Partial match queries: a limit process

Nicolas Broutin Ralph Neininger Henning Sulzbach

Background/Introduction

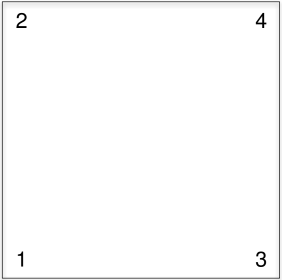
Data structures/Algorithms

- Analysis of costs/running times in natural conditions
- expected cost
- performance guarantee provided by concentration

Methodology

- complex “objects” that decompose recursively (tree like, or related)
- general approach for convergence using contractions

Searching geometric data and quadtrees



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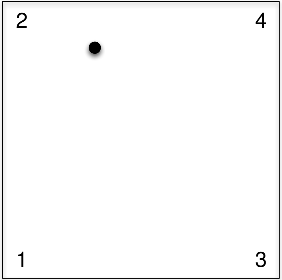

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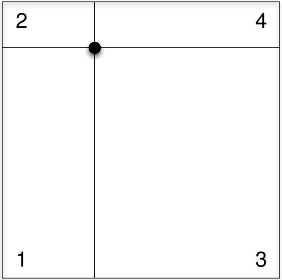

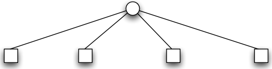

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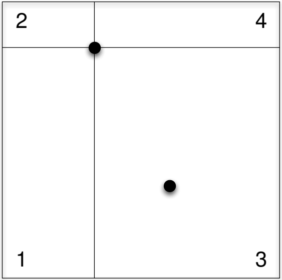

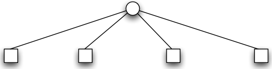

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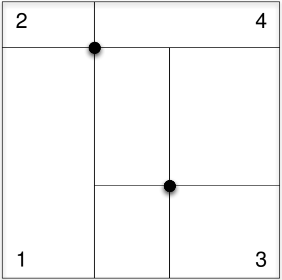

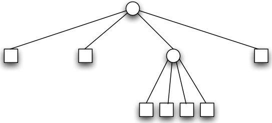

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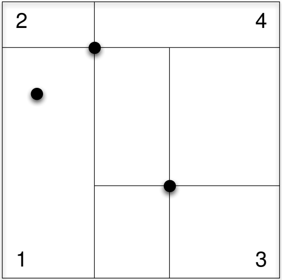

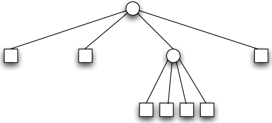

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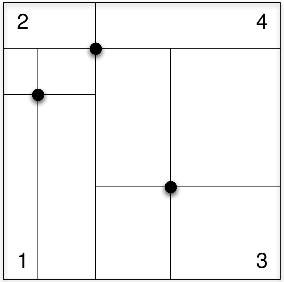

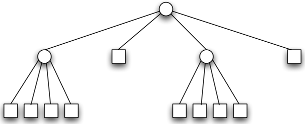

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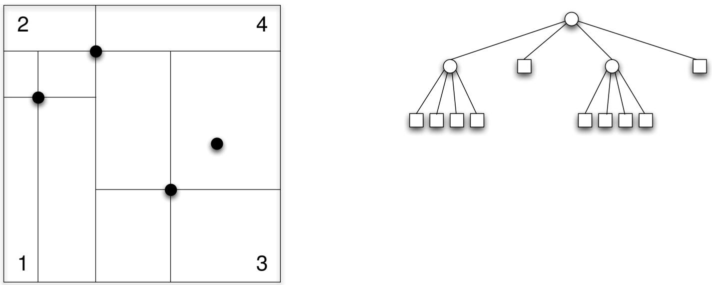

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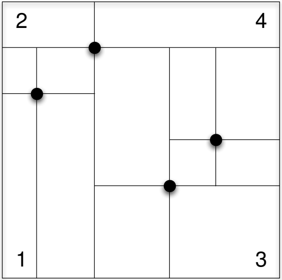

