1\,\underline{5}\,\underline{3}$ contains $\tau = 1\,3\,2$; $\rho = 6\,4\,3\,5\,1\,2$ avoids τ.



How to establish such continuum?

Definition.

Let ϑ be an ECO-operator for $\mathcal{C}$. A *generating tree* for $\mathcal{C}$ is a infinite rooted tree such that the vertices at level n are the objects of size n and their sons are the objects produced by ϑ.



A compact notation for generating trees is the notion of:

Definition.

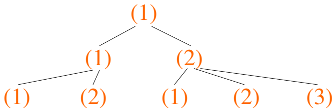
A *succession rule* is system $((r), \mathcal{S})$ consisting of an axiom (r) and a set of productions $\mathcal{S}$

$$\Omega = \begin{cases} (r) \\ (\ell) \rightsquigarrow (e_1), (e_2), \dots, (e_{k(\ell)}) \end{cases}$$

How to establish such continuum?

Definition.

Let ϑ be an ECO-operator for $\mathcal{C}$. A *generating tree* for $\mathcal{C}$ is a infinite rooted tree such that the vertices at level n are the objects of size n and their sons are the objects produced by ϑ.



A compact notation for generating trees is the notion of:

Definition.

A *succession rule* is system $((r), \mathcal{S})$ consisting of an axiom (r) and a set of productions $\mathcal{S}$

$$\Omega = \begin{cases} (r) \\ (\ell) \rightsquigarrow (e_1), (e_2), \ldots, (e_{k(\ell)}) \end{cases}$$

How to establish such continuum?

Definition.

Let ϑ be an ECO-operator for $\mathcal{C}$. A *generating tree* for $\mathcal{C}$ is a infinite rooted tree such that the vertices at level n are the objects of size n and their sons are the objects produced by ϑ.

$$\Omega_{Cat} = \begin{cases} (1) \\ (i) \rightsquigarrow (1), (2), \ldots, (i), (i+1) \end{cases}$$

A compact notation for generating trees is the notion of:

Definition.

A *succession rule* is system $((r), \mathcal{S})$ consisting of an axiom (r) and a set of productions $\mathcal{S}$

$$\Omega = \begin{cases} (r) \\ (\ell) \rightsquigarrow (e_1), (e_2), \ldots, (e_{k(\ell)}) \end{cases}$$

Examples

Catalan succession rule:

$$\Omega_{Cat} = \begin{cases} (1) \\ (i) \rightsquigarrow (1),(2),\dots,(i),(i+1) \end{cases}$$

Schröder succession rule:

$$\Omega_{Sep} = \begin{cases} (2) \\ (j) \rightsquigarrow (2),(3),\dots,(j),(j+1),(j+1) \end{cases}$$

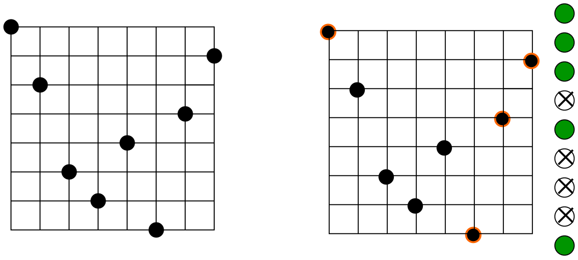
Baxter succession rule:

$$\Omega_{Bax} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1),\dots,(h,k+1) \\ \qquad\qquad (h+1,1),\dots,(h+1,k) \end{cases}$$

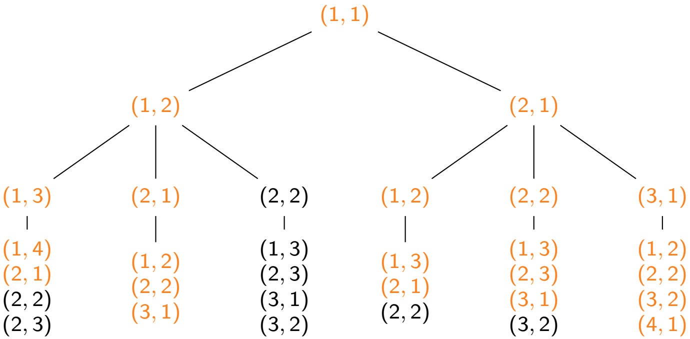
Baxter permutations

Definition. A *Baxter permutation* π is a permutation avoiding the generalized permutation patterns 2-41-3 and 3-14-2.

Each Baxter permutation of length $n+1$ is obtained by adding the rightmost point just above a right-to-left maximum or just below a right-to-left minimum of a Baxter permutation π of length n.



Comparison of the generating trees

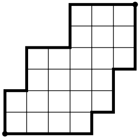


$$\Omega_{Cat} = \begin{cases} (1) \\ (i) \rightsquigarrow (1), (2), \ldots, (i), (i+1) \end{cases}$$

Baxter slicings

Definition.

