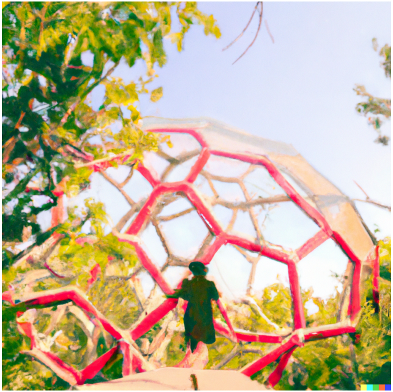
Flip Graphs and Matroids

Jean Cardinal, ULB



Outline

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Flip Graphs

Graph on a set of combinatorial objects, such that two adjacent objects differ by a single, reversible, exchange operation between elements composing the structure.

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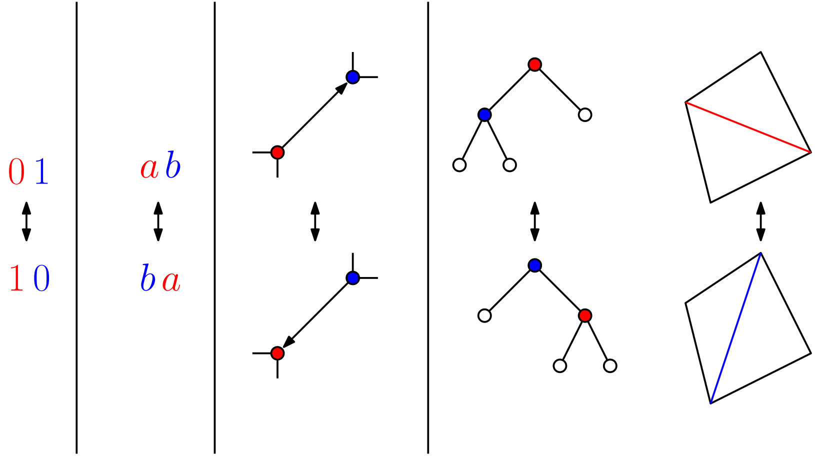
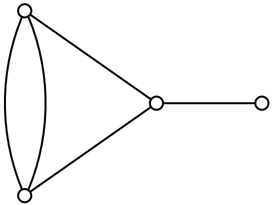
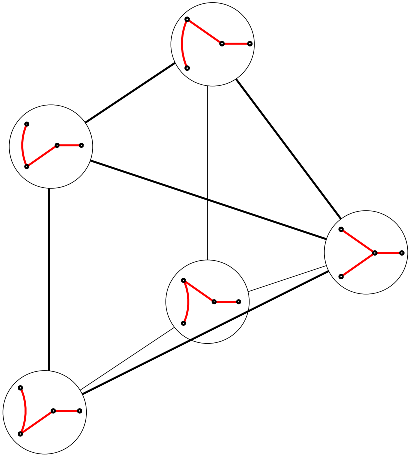
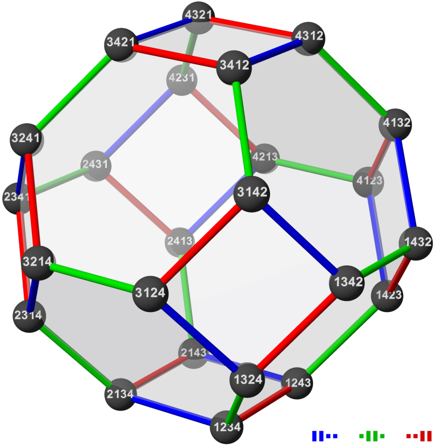