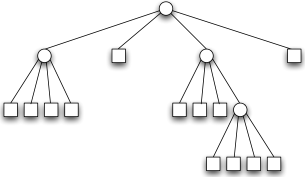

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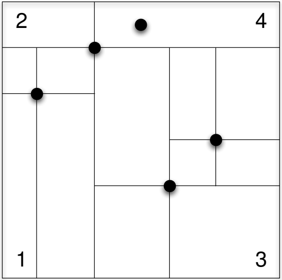

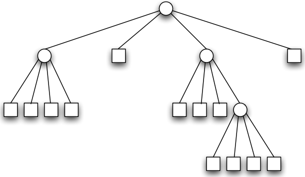

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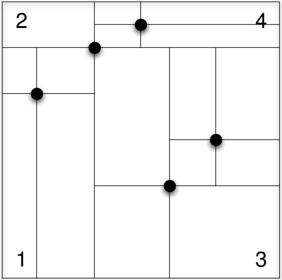

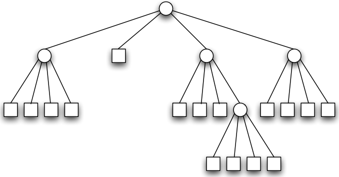

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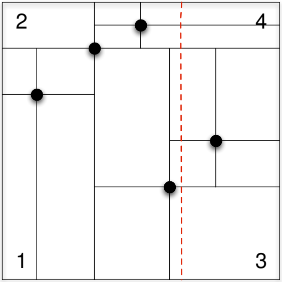

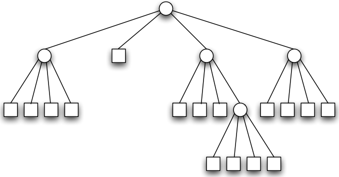

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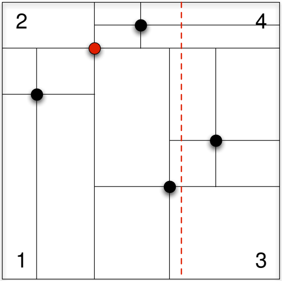

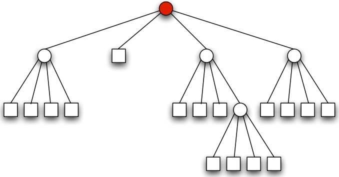

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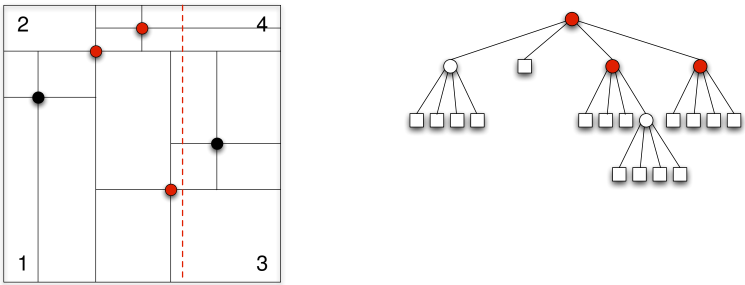

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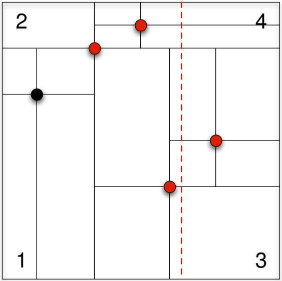
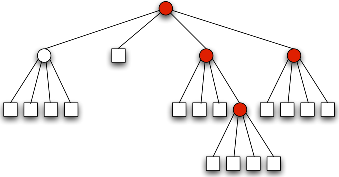




Model and Previous results

Point set = $\{(U_i, V_i), i \geq 1\}$ iid uniform in $[0,1]^2$

$C_n(s)$ the number of lines intersecting $\{x = s\}$ in a quadtree of size n

Theorem (Flajolet, Gonnet, Puech and Robson (1993))

For ξ uniform independent of $\{(U_i, V_i), i \geq 1\}$

$$\mathbf{E}\left[C_n(\xi)\right] \sim \kappa n^{\beta} \qquad \text{where} \quad \kappa = \frac{\Gamma(2\beta+2)}{2\Gamma(\beta+1)^2}, \quad \beta = \frac{\sqrt{17}-3}{2}$$

Theorem (Chern and Hwang (2003))