A *parallelogram polyomino* P is a set of cells in the Cartesian plane whose boundary is given by two non-intersecting lattice paths. The size of P is its semi-perimeter minus 1.



The number of parallelogram polyominoes of size n is the nth Catalan number.

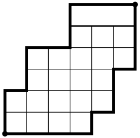
Definition.

A *Baxter slicing* is a parallelogram polyomino P of size n whose interior is divided in n blocks of width or height 1 such that removing the most outer block it remains a Baxter slicing of size $n-1$.

Baxter slicings

Definition.

A *parallelogram polyomino* P is a set of cells in the Cartesian plane whose boundary is given by two non-intersecting lattice paths. The size of P is its semi-perimeter minus 1.



The number of parallelogram polyominoes of size n is the nth Catalan number.

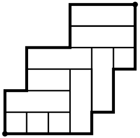
Definition.

A *Baxter slicing* is a parallelogram polyomino P of size n whose interior is divided in n blocks of width or height 1 such that removing the most outer block it remains a Baxter slicing of size $n-1$.

Baxter slicings

Definition.

A *parallelogram polyomino* P is a set of cells in the Cartesian plane whose boundary is given by two non-intersecting lattice paths. The size of P is its semi-perimeter minus 1.



The number of parallelogram polyominoes of size n is the nth Catalan number.

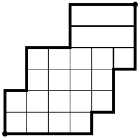
Definition.

A *Baxter slicing* is a parallelogram polyomino P of size n whose interior is divided in n blocks of width or height 1 such that removing the most outer block it remains a Baxter slicing of size $n-1$.

Baxter slicings

Definition.

A *parallelogram polyomino* P is a set of cells in the Cartesian plane whose boundary is given by two non-intersecting lattice paths. The size of P is its semi-perimeter minus 1.



The number of parallelogram polyominoes of size n is the nth Catalan number.

Definition.

A *Baxter slicing* is a parallelogram polyomino P of size n whose interior is divided in n blocks of width or height 1 such that removing the most outer block it remains a Baxter slicing of size $n-1$.

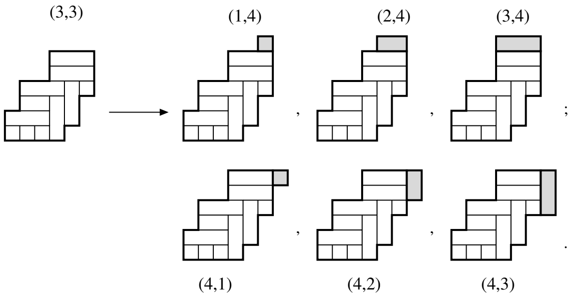
Baxter slicings

Theorem.

Baxter slicings grow according to

$$\Omega_{Bax} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1),\ldots,(h,k+1) \\ \qquad\qquad (h+1,1),\ldots,(h+1,k) \end{cases}$$

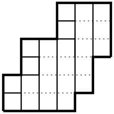
Hence, they are enumerated by Baxter numbers.



Catalan and Schröder slicings

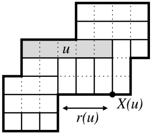
Definition.

A *Catalan slicing* is a Baxter slicing having all horizontal blocks of width 1.



Definition.

A *Schröder slicing* is a Baxter slicing having the width of any horizontal block u limited by $r(u) + 1$.

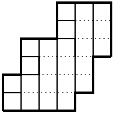


Every Catalan slicing is a Schröder slicing. The new Schröder family of slicings restricts the Baxter family and includes the Catalan family.

Catalan and Schröder slicings

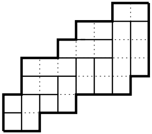
Definition.

A *Catalan slicing* is a Baxter slicing having all horizontal blocks of width 1.



Definition.

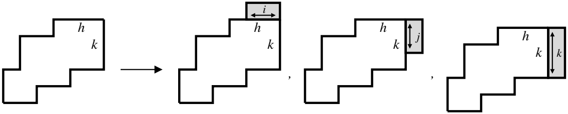
A *Schröder slicing* is a Baxter slicing having the width of any horizontal block u limited by $r(u) + 1$.



Every Catalan slicing is a Schröder slicing. The new Schröder family of slicings restricts the Baxter family and includes the Catalan family.

