Bipartite subfamilies of planar graphs

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The material of this talk

1.– **Background**

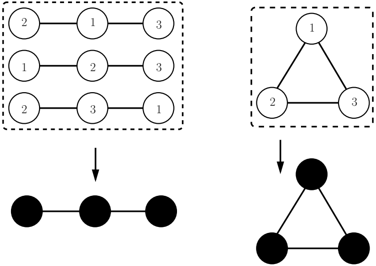
2.– **Graph decompositions. First results**

3.– **The bipartite framework**

Background

Objects: graphs

Labelled Graph= labelled vertices+edges.
Unlabelled Graph= labelled one up to permutation of labels.
Simple Graph= NO multiples edges, NO loops.



Question: How many graphs with n vertices are in the family?

The counting series

Strategy: **Encapsulate these numbers → Counting series**

- **Labelled** framework: **exponential generating functions**

$$A(x) = \sum_{a \in \mathcal{A}} \frac{x^{|a|}}{|a|!} = \sum_{n \geq 0} \frac{|\mathcal{A}_n|}{n!} x^n$$

- **Unlabelled** framework: **cycle index sums**

$$Z_{\mathcal{A}}(s_1, s_2, \ldots) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\substack{(\sigma,g) \in \mathfrak{S}_n \times \mathcal{A}_n \\ \sigma \cdot g = g}} s_1^{c_1} s_2^{c_2} \cdots s_n^{c_n},$$

$$\widetilde{A}(x) = Z_{\mathcal{A}}(x, x^2, x^3, \ldots) = \sum_{n \geq 0} |\widetilde{\mathcal{A}}_n| x^n.$$

The symbolic method

COMBINATORIAL RELATIONS between CLASSES

$$\Updownarrow$$

EQUATIONS between GENERATING FUNCTIONS

Class	Labelled setting	Unlabelled setting
$\mathcal{C} = \mathcal{A} \cup \mathcal{B}$	$C(x) = A(x) + B(x)$	$\widetilde{C}(x) = \widetilde{A}(x) + \widetilde{B}(x)$
$\mathcal{C} = \mathcal{A} \times \mathcal{B}$	$C(x) = A(x) \cdot B(x)$	$\widetilde{C}(x) = \widetilde{A}(x) \cdot \widetilde{B}(x)$
$\mathcal{C} = \mathrm{Set}(\mathcal{B})$	$C(x) = \exp(B(x))$	$\widetilde{C}(x) = \exp\left(\sum_{i\geq 1} \frac{1}{i} \widetilde{B}(x^i)\right)$
$\mathcal{C} = \mathcal{A} \circ \mathcal{B}$	$C(x) = A(B(x))$	$\widetilde{C}(x) = Z_{\mathcal{A}}(\widetilde{B}(x), \widetilde{B}(x^2), \ldots)$

Singularity analysis on generating functions

GFs: analytic functions in a neighbourhood of the origin.

The smallest singularity of $A(z)$ determines the asymptotics of the coefficients of $A(z)$.

- POSITION: exponential growth ρ.
- NATURE: subexponential growth
- Transfer Theorems: Let $\alpha \notin \{0, -1, -2, \ldots\}$. If

$$A(z) = a \cdot (1 - z/\rho)^{-\alpha} + o((1 - z/\rho)^{-\alpha})$$

then

$$a_n = [z^n]A(z) \sim \frac{a}{\Gamma(\alpha)} \cdot n^{\alpha-1} \cdot \rho^{-n}(1 + o(1))$$