Spanning trees



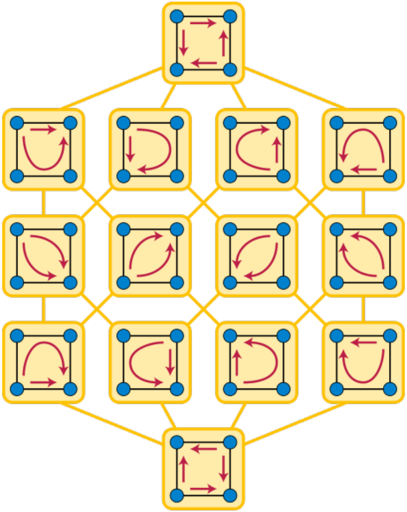


Permutations



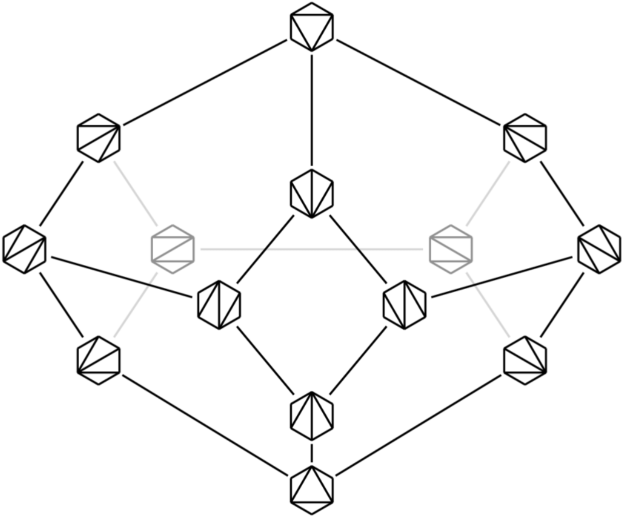
(T. Piesk, Creative Commons)

Acyclic orientations



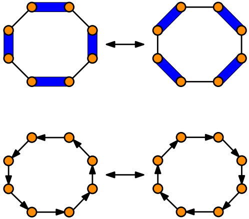
(D. Eppstein, Wikimedia commons)

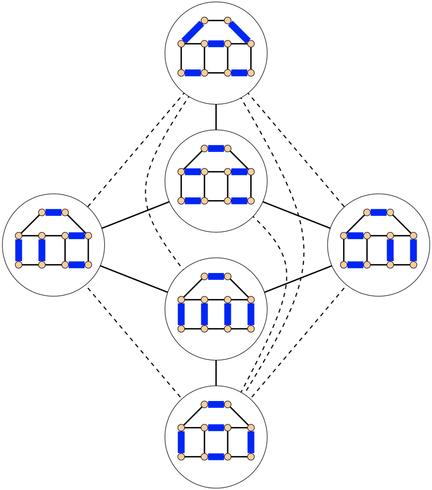
Triangulations



(Fomin, Zelevinsky)

Perfect matchings





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Polytopal flip graphs

Many flip graphs are skeletons of polytopes:

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Polymatroidal flip graphs

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Spanning tree polytopes	Matroids

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Spanning tree polytopes	Matroids
Permutohedra	Polymatroids
Associahedra	
Graphical zonotopes	

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Spanning tree polytopes	Matroids
Permutohedra Associahedra Graphical zonotopes	Polymatroids
Perfect matching polytope	Matroid intersections

Flip distances

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Given two combinatorial objects of the same size, can we efficiently compute the flip distance between them?

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Geodesics vs. Combinatorial reconfiguration formulation

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Geodesics vs. Combinatorial reconfiguration formulation
https://reconf.wikidot.com/

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Geodesics vs. Combinatorial reconfiguration formulation
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What is the complexity of computing the rotation distance between two binary trees?

Diameter

What is the diameter of the polytope?

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Santos 2012

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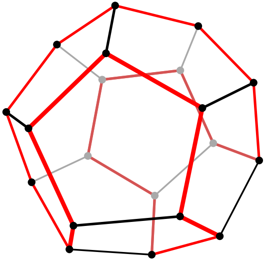
Santos 2012

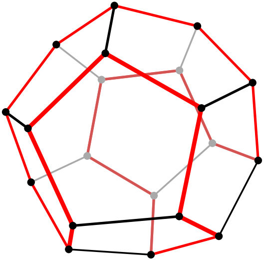
Polynomial Hirsch conjecture: The diameter of dimension n polytopes with f faces is at most some polynomial in n and f.