New Schröder succession rule

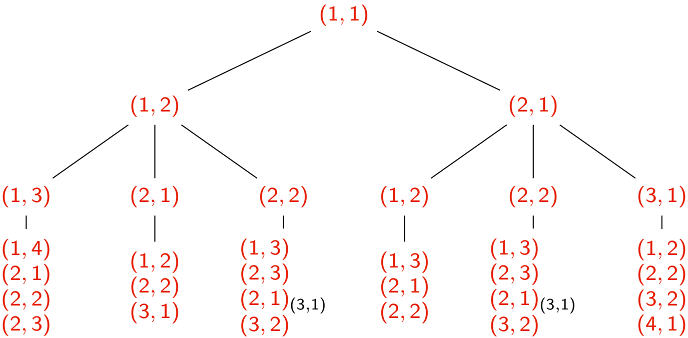
$$\Omega_{Sch} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1),(2,k+1),\ldots,(h,k+1), \\ \qquad\qquad (2,1),(2,2),\ldots,(2,k-1),(h+1,k) \end{cases}$$



Theorem.
The enumeration sequence associated with this new rule Ω_{Sch} is that of Schröder numbers.

- The rules Ω_{Sch} and Ω_{Sep} produce isomorphic generating trees.

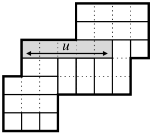
Comparison of the generating trees



$$\Omega_{Sch} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1),(2,k+1),\ldots,(h,k+1), \\ \qquad\qquad (2,1),(2,2),\ldots,(2,k-1),(h+1,k) \end{cases}$$

Row-restricted slicings

Definition. A *m-row-restricted slicing* is a Baxter slicing having the width of any horizontal block u limited by m, where $m \geq 1$.



$$\Omega_{row}^{(m)} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1),(2,k+1),\dots,(h,k+1), \\ \qquad (h+1,1),\dots,(h+1,k), \quad \text{if } h<m, \\ \qquad (m,1),\dots,(m,k), \ \text{if } h=m. \end{cases}$$

System for m-row-restricted slicings

The generating function of m-row-restricted slicings is given by $G_1(1,1) + \ldots + G_m(1,1)$, where each $G_i(u,v) = \sum_\alpha u^i v^{k(\alpha)} x^{n(\alpha)}$ is defined by

$$\begin{cases} G_1(u,v) = xuv + xuv(G_1(1,v) + G_2(1,v) + \ldots + G_m(1,v)) \\ \vdots \\ G_i(u,v) = \frac{xu^i v}{1-v}(G_{i-1}(1,1) - G_{i-1}(1,v)) + xu^i v(G_i(1,v) + \ldots + G_m(1,v)) \\ \vdots \\ G_m(u,v) = \frac{xu^m v}{1-v}(G_m(1,1) - G_m(1,v) + G_{m-1}(1,1) - G_{m-1}(1,v)) + xu^m v G_m(1,v) \end{cases}$$

System for m-row-restricted slicings

The generating function of m-row-restricted slicings is given by $G_1(1,1) + \ldots + G_m(1,1)$, where each $G_i(u,v) = \sum_\alpha u^i v^{k(\alpha)} x^{n(\alpha)}$ is defined by

$$\begin{cases} G_1(u,v) = xuv + xuv(G_1(1,v) + G_2(1,v) + \ldots + G_m(1,v)) \\ \vdots \\ G_i(u,v) = \frac{xu^i v}{1-v}(G_{i-1}(1,1) - G_{i-1}(1,v)) + xu^i v(G_i(1,v) + \ldots + G_m(1,v)) \\ \vdots \\ G_m(u,v) = \frac{xu^m v}{1-v}(G_m(1,1) - G_m(1,v) + G_{m-1}(1,1) - G_{m-1}(1,v)) + xu^m vG_m(1,v) \end{cases}$$

This system can be rewritten

- without u in $H_i(v) \equiv G_i(1,v)$;
- in the form of a matrix equation.

System for m-row-restricted slicings

$$\mathbf{K}_m(v)\mathbf{H}_m(v) = \mathbf{B}_m(v)\mathbf{H}_m(1) + \mathbf{C}_m(v)$$

$$\mathbf{K}_m(v) = \begin{pmatrix} 1-xv & -xv & -xv & -xv & \cdots & -xv \\ \frac{xv}{1-v} & 1-xv & -xv & -xv & \cdots & -xv \\ 0 & \frac{xv}{1-v} & 1-xv & -xv & \cdots & -xv \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{xv}{1-v} & 1-xv & -xv \\ 0 & 0 & 0 & \cdots & \frac{xv}{1-v} & 1-xv+\frac{xv}{1-v} \end{pmatrix}, \mathbf{C}_m(v) = \begin{pmatrix} xv \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\mathbf{H}_m(v) = \begin{pmatrix} H_1(v) \\ \vdots \\ H_m(v) \end{pmatrix} \text{ and } \mathbf{B}_m(v) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{xv}{1-v} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{xv}{1-v} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{xv}{1-v} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{xv}{1-v} & \frac{xv}{1-v} \end{pmatrix}.$$

System for m-row-restricted slicings

Let $\mathbf{K}^*_m(v) = |\mathbf{K}_m(v)|\mathbf{K}_m^{-1}(v)$. Multiplying on the left by $\mathbf{K}^*_m(v)$ gives

$$|\mathbf{K}_m(v)|\mathbf{H}_m(v) = \mathbf{K}^*_m(v)\left[\mathbf{B}_m(v)\mathbf{H}_m(1) + \mathbf{C}_m(v)\right].$$