Our starting point

Asymptotic enumeration and limit laws of planar graphs (Giménez, Noy)

$$g_1 \cdot n^{-7/2} \cdot \gamma_1^n \cdot n! \cdot (1 + o(1))$$

Asymptotic enumeration and limit laws of series-parallel graphs (Bodirsky, Giménez, Kang, Noy)

$$g_2 \cdot n^{-5/2} \cdot \gamma_2^n \cdot n! \cdot (1 + o(1))$$

Our starting point

$$g_1 \cdot n^{-7/2} \cdot \gamma_1^n \cdot n! \cdot (1 + o(1))$$

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$$\Downarrow$$

THE SUBEXPONENTIAL TERM GIVES THE “PHYSICS” OF THE GRAPHS

$$\Updownarrow$$

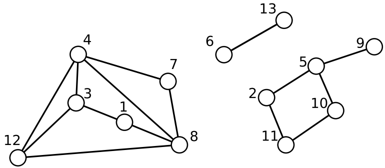
GENERAL FRAMEWORK TO UNDERSTAND THIS EXPONENT

Graph decompositions.
First results

General graphs from connected graphs

Let $\mathcal{C}$ be a family of *connected* graphs.

$\mathcal{G}$: graphs such that their *connected components* are in $\mathcal{C}$.



General graphs from connected graphs

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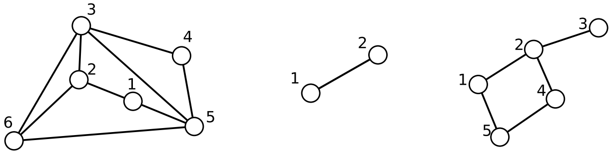
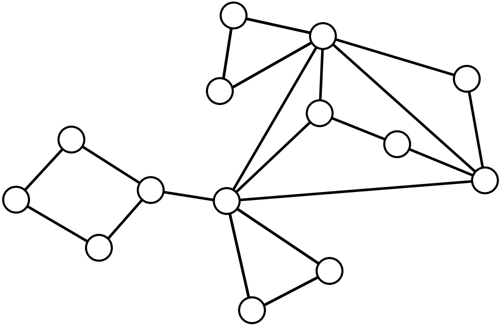


$$\mathcal{G} = \mathrm{Set}(\mathcal{C}) \Longrightarrow G(x) = \exp(C(x))$$

Connected graphs from 2-connected graphs

Let $\mathcal{B}$ be a family of *2-connected* graphs.

$\mathcal{C}$: *connected* graphs with *blocks* in $\mathcal{B}$.



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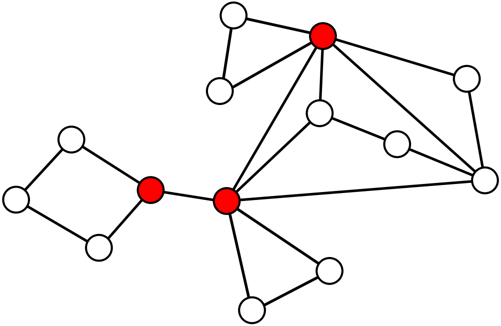
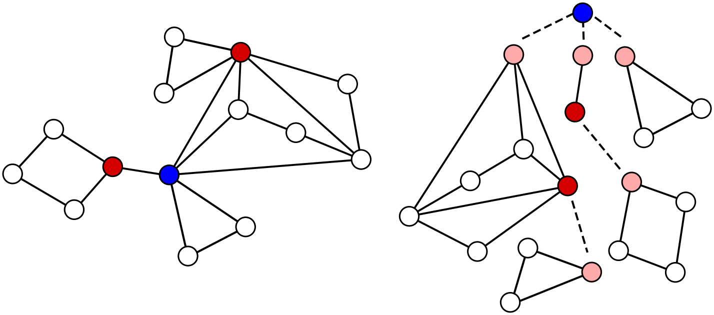


Connected graphs from 2-connected graphs



A *vertex-rooted* connected graph is a *tree* of rooted blocks.

$$\mathcal{C}^\bullet = v \times \mathrm{Set}(\mathcal{B}'(v \leftarrow \mathcal{C}^\bullet)) \Longrightarrow C^\bullet(x) = x \exp B'(C^\bullet(x))$$

2-connected graphs from 3-connected graphs

Decomposition in 3-connected components is slightly harder.