Let $\phi(z) = (z+1)(z+2) - 4$ and $\beta > \beta'$ the roots of ϕ. For ξ uniform independent of $\{(U_i, V_i), i \geq 1\}$, one has the exact expression

$$\mathbf{E}\left[C_n(\xi)\right] = \sum_{1 \leq k \leq n} \binom{n}{k} (-1)^{k+1} \frac{2(1-\beta)_{k-1}(1-\beta')_{k-1}}{k!(k+1)!}$$

Corollary (Chern and Hwang (2003))

For ξ uniform independent of $\{(U_i, V_i), i \geq 1\}$

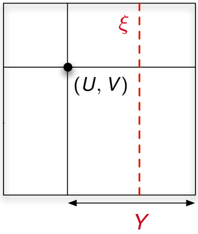
$$\mathbf{E}\left[C_n(\xi)\right] = \kappa n^{\beta} - 1 + O(n^{\beta-1})$$

Idea of the method / heuristic for the constants

Recursive decomposition

We have $Y \overset{d}{=} \max\{U_1, U_2\}$ and
$(I, J) = \text{Mult}(\text{Bin}(n-1, Y); V, (1-V))$ then

$$C_n(\xi) \overset{d}{=} 1 + C_I(\xi') + C_J(\xi')$$



$$\Rightarrow \quad \mathbf{E}[C_n(\xi)] \approx 2\mathbf{E}[C_{nYV}(\xi')]$$

Plugging $\mathbf{E}[C_n(\xi)] = \kappa n^\beta$ yields

$$1 = 2\mathbf{E}[Y^\beta V^\beta] = 2\mathbf{E}[Y^\beta] \cdot \mathbf{E}[V^\beta] = \frac{4}{(\beta+2)(\beta+1)} \quad \Rightarrow \quad \beta = \frac{\sqrt{17}-3}{2}$$

About the variance $\mathbf{Var}\,(C_n(\xi))$

Even when conditioning on the first point, the two terms are still dependent on the query line

The cost at a fixed query line

Idea:

- if the query line is fixed at $s \in (0,1)$, then we do have independence
- however, its relative position changes in the subproblems
- $\Rightarrow$ consider the entire process $(C_n(s), s \in (0,1))$

Theorem (Flajolet, Labelle, Laforest and Salvy 1995)

$$\mathbf{E}\left[C_n(0)\right] = \Theta(n^{\sqrt{2}-1}) = o(n^\beta)$$

Note: in particular, $\mathbf{E}[C_n(U_1)] = o(n^\beta)$, and $C_n(s)$ is not concentrated.

Theorem (Curien and Joseph (2011))

For every fixed $s \in (0,1)$, one has

$$\mathbf{E}\left[C_n(s)\right] \sim K_1 (s(1-s))^{\beta/2} n^\beta, \qquad K_1 = \frac{\Gamma(2\beta+2)\Gamma(\beta+2)}{2\Gamma(\beta+1)^3\Gamma(\beta/2+1)^2}.$$

Main result

Theorem

There exists a random continuous function Z such that, as $n \to \infty$,

$$\left(\frac{C_n(s)}{K_1 n^{\beta}}, s \in [0,1] \right) \xrightarrow{d} (Z(s), s \in [0,1]). \qquad (1)$$

This convergence in distribution holds in the Banach space $(\mathcal{D}[0,1], \|\cdot\|)$ of right-continuous functions with left limits (càdlàg) equipped with the supremum norm.

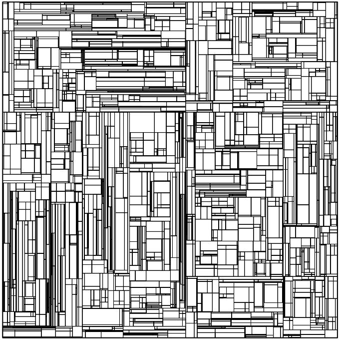
Proposition

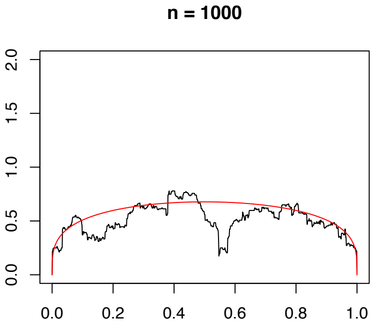
The distribution of the random function Z in (1) is a fixed point of the following equation

$$Z(s) \stackrel{d}{=} \mathbf{1}_{\{s<U\}} \left[(UV)^{\beta} Z^{(1)} \left(\frac{s}{U} \right) + (U(1-V))^{\beta} Z^{(2)} \left(\frac{s}{U} \right) \right] + \mathbf{1}_{\{s \geq U\}} \left[((1-U)V)^{\beta} Z^{(3)} \left(\frac{s-U}{1-U} \right) + ((1-U)(1-V))^{\beta} Z^{(4)} \left(\frac{s-U}{1-U} \right) \right],$$