- The RHS of the mth equation is a linear combination of all the m unknows $H_1(1), \dots, H_m(1)$;
- The equation $|\mathbf{K}_m(v)| = 0$ has $m-2$ solutions in v which are finite at $x = 0$. (N. R. Beaton)

System for *m*-row-restricted slicings

Let $\mathbf{K}^*_m(v) = |\mathbf{K}_m(v)|\mathbf{K}_m^{-1}(v)$. Multiplying on the left by $\mathbf{K}^*_m(v)$ gives

$$|\mathbf{K}_m(v)|\mathbf{H}_m(v) = \mathbf{K}^*_m(v)\left[\mathbf{B}_m(v)\mathbf{H}_m(1) + \mathbf{C}_m(v)\right].$$

- The RHS of the mth equation is a linear combination of all the m unknows $H_1(1), \dots, H_m(1)$;
- The equation $|\mathbf{K}_m(v)| = 0$ has $m-2$ solutions in v which are finite at $x = 0$. (N. R. Beaton)

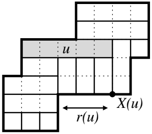
Conjecture.

For all $m \geq 0$, the generating functions of m-row-restricted slicings are algebraic.

- It holds for small value of m ($m \leq 5$).

Skinny slicings

Definition. A *m-skinny slicing* is a Baxter slicing having the width of any horizontal block u limited by $r(u)+m$.



$$\Omega_{sk}^{(m)} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1),(2,k+1),\dots,(h,k+1), \\ \qquad (h+1,1),\dots,(h+1,k-1),(h+1,k), \quad \text{if } h<m, \\ \qquad (m+1,1),\dots,(m+1,k-1),(h+1,k), \quad \text{if } h\geq m. \end{cases}$$

System for m-skinny slicings

$$
\begin{cases}
F_1(u,v) = & xuv + xuv(F_1(1,v) + F_2(1,v) + \ldots + F_m(1,v)) \\
F_2(u,v) = & \frac{xu^2v}{1-v}(F_1(1,1) - F_1(1,v)) + xu^2v(F_2(1,v) + \ldots + F_m(1,v)) \\
\vdots & \\
F_i(u,v) = & \frac{xu^iv}{1-v}(F_{i-1}(1,1) - F_{i-1}(1,v)) + xu^iv(F_i(1,v) + \ldots + F_m(1,v)) \\
\vdots & \\
F_m(u,v) = & \frac{xu^mv}{1-v}(F_{m-1}(1,1) - F_{m-1}(1,v)) + \frac{xu^{m+1}}{1-v}(vF_m(1,1) - F_m(1,v)) + xuF_m(u,v) \\
 & + \frac{xuv}{1-u}(u^{m-1}F_m(1,v) - F_m(u,v)),
\end{cases}
$$

where $F_i(u,v) = \sum_\alpha u^i v^{k(\alpha)} x^{n(\alpha)}$.

- The generating function of m-skinny slicings is given by $F_1(1,1) + \ldots + F_m(1,1)$.

	0	1	2	3	4	5	$\cdots$	∞
m-row-restricted slicings	$\frac{1}{1-x}$	$\frac{1-\sqrt{1-4x}}{2x}$	alg.	alg.	alg.	alg.	$\cdots$	D-fin.
m-skinny slicings	alg.	$\frac{1-x-\sqrt{1-6x+x^2}}{2x}$	alg.	alg.	?	?	$\cdots$	D-fin.

Goal 1.

To provide a continuum from Catalan to Baxter through Schröder.

Catalan sequence:
1,2,5,14,42,...(A000108)
$\mathcal{AV}$(132),Dyck paths,...

Schröder sequence:
1,2,6,22,90,...(A006318)
Schröder paths, separable permutations,...

Baxter sequence:
1,2,6,22,92,...(A001181)
$\mathcal{AV}$(2-41-3, 3-14-2),...

Semi-Baxter sequence:
1,2,6,23,104,...(A117106)
plane permutations, $\mathcal{AV}$(2-41-3),...

Factorial sequence:
1,2,6,24,120,...(A000142)
permutations,...

Goal 2.

To provide a continuum from Baxter to Factorial through semi-Baxter.

Permutations

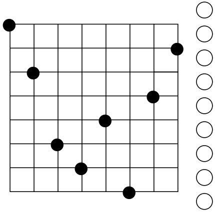
The number of permutations of length n is $n!$.

- For $n \geq 2$, factorial numbers satisfy:

$$f_n = n\, f_{n-1}, \text{ with } f_1 = 1.$$

- Succession rule:

$$\Omega = \begin{cases} (1) \\ (n) \to (n+1)^{n+1} \end{cases}$$



Permutations

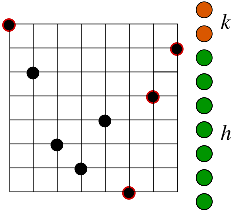
The number of permutations of length n is $n!$.