Let $\mathcal{T}$ be a family of *3-connected* graphs: $T(x, z)$.

We define $\mathcal{B}$ as those *2-connected* graphs such that can be obtained from *series*, *parallel*, and $\mathcal{T}$-compositions.

$$D(x, y) = (1 + y) \exp\left(\frac{xD^2}{1 + xD} + \frac{1}{2x^2}\frac{\partial T}{\partial z}(x, D)\right) - 1$$

$$\frac{\partial B}{\partial y}(x, y) = \frac{x^2}{2}\left(\frac{1 + D(x, y)}{1 + y}\right)$$

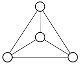
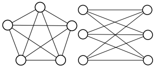
D is the GF for networks (essentially edge-rooted 2-connected graphs without the edge root).

A set of equations

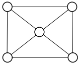
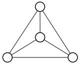
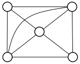
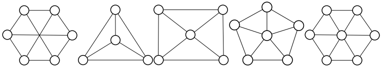
$$\begin{cases} \frac{1}{2x^2D}\frac{\partial T}{\partial z}(x,D) - \log\left(\frac{1+D}{1+y}\right) + \frac{xD^2}{1+xD} = 0 \\ \frac{\partial B}{\partial y}(x,y) = \frac{x^2}{2}\left(\frac{1+D(x,y)}{1+y}\right) \end{cases}$$

$$\begin{cases} C^\bullet(x) = x\exp\left(B'(C^\bullet(x))\right) \\ G(x) = \exp(C(x)) \end{cases}$$

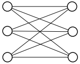
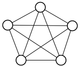
Examples of families & excluded minors (I)

- Series-parallel graphs
  - Excluded minors: 
  - $\mathcal{T}$: None.
  - $T(x, z) = 0$.
- Planar graphs
  - Excluded minors: 
  - $\mathcal{T}$: 3-connected planar graphs.
  - $T(x, z)$: *The number of labelled 2-connected planar graphs* **(Bender, Gao, Wormald, 2002)**

Examples of families & excluded minors (II)

- W_4-free
  - Excluded minors: 
  - $\mathcal{T}$: 
  - $T(x,z) = \frac{1}{4!} x^4 z^6$.
- K_5^--free
  - Excluded minors: 
  - $\mathcal{T}$:  , ...
  - $T(x,z) = \frac{70}{6!} x^6 z^9 - \frac{1}{2} x \left(\log(1 - xz^2) + 2xz^2 + x^2 z^4 \right)$.

Examples of families & excluded minors (III)

- $K_{3,3}$-free **(Gerke, Giménez, Noy, Weibl, 2006)**
  - Excluded minors: 
  - 3-connected components: , 3-connected planar graphs.
  - $T(x, z) = \dots$.
- If $\mathcal{G} = \mathrm{Ex}(\mathcal{M})$ and all the excluded minors $\mathcal{M}$ are 3-connected, then $\mathcal{G}$ can be expressed in terms of its 3-connected graphs.

RESULT: asymptotic enumeration

If either $\frac{\partial T}{\partial z}(x, z)$

- has no singularity, or
- the singularity type is $(1 - z/z_0)^\alpha$ with $\alpha < 1$,

then the situation is alike to the **series-parallel case**:

$$D(x) \sim d \cdot (1 - x/x_0)^{1/2} \qquad d_n \sim d \cdot n^{-3/2} \cdot x_0^{-n} \cdot n!$$

$$B(x) \sim b \cdot (1 - x/x_0)^{3/2} \qquad b_n \sim b \cdot n^{-5/2} \cdot x_0^{-n} \cdot n!$$

$$C(x) \sim c \cdot (1 - x/\rho)^{3/2} \qquad c_n \sim c \cdot n^{-5/2} \cdot \rho^{-n} \cdot n!$$

$$G(x) \sim g \cdot (1 - x/\rho)^{3/2} \qquad g_n \sim g \cdot n^{-5/2} \cdot \rho^{-n} \cdot n!$$

RESULT: asymptotic enumeration (II)

If $\frac{\partial T}{\partial z}(x,z)$ has singularity type $(1 - z/z_0)^{3/2}$, then 3 different situations may happen.