Hamiltonicity



Is the skeleton of the polytope Hamiltonian? Hamilton 1856

Hamiltonicity



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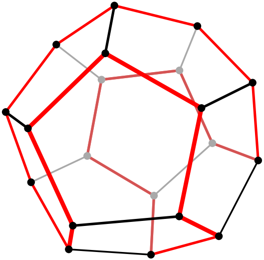

Is the skeleton of the polytope Hamiltonian? Hamilton 1856

Is there a Gray code for the combinatorial objects?

Again, two versions:

Combinatorial Does there always exist a Hamiltonian cycle?

Computational Can we compute it efficiently, say with bounded delay?

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Matroids

A matroid can also be defined as $M = (E, \mathcal{B})$, where $\mathcal{B}$ is a set of bases, satisfying the basis exchange axiom:

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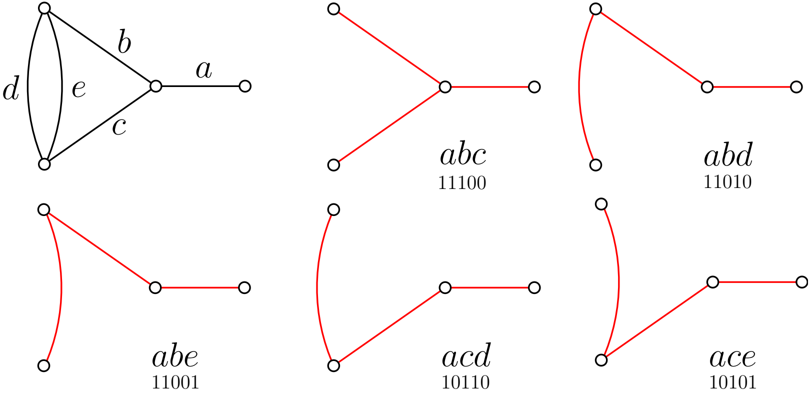
A matroid can also be defined as $M = (E, \mathcal{B})$, where $\mathcal{B}$ is a set of bases, satisfying the basis exchange axiom:

If A and B are two distinct bases, then for any element $a \in A \setminus B$, there exists an element $b \in B \setminus A$ such that $A \setminus \{a\} \cup \{b\} \in \mathcal{B}$.
Whitney 1935, Nakasawa 1935-38, McLane 1936, Rado 1940s, Tutte 1950s

Bases

The bases of M are its maximal independent sets.



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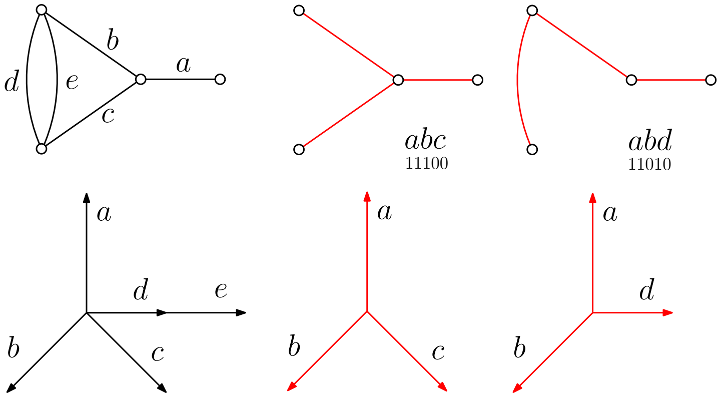

Matroid polytopes

The polytope of M is the convex hull of the indicator vectors of the bases of M:

$$P_M = \mathrm{conv}\{e_B : B \in \mathcal{B}\}$$

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Theorem

A 0/1 polytope P is the polytope of a matroid if and only if:

- *every edge of P is a translate of $e_i - e_j$, for some i, j,*
- *there exists a submodular rank function $r : 2^E \mapsto \mathbb{N}$ s.t.:*

$$P = P_r := \{x \in \mathbb{R}^E : \sum_{i \in U} x_i \leq r(U)\ \forall U \subset E \wedge \sum_{i \in E} x_i = r(E)\}.$$

Gel'fand, Goresky, MacPherson, Serganova 1987

Distances and Hamiltonicity

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Edmonds 1965

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- The diameter $\delta(P_M)$ is therefore (half) the maximum symmetric difference between two bases.
- Can be computed in polynomial time using the Matroid Union theorem and Edmonds' Matroid partition algorithm.

  Edmonds 1965
- It is known that any 0/1 polytope is Hamilton-connected

  Naddef-Pulleyblank 1984
- Efficient Gray codes using linear optimization as a black box

  Merino-Mütze 2023

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Polymatroids

Theorem

A polytope P is a polymatroid if and only if:

- *every edge of P is parallel to $e_i - e_j$, for some i, j,*
- *there exists a submodular function $f : 2^E \mapsto \mathbb{R}$ s.t.:*

$$P = P_f := \{x \in \mathbb{R}^E : \sum_{i \in U} x_i \leq f(U) \ \forall U \subset E \wedge \sum_{i \in E} x_i = f(E)\}.$$

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- Greedy optimization algorithm
- Aka generalized permutahedra, or submodular polyhedra

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Acyclic orientations and graphical zonotopes

Given a simple, connected graph $G = ([n], E)$, let $f : 2^{[n]} \to \mathbb{N}$,

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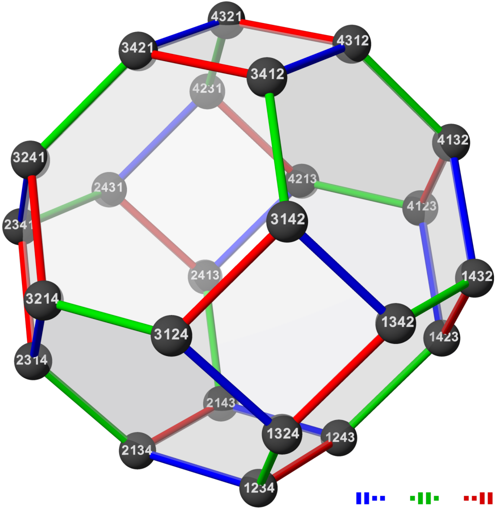
- The skeleton of P_f is the flip graph on acyclic orientations of G.
- Distances and diameter: Easy.
- Hamiltonicity: not always. When exactly is an open problem.

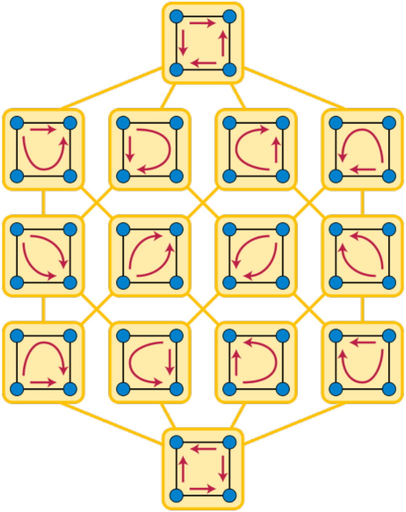
Example: Permutahedron

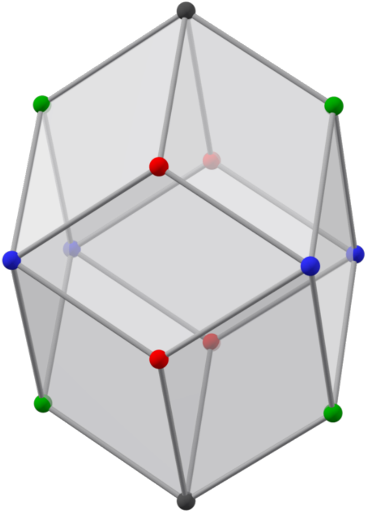
When G is the complete graph, we obtain all permutations.