- For $n \geq 2$, factorial numbers satisfy:

$$f_n = n\, f_{n-1}, \text{ with } f_1 = 1.$$

- Succession rule:

$$\Omega = \begin{cases} (1) \\ (n) \rightarrow (n+1)^{n+1} \end{cases}$$



Permutations

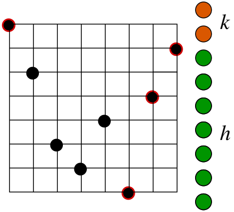
The number of permutations of length n is $n!$.

- For $n \geq 2$, factorial numbers satisfy:

$$f_n = n\, f_{n-1}, \text{ with } f_1 = 1.$$

- Succession rule:

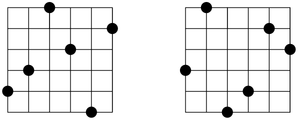
$$\Omega_{Fac} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1, h+k), \ldots, (h, k+1) \\ \qquad\qquad (h+k, 1), \ldots, (h+1, k) \end{cases}$$



Semi-Baxter permutations

Definition.

A *semi-Baxter permutation* π is a permutation avoiding the generalized permutation pattern 2-41-3.



Theorem.

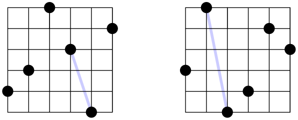
Semi-Baxter permutations grow according to

$$\Omega_{semi} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1),\ldots,(h,k+1) \\ \qquad\qquad (h+k,1),\ldots,(h+1,k) \end{cases}$$

Semi-Baxter permutations

Definition.

A *semi-Baxter permutation* π is a permutation avoiding the generalized permutation pattern 2-41-3.



Theorem.

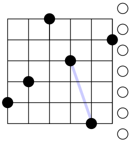
Semi-Baxter permutations grow according to

$$\Omega_{semi} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1),\dots,(h,k+1) \\ \qquad\qquad (h+k,1),\dots,(h+1,k) \end{cases}$$

Semi-Baxter permutations

Definition.

A *semi-Baxter permutation* π is a permutation avoiding the generalized permutation pattern 2-41-3.



Theorem.

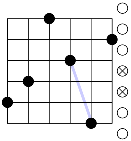
Semi-Baxter permutations grow according to

$$\Omega_{semi} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1), \ldots, (h,k+1) \\ \qquad\qquad (h+k,1), \ldots, (h+1,k) \end{cases}$$

Semi-Baxter permutations

Definition.

A *semi-Baxter permutation* π is a permutation avoiding the generalized permutation pattern 2-41-3.



Theorem.

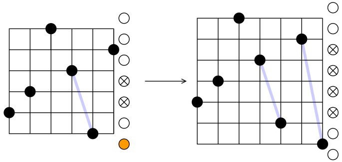
Semi-Baxter permutations grow according to

$$\Omega_{semi} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1),\ldots,(h,k+1) \\ \qquad\qquad (h+k,1),\ldots,(h+1,k) \end{cases}$$

Semi-Baxter permutations

Definition.

A *semi-Baxter permutation* π is a permutation avoiding the generalized permutation pattern 2-41-3.



Theorem.

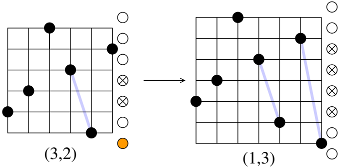
Semi-Baxter permutations grow according to

$$\Omega_{semi} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1),\ldots,(h,k+1) \\ \qquad\qquad (h+k,1),\ldots,(h+1,k) \end{cases}$$

Semi-Baxter permutations

Definition.

A *semi-Baxter permutation* π is a permutation avoiding the generalized permutation pattern 2-41-3.



Theorem.

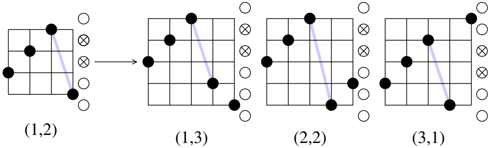
Semi-Baxter permutations grow according to

$$\Omega_{semi} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1),\dots,(h,k+1) \\ \qquad\qquad (h+k,1),\dots,(h+1,k) \end{cases}$$

Semi-Baxter permutations

Definition.

A *semi-Baxter permutation* π is a permutation avoiding the generalized permutation pattern 2-41-3.



Theorem.

Semi-Baxter permutations grow according to

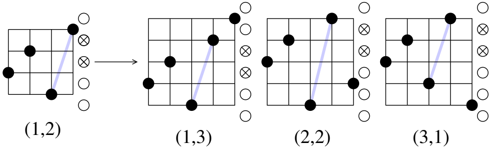
$$\Omega_{semi} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1),\dots,(h,k+1) \\ \qquad\qquad (h+k,1),\dots,(h+1,k) \end{cases}$$

Plane permutations

Definition.