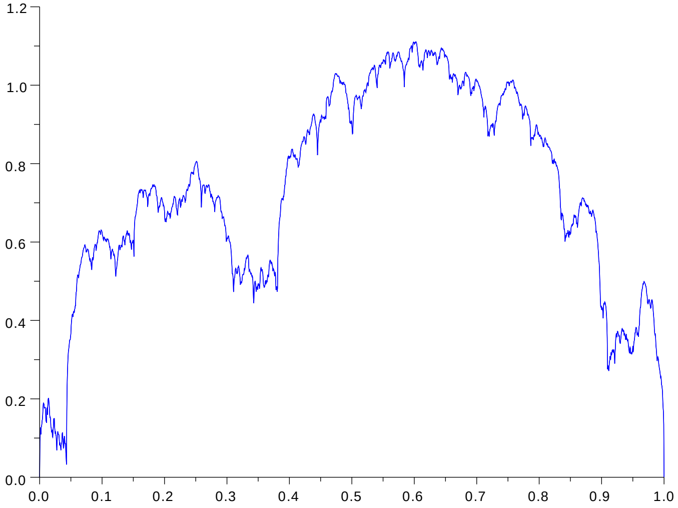
where U and V are independent $[0,1]$-uniform random variables and $Z^{(i)}$, $i = 1, \dots, 4$ are independent copies of the process Z, which are also independent of U and V. Furthermore, Z in (1) is the only solution such that $\mathbf{E}[Z(s)] = (s(1-s))^{\beta/2}$ for all $s \in [0,1]$ and $\mathbf{E}[\|Z\|^2] < \infty$.

What does it look like I





What does it look like II



Moments and supremum

Theorem

We have for all $s \in (0,1)$*, as* $n \to \infty$*,*

$$\mathbf{Var}\,(C_n(s)) \sim \left(2\mathrm{B}(\beta+1,\beta+1)\frac{2\beta+1}{3(1-\beta)} - 1\right)(s(1-s))^{\beta} n^{2\beta}.$$

Here, $B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$ *denotes the Eulerian beta integral* ($a, b > 0$)*.*

Theorem

Let $S_n = \sup_{s\in[0,1]} C_n(s)$*. Then, as* $n \to \infty$*,*

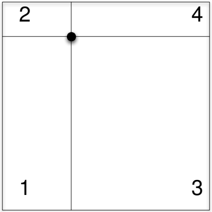
$$n^{-\beta} S_n \xrightarrow{d} S = \sup_{s\in[0,1]} Z(s) \qquad \textit{and} \qquad \mathbf{E}[S_n] \sim n^{\beta}\mathbf{E}[S], \qquad \mathbf{Var}(S_n) \sim n^{2\beta}\mathbf{Var}(S).$$

Convergence in distribution by contraction I.

Cost of the construction of the quadtree / path length

$P_n = \sum_{i=1}^{n} D_i$ with D_i the depth of the i-th inserted point

- I_n^r the number of points inside the r-th child cell
- Q^r the volume or the r-th child cell



We have

$$P_n \stackrel{d}{=} \sum_{r=1}^{4} P_{I_n^r} + n - 1 \quad \text{and write} \quad X_n = \frac{P_n - \alpha n \log n}{n}$$

$$(I_n^1, \ldots, I_n^4) \stackrel{d}{=} \mathrm{Mult}(n-1; UV, U(1-V), (1-U)(1-V), (1-U)V).$$

Shifting and rescaling we obtain:

$$\underbrace{\frac{P_n - \alpha n \log n}{n}}_{X_n} = \sum_{r=1}^{4} \underbrace{\left(\frac{I_n^r}{n}\right)}_{A_n^r} \frac{P_{I_n^r} - \alpha I_n^r \log I_n^r}{I_n^r} + \underbrace{\frac{n-1}{n} - \frac{\alpha \log n}{n} + \alpha \sum_{r=1}^{4} \left(\frac{I_n^r}{n}\right) \log\left(\frac{I_n^r}{n}\right)}_{b_n}$$

Convergence in distribution by contraction II.