Example: Bilinski dodecahedron

When G is a 4-cycle.





Hypergraphic polytopes

Given a hypergraph $H = (V, \mathcal{E})$, where $\mathcal{E} \subseteq 2^V \setminus \{\emptyset\}$, let $f_H : 2^V \to \mathbb{N}$ be defined as

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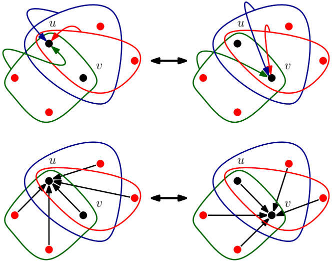
- Minkowski sum of standard simplices
- Vertices $\leftrightarrow$ Acyclic orientations of hypergraphs, edges $\leftrightarrow$ flips

  Benedetti, Bergeron, Machacek 2018, C., Hoang, Merino, Mička, Mütze 2023

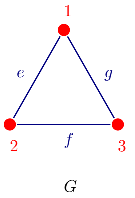


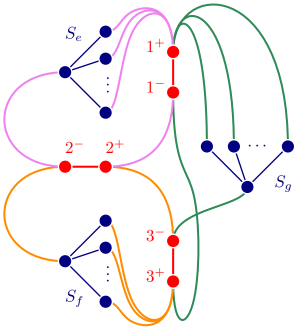
Flip distances in hypergraphic polytopes

Theorem

Computing the flip distance between two acyclic orientations of hypergraph H is APX-hard even when the input hypergraph $H = (V, \mathcal{E})$ is known to have bounded maximum degree and be such that $|e| \leq 3$ for every $e \in \mathcal{E}$.

C., Steiner 2023





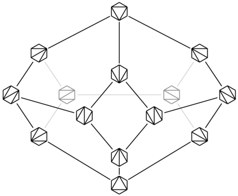
Associahedra are hypergraphic

Let $H = ([n], \mathcal{E})$ be the set of intervals in $[n]$:

$$\mathcal{E} := \{\{i, i+1, \ldots, j\} : 1 \leq i < j \leq n\}.$$

Then the hypergraphic polytope of H is Loday's associahedron.

Loday 2004



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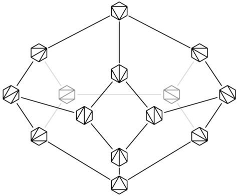

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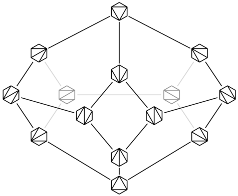

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Sleator, Tarjan, Thurston 1988, Pournin 2014

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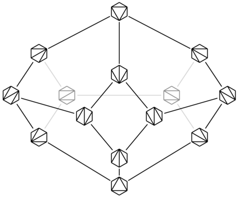

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- Hamiltonicity: Yes.

  Lucas 1987, Lucas, Roelants van Baronaigien, Ruskey 1993

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Graph associahedra and elimination trees

When $H = (V, \mathcal{E})$ is the graphical building set of a graph $G = (V, E)$:

$$\mathcal{E} := \{S \subseteq V : G[S] \text{ is connected}\},$$

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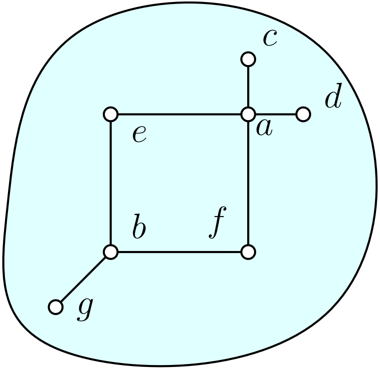
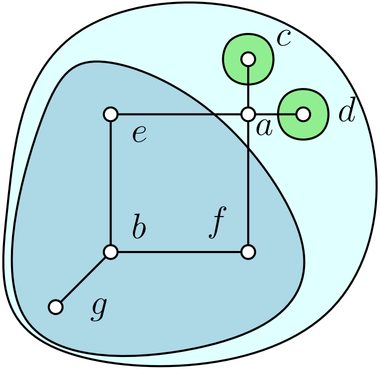
- Vertices of P_H are one-to-one with elimination trees of G,
- and the skeleton of P_H is the rotation graph on elimination trees of G.