A *plane permutation* π is a permutation avoiding the generalized permutation pattern 2-14-3.

- Enumerating plane permutations: open problem by Bousquet -Mélou and Butler.

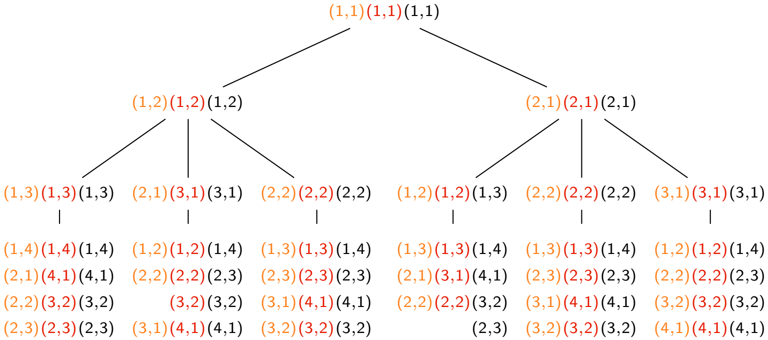


Theorem.

Plane permutations grow according to

$$\Omega_{semi} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1),\dots,(h,k+1) \\ \qquad\qquad (h+k,1),\dots,(h+1,k) \end{cases}$$

Comparison of the generating trees



$$\Omega_{semi} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1),\dots,(h,k+1) \\ \qquad\qquad\ (h+k,1),\dots,(h+1,k) \end{cases}$$

Enumerative properties

From Ω_{semi}, $S(x;y,z) \equiv S(y,z) = \sum_{n,h,k\geq 1} S_{h,k} x^n y^h z^k$ satisfies:

$$S(y,z) = xyz + \frac{xyz}{1-y}\left(S(1,z) - S(y,z)\right) + \frac{xyz}{z-y}\left(S(y,z) - S(y,y)\right)$$

- Set $y = 1 + a$. Write the *kernel form*:

$$K(a,z)S(1{+}a,z) = xz(1{+}a) + \frac{xz(1+a)}{a}S(1,z) - \frac{xz(1+a)}{z-1-a}S(1{+}a,1{+}a)$$

- By exploiting transformations that leave $K(a,z)$ unchanged, we obtain a system of 5 equations in 6 overlapping unknowns.

- Set Z_+ be such that $K(a,Z_+) = 0$. Eliminating overlapping unknowns, yields:

$$S(1+a,1+a) - \frac{(1+a)^2 x}{a^4} S\left(1, 1+\bar{a}\right) - P(a,Z_+) = 0.$$

Enumerative properties

From Ω_{semi}, $S(x;y,z) \equiv S(y,z) = \sum_{n,h,k\geq 1} S_{h,k} x^n y^h z^k$ satisfies:

$$S(y,z) = xyz + \frac{xyz}{1-y}\left(S(1,z) - S(y,z)\right) + \frac{xyz}{z-y}\left(S(y,z) - S(y,y)\right)$$

Theorem.

Let $W(x;a) \equiv W$ be such that

$$W = x\bar{a}(1+a)(W+1+a)(W+a).$$

The series solution $S(y,z)$ satisfies

$$S(1+a,1+a) = \Omega_{\geq}[P(a,W+1+a)], \text{ where}$$

$$P(a,W+1+a) = (1+a)^2 x + \left(\bar{a}^5 + \bar{a}^4 + 2 + 2a\right) x W - (\bar{a}^5 + \bar{a}^4 - \bar{a}^3 + \bar{a}^2 + \bar{a} - 1) x W^2 - \left(\bar{a}^4 - \bar{a}^2\right) x W^3.$$

Enumerative properties

Corollary.

For all $n \geq 1$, the semi-Baxter numbers SB_n satisfy:

$$SB_{n+1} = \frac{1}{n}\sum_{j=0}^{n}\binom{n}{j}\left[2\binom{n+1}{j+2}\binom{n+j+2}{n+2} + \binom{n}{j+1}\binom{n+j+2}{n-3} + 3\binom{n}{j+4}\binom{n+j+4}{n+1}\right.$$
$$\left. +2\frac{nj-j^2-n^2-8j+4n-15}{(n+1)(j+5)}\binom{n}{j+2}\binom{n+j+4}{n} + \frac{2n}{j+3}\binom{n}{j+2}\binom{n+j+2}{n}\right]$$

Conjecture. (PhD thesis by D. Bevan)

For $n \geq 2$,

$$SB_n = \frac{24((5n^3-5n+6)a_{n+1} - (5n^2+15n+18)a_n)}{5(n-1)n^2(n+2)^2(n+3)^2(n+4)},$$

where $a_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}$ is the nth Apéry number.