General problem:

A recursive family of equations $X_n \overset{d}{=} \sum_{r=1}^{4} A_n^r \cdot X_{I_n^r}^r + b_n$ with

- $(A_n^1, \ldots, A_n^4, I_n^1, \ldots, I_n^4, b_n)$ independent of $((X^1), \ldots, (X^4))$
- $(X_n^r, n \geq 1)$ iid copies of (X)

The equation "converges" to a limit equation:

$$A_n^r = \frac{I_n^r}{n} \to \mathrm{Leb}(Q_r)$$

$$b_n = \frac{n-1}{n} - \frac{\alpha \log n}{n} + \alpha \sum_{r=1}^{4} \left(\frac{I_n^r}{n}\right) \log\left(\frac{I_n^r}{n}\right) \to 1 + \alpha \sum_{r=1}^{4} \mathrm{Leb}(Q_r) \log \mathrm{Leb}(Q_r)$$

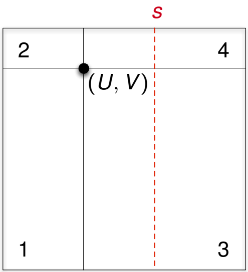
$$X \overset{d}{=} \sum_{r=1}^{4} \mathrm{Leb}(Q_r) \cdot X^r + 1 + \alpha \sum_{r=1}^{4} \mathrm{Leb}(Q_r) \log \mathrm{Leb}(Q_r) \qquad (2)$$

Formalization: (2) a transfer map on a **space of probability measures on** $\mathbb{R}$.

$d_2(\phi, \varphi) = \inf\{\|X - Y\|_2 : \mathscr{L}(X) = \phi, \mathscr{L}(Y) = \varphi\}$

- on $\mathscr{M}_2 = \{\text{probability measures } \mu : \int x^2 d\mu < \infty\}$ no contraction (can shift!)
- on $\mathscr{M}_2^0 = \{\mu \in \mathscr{M}_2 : \int x d\mu = 0\}$ contraction

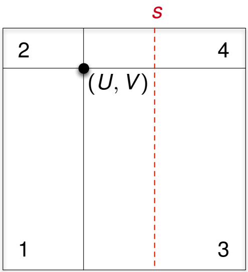
Convergence for partial match processes



$$(I_n^{(1)}, \dots, I_n^{(4)}) \stackrel{d}{=} \mathrm{Mult}(n-1; UV, U(1-V), (1-U)(1-V), (1-U)V)$$

$$C_n(s) \stackrel{d}{=} 1 + \mathbf{1}_{\{s<U\}} \left[C^{(1)}_{I_n^{(1)}}\left(\frac{s}{U}\right) + C^{(2)}_{I_n^{(2)}}\left(\frac{s}{U}\right) \right] + \mathbf{1}_{\{s \geq U\}} \left[C^{(3)}_{I_n^{(3)}}\left(\frac{1-s}{1-U}\right) + C^{(4)}_{I_n^{(4)}}\left(\frac{1-s}{1-U}\right) \right]$$

Convergence for partial match processes



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Heuristic: If $n^{-\beta} C_n(\cdot)$ converges, we should have $n^{-\beta} C_n(\cdot) \to Z(\cdot)$ satisfying

$$Z(s) \stackrel{d}{=} \mathbf{1}_{\{s<U\}} \left[(UV)^\beta Z^{(1)}\left(\frac{s}{U}\right) + (U(1-V))^\beta Z^{(2)}\left(\frac{s}{U}\right) \right] + \mathbf{1}_{\{s\ge U\}} \left[((1-U)V)^\beta Z^{(3)}\left(\frac{s-U}{1-U}\right) + ((1-U)(1-V))^\beta Z^{(4)}\left(\frac{s-U}{1-U}\right) \right]$$

Convergence in $\mathcal{D}[0,1]$ by contraction arguments I.

