An introduction to free probability
2. Noncrossing partitions and free cumulants

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Definition. A **partition** of a set X is a family π of subsets of X such that $\bigcup \pi = X$ and if $U, V \in \pi$ then either $U = V$ or $U \cap V = \emptyset$. Elements of π are called *blocks* of π.

The class of partitions of the set $\{1, 2, \ldots, n\}$ will be denoted $P(n)$.
The cardinality of $P(n)$ is counted by **Bell numbers** B_n:

$1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \ldots$ (sequence A000110 in OEIS).

Recurrence relation: $B_0 = 1$,

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

The exponential generating function:

$$B(z) = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = \exp(e^z - 1).$$

The number of partitions in $P(n)$ consisting on k blocks: **Stirling numbers of the second kind**: $\left\{ {n \atop k} \right\}$.

Definition. A partition $\pi \in P(n)$ is called **noncrossing** if for every $1 \leq k_1 < k_2 < k_3 < k_4 \leq n$ we have implication:

$$k_1, k_3 \in U \in \pi, \;\; k_2, k_4 \in V \in \pi \implies U = V.$$

$NC(n)$-the class of noncrossing partitions of the set $\{1, 2, \ldots, n\}$.
The number of elements in $NC(n)$: the **Catalan numbers**: $\binom{2n+1}{n}\frac{1}{2n+1}$:

$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \ldots$ (sequence A000108 in OEIS).

They satisfy recurrence: $C_0 = 1$ and

$$C_{n+1} = \sum_{i=0}^{n} C_i \, C_{n-i} \quad \text{for } n \geq 0.$$

The generating function:

$$C(z) = \sum_{n=0}^{\infty} C_n z^n = \frac{2}{1+\sqrt{1-4z}}.$$

The number of $\pi \in NC(n)$ having k blocks: the **Narayana numbers**:

$$\frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

For $\pi \in NC(n)$ define sequence $\Lambda(\pi) = (x_1, x_2, \ldots, x_n)$ as follows:

$$x_k = \begin{cases} |U|-1 & \text{if } k \text{ is the first element of a block } U \in \pi, \\ -1 & \text{otherwise.} \end{cases}$$

Note that the sequence $\Lambda(\pi)$ has the following properties:

1. $x_k \in \{-1, 0, 1, 2, 3, \ldots\}$,
2. $x_1 + x_2 + \ldots + x_k \geq 0$ for $1 \leq k \leq n$,
3. $x_1 + x_2 + \ldots + x_n = 0$.

Proposition. The map Λ is a bijection of $NC(n)$ onto the class of sequences satisfying (1-2-3).

Classical cumulants

Let X be a random variable, μ its distribution, a probability measure on $\mathbb{R}$. We assume that X is bounded. Moments of X, μ:

$$\mathbf{m}_n(X) = \mathbf{m}_n(\mu) := E(X^n) = \int_{\mathbb{R}} t^n \, d\mu(t).$$

Cumulants $\kappa_n(\mu) = \kappa_n$ of X and μ are defined as

$$\log(\mathrm{E}(e^{tX})) = \sum_{n=1}^{\infty} \kappa_n \frac{t^n}{n!}$$

Then for independent random variables $X \sim \mu$, $Y \sim \nu$ we have

$$\kappa_n(X + Y) = \kappa_n(X) + \kappa_n(Y) \tag{1}$$

or

$$\kappa_n(\mu * \nu) = \kappa_n(\mu) + \kappa_n(\nu).$$

Relation between moments and cumulants:

$$\mathbf{m}_n(\mu) = \sum_{\pi \in P(n)} \prod_{V \in \pi} \kappa_{|V|}(\mu). \qquad (2)$$

Examples:

The **normal distribution** $\mathcal{N}(a, \sigma^2)$, with density

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-a)^2}{2\sigma^2}\right)$$

we have $\kappa_1 = a, \kappa_2 = \sigma^2$ and $\kappa_n = 0$ for $n \geq 3$.

The **Poisson distribution**

$$\sum_{k=0}^{\infty} \frac{\lambda^k \exp(-\lambda)}{k!} \delta_k,$$

so that $\Pr(X = k) = \frac{\lambda^k \exp(-\lambda)}{k!}$, we have $\kappa_n = \lambda$ for all $n \geq 1$.