P-recursiveness

The numbers SB_n are recursively defined by $SB_0 = 0$, $SB_1 = 1$ and for $n \geq 2$,

$$SB_n = \frac{11n^2+11n-6}{(n+4)(n+3)} SB_{n-1} + \frac{(n-3)(n-2)}{(n+4)(n+3)} SB_{n-2}.$$

It holds for Baxter numbers that $B_0 = 0$, $B_1 = 1$ and for $n \geq 2$,

$$B_n = \frac{7n^2+7n-2}{(n+3)(n+2)} B_{n-1} + \frac{8(n-2)(n-1)}{(n+3)(n+2)} B_{n-2}.$$

P-recursiveness

The numbers SB_n are recursively defined by $SB_0 = 0$, $SB_1 = 1$ and for $n \geq 2$,

$$SB_n = \frac{11n^2+11n-6}{(n+4)(n+3)} SB_{n-1} + \frac{(n-3)(n-2)}{(n+4)(n+3)} SB_{n-2}.$$

It holds for Baxter numbers that $B_0 = 0$, $B_1 = 1$ and for $n \geq 2$,

$$B_n = \frac{7n^2+7n-2}{(n+3)(n+2)} B_{n-1} + \frac{8(n-2)(n-1)}{(n+3)(n+2)} B_{n-2}.$$

- $SB_n \underset{n\to\infty}{\sim} A\,\frac{\mu^n}{n^6}\left(1 + O\left(\frac{1}{n}\right)\right)$, where $\mu = \frac{11}{2} + \frac{5}{2}\sqrt{5}$ and $A \approx 94.34$

Another occurrence

Definition.

An inversion sequence is an integer sequence $(e_1, e_2, \dots, e_n)$ satisfying $0 \leq e_i < i$ for all $i \in \{1, 2, \dots, n\}$.

Example. $(0, 1, 2)$ is an inversion sequence, $(0, 2, 1)$ is not.

The inversion sequence $e = (0, 0, \underline{2}, \underline{1}, 4, \underline{1}, 3, 7)$ avoids 210, but contains 100.

Theorem. (Conjectured by Martinez and Savage[1])

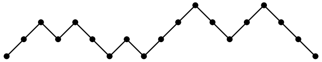
The family of inversion sequences avoiding 210 and 100 is enumerated by semi-Baxter numbers.

[1] *Patterns in Inversion Sequences II: Inversion Sequences Avoiding Triples of Relations*, online available on *Arxiv1609.08106*.

Factorial paths

Definition.

A *factorial path* is a Dyck path P in which every free (not lying in a valley) up steps U has a label in $[1, e+1]$, where e is the number of down steps preceeding U in P.



Theorem.

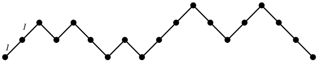
Factorial paths satisfy the recursive relation for factorial numbers

$$f_n = n\, f_{n-1}, \text{ where } f_1 = 1.$$

Factorial paths

Definition.

A *factorial path* is a Dyck path P in which every free (not lying in a valley) up steps U has a label in $[1, e+1]$, where e is the number of down steps preceeding U in P.



Theorem.

Factorial paths satisfy the recursive relation for factorial numbers

$$f_n = n\, f_{n-1}, \text{ where } f_1 = 1.$$

Factorial paths

Definition.

A *factorial path* is a Dyck path P in which every free (not lying in a valley) up steps U has a label in $[1, e+1]$, where e is the number of down steps preceeding U in P.

Theorem.

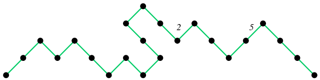
Factorial paths satisfy the recursive relation for factorial numbers

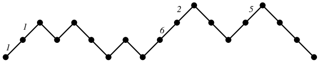
$$f_n = n\, f_{n-1}, \text{ where } f_1 = 1.$$

Factorial paths

Definition.

A *factorial path* is a Dyck path P in which every free (not lying in a valley) up steps U has a label in $[1, e+1]$, where e is the number of down steps preceeding U in P.





Theorem.

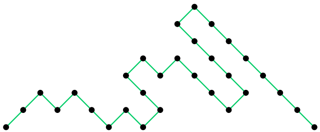
Factorial paths satisfy the recursive relation for factorial numbers

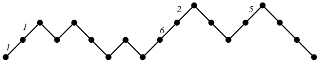
$$f_n = n\, f_{n-1}, \text{ where } f_1 = 1.$$

Factorial paths

Definition.

A *factorial path* is a Dyck path P in which every free (not lying in a valley) up steps U has a label in $[1, e+1]$, where e is the number of down steps preceeding U in P.





Theorem.