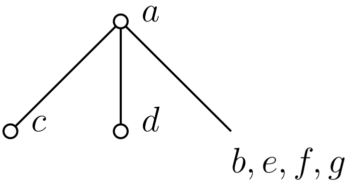
Elimination trees



a, b, c, d, e, f, g

Elimination trees





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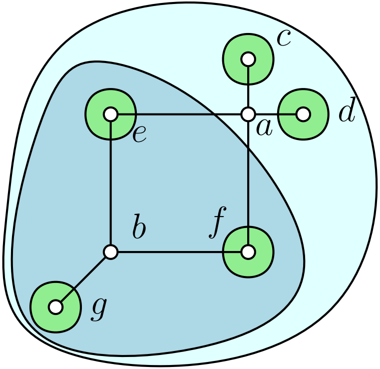
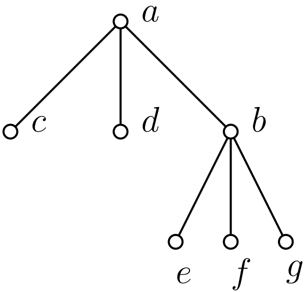
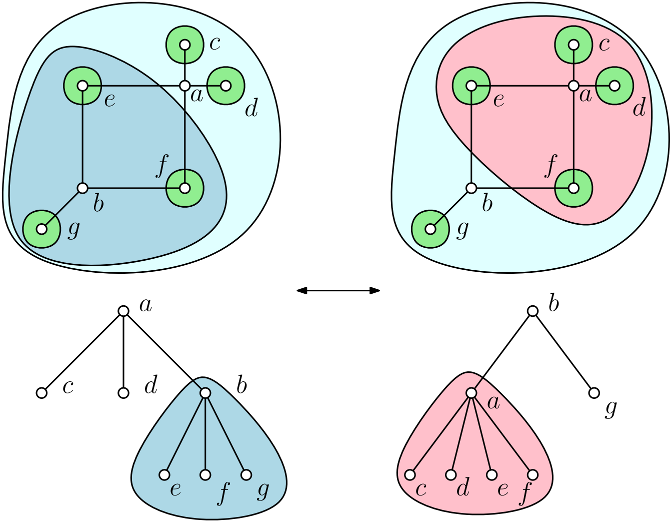


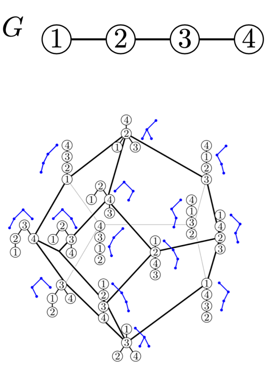


Rotations in elimination trees

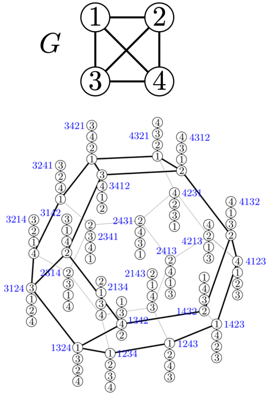


Graph Associahedra

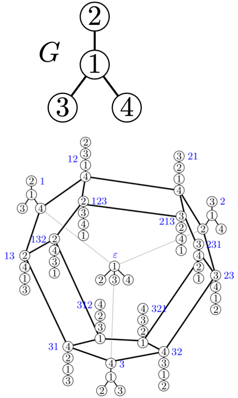
associahedron



permutahedron



stellohedron



Distances and diameters of graph associahedra

- Distances: Computing rotation distances is NP-hard

  Ito, Kakimura, Kamiyama, Kobayashi, Maezawa, Nozaki, Okamoto 2023

Distances and diameters of graph associahedra

- Distances: Computing rotation distances is NP-hard

  Ito, Kakimura, Kamiyama, Kobayashi, Maezawa, Nozaki, Okamoto 2023
- ... unless the graph is a star or a complete split graph.