Definition. A (noncommutative) probability space is a pair $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is a complex unital $*$-algebra and ϕ is a state on $\mathcal{A}$, i.e. a linear map $\mathcal{A} \to \mathbb{C}$ such that $\phi(\mathbf{1}) = 1$ and $\phi(a^*a) \geq 0$ for all $a \in \mathcal{A}$.

Definition: A family $\{\mathcal{A}_i\}_{i \in I}$ of unital (i.e. $\mathbf{1} \in \mathcal{A}_i$) subalgebras is called *free* if

$$\phi(a_1 a_2 \dots a_m) = 0$$

whenever $m \geq 1$, $a_1 \in \mathcal{A}_{i_1}, \dots, a_m \in \mathcal{A}_{i_m}$, $i_1, \dots, i_m \in I$, $i_1 \neq i_2 \neq \dots \neq i_m$ and $\phi(a_1) = \dots = \phi(a_m) = 0$.

Main example: Unital free product. Let $(\mathcal{A}_i, \phi_i)$, $i \in I$, noncommutative probability spaces. Put $\mathcal{A}_i^0 := \mathrm{Ker}\phi_i$. Then the unital free product $\mathcal{A} = *_{i \in I} \mathcal{A}_i$ can be represented as

$$\mathcal{A} := \mathbb{C}\mathbf{1} \oplus \bigoplus_{\substack{m \geq 1 \\ i_1, \dots, i_m \in I \\ i_1 \neq i_2 \neq \dots \neq i_m}} \mathcal{A}_{i_1}^0 \otimes \mathcal{A}_{i_2}^0 \otimes \dots \otimes \mathcal{A}_{i_m}^0 = \mathbb{C}\mathbf{1} \oplus \mathcal{A}^0. \tag{3}$$

with the state defined by $\phi(\mathbf{1}) = 1$ and $\phi(a) = 0$ for $a \in \mathcal{A}_0$. Then $\{\mathcal{A}_i\}_{i \in I}$ is a free family in $(\mathcal{A}, \phi)$

Suppose $a_k \in \mathcal{A}_1$, $b_k \in \mathcal{A}_2$. We write $a_k = \alpha_k \mathbf{1} + a_k^0$, where $\alpha_k = \phi(a_k)$, $\phi(a_k^0) = 0$, $b_k = \beta_k \mathbf{1} + b_k^0$ where $\beta_k = \phi(b_k)$, $\phi(b_k^0) = 0$, Then

$$\phi(a_1 b_1) = \phi\left((\alpha_1 \mathbf{1} + a_1^0)(\beta_k \mathbf{1} + b_k^0)\right)$$

$$= \alpha_1\beta_1 + \alpha_1\phi(b_1^0) + \beta_1\phi(a_1^0) + \phi\left(a_1^0 b_k^0\right) = \alpha_1\beta_1 = \phi(a_1)\phi(b_1).$$

In a similar way:

$$\phi(a_1 b_1 a_2) = \phi(a_1 a_2)\phi(b_1)$$

and

$$\phi(a_1 b_1 a_2 b_2) = \phi(a_1 a_2)\phi(b_1)\phi(b_2) + \phi(a_1)\phi(a_2)\phi(b_1 b_2)$$
$$-\phi(a_1)\phi(a_2)\phi(b_1)\phi(b_2).$$

Proposition. Assume, that $a \in \mathcal{A}_1$, $b \in \mathcal{A}_2$, and $\mathcal{A}_1, \mathcal{A}_2$ are free. Then the moments $\phi((a+b)^n)$ of $a+b$ depend only on the moments $\phi(a^n)$ of a and the moments $\phi(b^n)$ of b.

Distribution of a self-adjoint element $a = a^* \in \mathcal{A}$

is the probability measure μ on $\mathbb{R}$ satisfying:

$$\phi(a^n) = \int_{\mathbb{R}} t^n \, d\mu(t), \qquad n = 1, 2, \ldots,$$

so that $\phi(a^n)$ are *moments* of μ.

If a, b are free and the distribution of a, b is μ, ν respectively then the distribution of $a+b$ will be denoted $\mu \boxplus \nu$- the *additive free convolution.*

We want to compute the moments $\phi((a+b)^n)$ from $\phi(a^n)$ and $\phi(b^n)$.