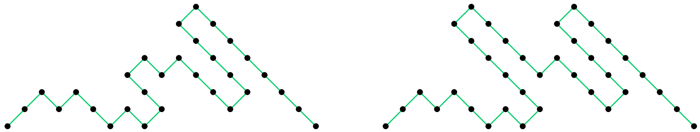
Factorial paths satisfy the recursive relation for factorial numbers

$$f_n = n\, f_{n-1}, \text{ where } f_1 = 1.$$

Factorial paths

Definition.

A *factorial path* is a Dyck path P in which every free (not lying in a valley) up steps U has a label in $[1, e+1]$, where e is the number of down steps preceeding U in P.



Theorem.

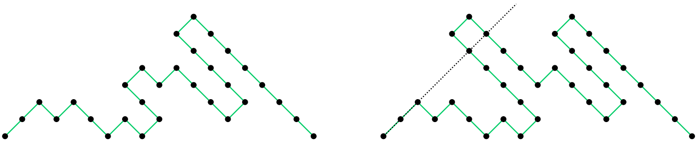
Factorial paths satisfy the recursive relation for factorial numbers

$$f_n = n\, f_{n-1}, \text{ where } f_1 = 1.$$

Factorial paths

Definition.

A *factorial path* is a Dyck path P in which every free (not lying in a valley) up steps U has a label in $[1, e+1]$, where e is the number of down steps preceeding U in P.



Theorem.

Factorial paths satisfy the recursive relation for factorial numbers

$$f_n = n\, f_{n-1}, \text{ where } f_1 = 1.$$

Semi-Baxter paths

Definition.

A *semi-Baxter path* is a factorial path in which, for every pair of consecutive free up step (U', U''), the label of U'' is in $[1, h]$, where $h \geq 1$ is given by summing the label of U' with the number of down steps between U' and U''.

Theorem.

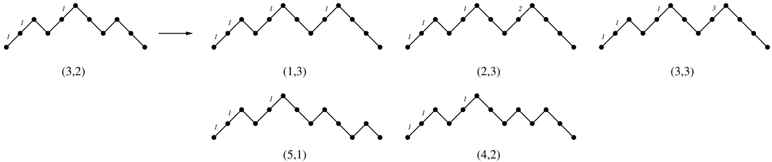
Semi-Baxter paths grow according to

$$\Omega_{semi} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1), \ldots, (h,k+1) \\ \qquad\qquad (h+k,1), \ldots, (h+1,k) \end{cases}$$

Semi-Baxter paths

Definition.

A *semi-Baxter path* is a factorial path in which, for every pair of consecutive free up step (U', U''), the label of U'' is in $[1, h]$, where $h \geq 1$ is given by summing the label of U' with the number of down steps between U' and U''.



Theorem.

Semi-Baxter paths grow according to

$$\Omega_{semi} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1), \ldots, (h,k+1) \\ \qquad\qquad (h+k,1), \ldots, (h+1,k) \end{cases}$$

Baxter paths

Definition.

A *Baxter path* is a factorial path in which, for every pair of consecutive free up step (U', U''), the label of U'' is in $[1, h]$, where $h \geq 1$ is given by summing the label of U' with the number of DU factors between U' and U''.

Theorem.

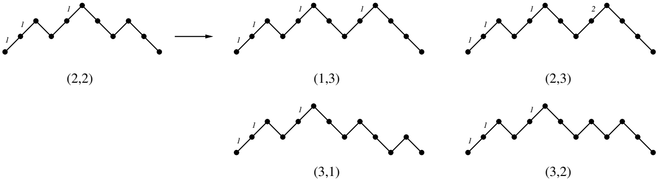
Baxter paths grow according to

$$\Omega_{Bax} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1), \ldots, (h,k+1) \\ \qquad\qquad (h+1,1), \ldots, (h+1,k) \end{cases}$$

Baxter paths

Definition.

A *Baxter path* is a factorial path in which, for every pair of consecutive free up step (U', U''), the label of U'' is in $[1, h]$, where $h \geq 1$ is given by summing the label of U' with the number of DU factors between U' and U''.



Theorem.

Baxter paths grow according to

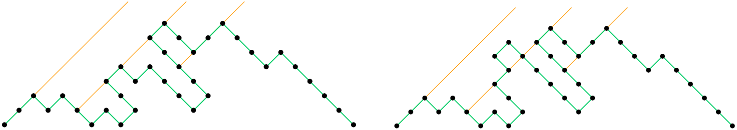
$$\Omega_{Bax} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1),\ldots,(h,k+1) \\ \qquad\qquad (h+1,1),\ldots,(h+1,k) \end{cases}$$

Further work

- Investigate skew representation of factorial paths:

It may suggest some constraints to impose on the family of factorial paths to discover other sequences generalizing Baxter.

- Steady paths



They are enumerated by $1, 2, 6, 23, 105, 549, \ldots$ (A113227) and are in simple bijection with $\mathcal{AV}$(1-34-2).

THANK YOU

for your kind attention