Neininger and Sulzbach (2011+)

Let (X_n) be $\mathcal{D}[0,1]$-valued random variables with

$$X_n \stackrel{d}{=} \sum_{r=1}^{K} A_n^{(r)} \circ X^{(r)}_{I_n^{(r)}} + b_n, \quad n \geq 1,$$

where

- $(A_n^{(1)}, \ldots, A_n^{(K)})$ are random linear and continuous operators on $\mathcal{D}[0,1]$
- b_n is a $\mathcal{D}[0,1]$-valued random variable
- $I_n^{(1)}, \ldots, I_n^{(K)}$ are random integers between 0 and $n-1$
- $(X_n^{(1)}), \ldots, (X_n^{(K)})$ are distributed like (X_n)
- $(A_n^{(1)}, \ldots, A_n^{(K)}, b_n, I_n^{(1)}, \ldots, I_n^{(K)}), (X_n^{(1)}), \ldots, (X_n^{(K)})$ are independent

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Example: here, because of the rescaling, we have

$$A_n^{(1)} : f \mapsto \mathbf{1}_{\{\,\cdot\, \leq U\}} \left(\frac{I_n^{(1)}}{n}\right)^{\beta} f\left(\frac{\cdot}{U}\right)$$

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Keep in mind

We want contraction in a **space of probability measures on $\mathcal{D}[0,1]$**.

Convergence in $\mathcal{D}[0,1]$ by contraction arguments II.

Neininger and Sulzbach (2011+)

For a random linear operator A write

$$\|A\|_2 := \mathbf{E}[\|A\|_{\mathrm{op}}^2]^{1/2} \quad \text{with} \quad \|A\|_{\mathrm{op}} := \sup_{\|x\|=1} \|A(x)\|$$

(A1) CONVERGENCE AND CONTRACTION (SIMPLIFIED).

- we have $\|A_n^{(r)}\|_2, \|b_n\|_2 < \infty$ for all $r = 1, \ldots, K$ and $n \geq 0$
- there exist random operators $A^{(1)}, \ldots, A^{(K)}$ on $\mathcal{D}[0,1]$ and a $\mathcal{D}[0,1]$-valued random variable b such that

$$\|b_n - b\|_2 + \sum_{r=1}^{K} \left(\|A_n^{(r)} - A^{(r)}\|_2 \right) \leq R(n) \qquad R(n) \to 0$$

- for all $\ell \in \mathbb{N}$,

$$L^* = \limsup_{n\to\infty} \mathbf{E}\left[\sum_{r=1}^{K} \|A_r^{(n)}\|_{\mathrm{op}}^2 \right] < 1.$$

(A2) EXISTENCE AND EQUALITY OF MOMENTS. $\mathbf{E}[\|X_n\|^2] < \infty$ for all n and $\mathbf{E}[X_{n_1}(t)] = \mathbf{E}[X_{n_2}(t)]$ for all $n_1, n_2 \in \mathbb{N}_0, t \in [0,1]$.

Convergence in $\mathcal{D}[0,1]$ by contraction arguments III.

Neininger and Sulzbach (2011+)

(A3) EXISTENCE OF A CONTINUOUS SOLUTION. There exists a solution X of the fixed-point equation

$$X \stackrel{d}{=} \sum_{r=1}^{K} A_r \circ X^{(r)} + b$$

with continuous paths, $\mathbf{E}[\|X\|^2] < \infty$ and $\mathbf{E}[X(t)] = \mathbf{E}[X_1(t)]$ for all $t \in [0,1]$.

(A4) PERTURBATION CONDITION. $X_n = W_n + h_n$ where $\|h_n - h\| \to 0$ with $h \in \mathcal{D}[0,1]$ and random variables W_n in $\mathcal{D}[0,1]$ such that there exists a sequence (r_n) with, as $n \to \infty$,

$$\mathbf{P}\left(W_n \notin \mathcal{D}_{r_n}[0,1]\right) \to 0.$$

Here, $\mathcal{D}_{r_n}[0,1] \subset \mathcal{D}[0,1]$ denotes the set of functions on the unit interval, for which there is a decomposition of $[0,1]$ into intervals of length as least r_n on which they are constant.

(A5) RATE OF CONVERGENCE. $R(n) = o\left(\log^{-m}(1/r_n)\right)$.