For $a \in \mathcal{A}$ we define its free cumulants $r_n(a)$ by the relation:

$$\phi(a^n) = \sum_{\pi \in NC(n)} \prod_{V \in \pi} r_{|V|}(a). \tag{4}$$

In particular

$$\phi(a) = r_1(a),$$

$$\phi(a^2) = r_1(a)^2 + r_2(a),$$

$$\phi(a^3) = r_1(a)^3 + 3r_1(a)r_2(a) + r_3(a),$$

The moment sequence $\phi(a^n)$ and the cumulant sequence $r_n(a)$ determine each other. We are going to prove

Theorem. If $a, b \in \mathcal{A}$ are free (i.e. belong to free subalgebras) then

$$r_n(a+b) = r_n(a) + r_n(b). \tag{5}$$

Examples.

1. Catalan numbers: if

$$\phi(a^n) = \binom{2n+1}{n} \frac{1}{2n+1} \quad \text{then} \quad r_n(a) = 1 \quad \text{for all } n \geq 1. \tag{6}$$

2. More generally: Fuss/Raney numbers: if

$$\phi(a^n) = \binom{pn+r}{n} \frac{r}{pn+r} \quad \text{then} \tag{7}$$

$$r_n(a) = \binom{(p-r)n+r}{n} \frac{r}{(p-r)n+r}. \tag{8}$$

W. Młotkowski, Fuss-Catalan numbers in noncommutative probability, Documenta Mathematica 15 (2010).

3. Aerated Catalan numbers: if

$$\phi(a^n) = \begin{cases} \binom{2k+1}{k}\frac{1}{2k+1} & \text{if } n = 2k, \\ 0 & \text{if } n \text{ odd}, \end{cases} \quad \text{then } r_n(a) = \begin{cases} 1 & \text{if } n = 2, \\ 0 & \text{if } n \neq 2. \end{cases} \tag{9}$$

4. More generally, aerated Fuss/Raney numbers, if

$$\phi(a^n) = \begin{cases} \binom{pk+r}{k}\frac{r}{pk+r} & \text{if } n = 2k, \\ 0 & \text{if } n \text{ is odd}, \end{cases} \tag{10}$$

then

$$r_n(a) = \begin{cases} \binom{(p-2r)k+r}{k}\frac{r}{(p-2r)k+r} & \text{if } n = 2k, \\ 0 & \text{if } n \text{ is odd}. \end{cases} \tag{11}$$

Free Gaussian law $\gamma_{a,r}$:

$$\frac{1}{2\pi r^2}\sqrt{4r^2-(x-a)^2}\chi_{[a-2r,a+2r]}(x)\,dx, \tag{12}$$

then $r_1(\gamma_{a,r})=a$, $r_2(\gamma_{a,r})=r^2$ and $r_n(\gamma_{a,r})=0$ for $r\geq 3$.
Free Poisson law ϖ_t:

$$\max\{1-t,0\}\delta_0+\frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\chi_{[(1-\sqrt{t})^2,(1+\sqrt{t})^2]}(x)\,dx \tag{13}$$

then $r_n(\varpi_t)=t$ for all $n\geq 1$.

Cuntz algebra

Let $\mathcal{H}$ be a Hilbert space and define the *full Fock space* of $\mathcal{H}$:

$$\mathcal{F}(\mathcal{H}) := \mathbb{C}\Omega \oplus \bigoplus_{m=1}^{\infty} \mathcal{H}^{\otimes m}.$$

Fix an orthonormal basis e_i, $i \in I$. Then the vectors

$$e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_m},$$

$m \geq 0$, $i_1, i_2, \ldots, i_m \in I$, form an orthonormal basis of $\mathcal{F}(\mathcal{H})$. The vector corresponding to the empty word ($m = 0$) will be denoted by Ω.
For $i \in I$ define operator ℓ_i:

$$\ell_i e_{i_1} \otimes \ldots \otimes e_{i_m} = e_i \otimes e_{i_1} \otimes \ldots \otimes e_{i_m}$$

in particular $\ell_i \Omega = e_i$, and its adjoint:

$$\ell_i^* e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_m} = \begin{cases} e_{i_2} \otimes \ldots \otimes e_{i_m} & \text{if } m \geq 1 \text{ and } i_1 = i \\ 0 & \text{otherwise.} \end{cases}$$

Note the relation

$$\ell_i^* \ell_j = \begin{cases} \mathbf{1} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Define:
$\mathcal{A}$- the unital algebra generated by all ℓ_i, ℓ_i^*, $i \in I$,
$\mathcal{A}_i$- the unital subalgebra generated by ℓ_i, ℓ_i^*.
For $a \in \mathcal{A}$ we put $\phi(a) := \langle a\Omega, \Omega \rangle$.
By (14), $\mathcal{A}_i$ is the linear span of

$$\{\ell_i^m (\ell_i^*)^n : m, n \geq 0\},$$

while $\mathcal{A}$ is the linear span of

$$\{\ell_{i_1} \ell_{i_2} \ldots \ell_{i_m} \ell_{j_1}^* \ell_{j_2}^* \ldots \ell_{j_n}^* : m, n \geq 0\}.$$

Proposition.