  C., Pournin, Valencia-Pabon 2023
- Diameter:
  - Tree associahedra have worst-case diameter $\Theta(n \log n)$

    C., Langerman, Perez-Lantero 2018

Distances and diameters of graph associahedra

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- Hamiltonicity: Always!
  Manneville-Pilaud 2015, C., Merino, Mütze 2023

Table of Contents

Associahedra

- Jean-Louis Loday. *Realization of the Stasheff polytope*. Arch. Math. (Basel), 83(3):267–278, 2004.
- Lionel Pournin. *The diameter of associahedra*. Adv. Math., 259:13–42, 2014.
- Vincent Pilaud, Francisco Santos, and G ünter M. Ziegler. *Celebrating Loday's associahedron*. Arch. Math. (Basel), 2023. To appear.

Polymatroids and generalized permutohedra

- Alexander Postnikov, Victor Reiner, and Lauren Williams. *Faces of generalized permutohedra*. Doc. Math., 13:207–273, 2008.
- Alexander Postnikov. *Permutohedra, associahedra, and beyond*. Int. Math. Res. Not. IMRN, 2009(6):1026–1106, 2009
- Marcelo Aguiar and Federico Ardila. *Hopf monoids and generalized permutahedra*. To appear in Mem. Amer. Math. Soc. 2017.

Acyclic orientations

- Carolina Benedetti, Nantel Bergeron, and John Machacek. *Hypergraphic polytopes: combinatorial properties and antipode*. J. Comb., 10(3):515–544, 2019.
- Vincent Pilaud. *Acyclic reorientation lattices and their lattice quotients*. Proceedings of the 34th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC), 2022.
- Jean Cardinal, Hung Phuc Hoang, Arturo I. Merino, Ondrej Micka, and Torsten Mütze. *Combinatorial generation via permutation languages. V. Acyclic orientations*. SIAM J. Discret. Math., 37(3):1509–1547, 2023.

Graph associahedra

- Michael P. Carr and Satyan L. Devadoss. *Coxeter complexes and graph-associahedra*. Topology Appl., 153(12):2155–2168, 2006.
- Thibault Manneville and Vincent Pilaud. *Graph properties of graph associahedra*. Sém. Lothar. Combin., 73:Art. B73d, 31, [2014–2016].
- Jean Cardinal, Arturo Merino, and Torsten Mütze. *Efficient generation of elimination trees and graph associahedra*. In Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), 2022.
- Jean Cardinal, Lionel Pournin, and Mario Valencia-Pabon. *Diameter estimates for graph associahedra*. Ann. Comb., 26(4):873–902, 2022

Computational complexity

- Takehiro Ito, Naonori Kakimura, Naoyuki Kamiyama, Yusuke Kobayashi, Shun-ichi Maezawa, Yuta Nozaki, and Yoshio Okamoto. *Hardness of finding combinatorial shortest paths on graph associahedra*. In Proc. 50th International Colloquium on Automata, Languages, and Programming, ICALP 2023.
- Jean Cardinal, Lionel Pournin, and Mario Valencia-Pabon. *The rotation distance of brooms*. European J. Combin., 2023. To appear.
- Jean Cardinal and Raphael Steiner. *Inapproximability of shortest paths on perfect matching polytopes*. In Integer Programming and Combinatorial Optimization - 24th International Conference, IPCO 2023.
- Jean Cardinal and Raphael Steiner, *Shortest paths on polymatroids and hypergraphic polytopes*, preprint on `https://arxiv.org/abs/2311.00779`.