Existence of a continuous solution

Define

- a complete tree $T = \bigcup_{n\geq 0}\{1,2,3,4\}^n$ with (U_u, V_u), $u \in \mathcal{T}$, iid uniform on $[0,1]$
- a starting function $h(s) = (s(1-s))^{\beta/2}$
- an iteration/mixing operator $G : [0,1]^2 \times \mathcal{C}[0,1]^4 \to \mathcal{C}[0,1]$

$$
\begin{aligned}
&G(x, y; f_1, f_2, f_3, f_4)(s) = \\
&\mathbf{1}_{\{s<x\}} \left[(xy)^\beta f_1\left(\frac{s}{x}\right) + (x(1-y))^\beta f_2\left(\frac{s}{x}\right)\right] \\
&+ \mathbf{1}_{\{s\geq x\}} \left[((1-x)y)^\beta f_3\left(\frac{s-x}{1-x}\right) + ((1-x)(1-y))^\beta f_4\left(\frac{s-x}{1-x}\right)\right]
\end{aligned}
$$

For every node $u \in \mathcal{T}$, let

$$
\begin{aligned}
Z_0^u &= h \\
Z_{n+1}^u &= G(U_u, V_u; Z_n^{u1}, Z_n^{u2}, Z_n^{u3}, Z_n^{u4})
\end{aligned}
$$

Lemma

$Z_n = Z_n^\varnothing$, $n \geq 0$, is a non-negative martingale

Existence of a continuous solution

Define

- a complete tree $T = \bigcup_{n\geq 0}\{1,2,3,4\}^n$ with (U_u, V_u), $u \in \mathcal{T}$, iid uniform on $[0,1]$
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$$\mathbf{1}_{\{s<x\}}\left[(xy)^\beta f_1\left(\frac{s}{x}\right) + (x(1-y))^\beta f_2\left(\frac{s}{x}\right)\right]$$
$$+\,\mathbf{1}_{\{s\geq x\}}\left[((1-x)y)^\beta f_3\left(\frac{s-x}{1-x}\right) + ((1-x)(1-y))^\beta f_4\left(\frac{s-x}{1-x}\right)\right]$$

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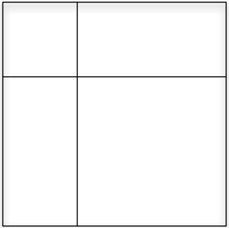
$$Z_{n+1}^u = G(U_u, V_u; Z_n^{u1}, Z_n^{u2}, Z_n^{u3}, Z_n^{u4})$$



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Existence of a continuous solution

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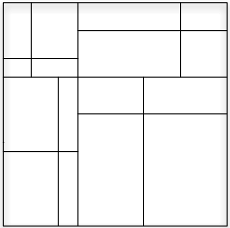


Lemma

$Z_n = Z_n^\varnothing$, $n \geq 0$, is a non-negative martingale

Uniform convergence of the mean

Proposition

There exists $\varepsilon > 0$ *such that*

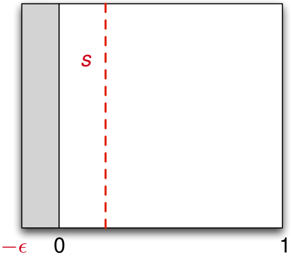
$$\sup_{s\in[0,1]} |t^{-\beta}\mathbf{E}[P_t(s)] - \mu_1(s)| = O(t^{-\varepsilon}).$$

$$\sup_{s\in[0,1]} |t^{-\beta}\mathbf{E}[P_t(s)] - \mu_1(s)| \le \sup_{s\le\delta} \left|t^{-\beta}\mathbf{E}[P_t(s)] - \mu_1(s)\right| + \sup_{s\in(\delta,1/2]} \left|t^{-\beta}\mathbf{E}[P_t(s)] - \mu_1(s)\right|.$$

Proposition (Almost monotonicity)

For any $s < 1/2$ *and* $\varepsilon \in [0, 1-2s)$, *we have*

$$\mathbf{E}[P_t(s)] \le \mathbf{E}\left[P_{t(1+\varepsilon)}\left(\frac{s+\varepsilon}{1+\varepsilon}\right)\right].$$



Thank you!