1. If $m + n > 0$ then

$$\phi\left(\ell_{i_1} \ell_{i_2} \ldots \ell_{i_m} \ell_{j_1}^* \ell_{j_2}^* \ldots \ell_{j_n}^*\right) = 0$$

2. The family $\{\mathcal{A}_i\}_{i \in I}$ is free in $(\mathcal{A}, \phi)$.

Lemma. Suppose that $x_1, x_2, \ldots, x_n \in \{-1, 0, 2, 3, \ldots\}$ and denote $\ell_i^{-1} := \ell_i^*$. Then

$$\phi\left(\ell_1^{x_n} \ldots \ell_1^{x_2} \ell_1^{x_1}\right) = 1$$

iff the sequence $(x_1, x_2, \ldots, x_n)$ satisfies conditions (1-2-3) from page 4 and

$$\phi\left(\ell_1^{x_n} \ldots \ell_1^{x_2} \ell_1^{x_1}\right) = 0$$

otherwise.

Proposition. Let

$$T_1 = \ell_1^* + \sum_{k=1}^{\infty} \alpha_k \ell_1^{k-1}$$

for some $\alpha_n \in \mathbb{C}$. Then α_k are free cumulants of T_1:

$$\phi(T_1^n) = \sum_{\pi \in NC(n)} \prod_{V \in \pi} \alpha_{|V|}.$$

More generally:

Lemma. Suppose that $x_1, x_2, \ldots, x_n \in \{-1, 0, 2, 3, \ldots\}$ and $i_1, i_2, \ldots, i_n \in I$. Then

$$\phi\left(\ell_{i_n}^{x_n} \ldots \ell_{i_2}^{x_2} \ell_{i_1}^{x_1}\right) = 1$$

iff the sequence $(x_1, x_2, \ldots, x_n)$ satisfies (1-2-3) from page 4, i.e. $(x_1, x_2, \ldots, x_n) = \Lambda(\pi)$ for some $\pi \in NC(n)$, and, moreover, if $p, q \in V \in \pi$ then $i_p = i_q$. Otherwise we have

$$\phi\left(\ell_{i_n}^{x_n} \ldots \ell_{i_2}^{x_2} \ell_{i_1}^{x_1}\right) = 0.$$

Theorem. (Voiculescu 1986) Let

$$T_1 = \ell_1^* + \sum_{k=1}^{\infty} \alpha_k \ell_1^{k-1},$$

$$T_2 = \ell_2^* + \sum_{k=1}^{\infty} \alpha_k \ell_2^{k-1},$$

and

$$T = \ell_1^* + \sum_{k=1}^{\infty} (\alpha_k + \beta_k)\, \ell_1^{k-1}.$$

Then for every $n \geq 0$

$$\phi(T^n) = \phi((T_1 + T_2)^n).$$

Proof. From the lemma we have

$$\phi\left((T_1+T_2)^n\right) = \sum_{\pi\in NC(n)} \sum_{\substack{V\in\pi \\ \gamma_V\in\{\alpha_{|V|},\beta_{|V|}\}}} \prod_{V\in\pi} \gamma_V$$

$$= \sum_{\pi\in NC(n)} \prod_{V\in\pi} \left(\alpha_{|V|}+\beta_{|V|}\right) = \phi(T^n).$$

Theorem. Suppose that $(\mathcal{A},\phi)$ is a probability space, $\mathcal{A}_1,\mathcal{A}_2$ are free subalgebras, $a\in\mathcal{A}_1$, $b\in\mathcal{A}_2$. Then for every $n\geq 1$ we have

$$r_n(a+b) = r_n(a) + r_n(b).$$

Proof. We can assume that $a = T_1$, $b = T_2$, the elements in the Cuntz algebra, with $\alpha_k = r_k(a)$, $\beta_k = r_k(b)$.