Case 1 (**Planar case**)

$$D(x) \sim d \cdot (1 - x/x_0)^{3/2} \qquad d_n \sim d \cdot n^{-5/2} \cdot x_0^{-n} \cdot n!$$

$$B(x) \sim b \cdot (1 - x/x_0)^{5/2} \qquad b_n \sim b \cdot n^{-7/2} \cdot x_0^{-n} \cdot n!$$

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RESULT: asymptotic enumeration (II)

If $\frac{\partial T}{\partial z}(x,z)$ has singularity type $(1-z/z_0)^{3/2}$, then 3 different situations may happen.

Case 2 (**Series-parallel case**)

$$D(x) \sim d \cdot (1-x/x_0)^{1/2} \qquad d_n \sim d \cdot n^{-3/2} \cdot x_0^{-n} \cdot n!$$
$$B(x) \sim b \cdot (1-x/x_0)^{3/2} \qquad b_n \sim b \cdot n^{-5/2} \cdot x_0^{-n} \cdot n!$$
$$C(x) \sim c \cdot (1-x/\rho)^{3/2} \qquad c_n \sim c \cdot n^{-5/2} \cdot \rho^{-n} \cdot n!$$
$$G(x) \sim g \cdot (1-x/\rho)^{3/2} \qquad g_n \sim g \cdot n^{-5/2} \cdot \rho^{-n} \cdot n!$$

RESULT: asymptotic enumeration (II)

If $\frac{\partial T}{\partial z}(x, z)$ has singularity type $(1 - z/z_0)^{3/2}$, then 3 different situations may happen.

Case 3 (**Mixed case**)

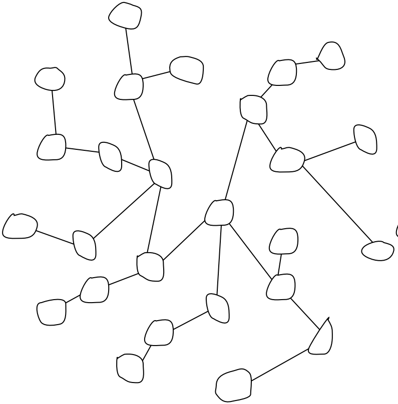
$$D(x) \sim d \cdot (1 - x/x_0)^{3/2} \qquad d_n \sim d \cdot n^{-5/2} \cdot x_0^{-n} \cdot n!$$

$$B(x) \sim b \cdot (1 - x/x_0)^{5/2} \qquad b_n \sim b \cdot n^{-7/2} \cdot x_0^{-n} \cdot n!$$

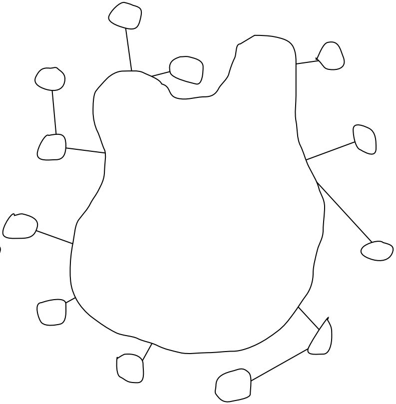
$$C(x) \sim c \cdot (1 - x/\rho)^{3/2} \qquad c_n \sim c \cdot n^{-5/2} \cdot \rho^{-n} \cdot n!$$

$$G(x) \sim g \cdot (1 - x/\rho)^{3/2} \qquad g_n \sim g \cdot n^{-5/2} \cdot \rho^{-n} \cdot n!$$

2 different pictures



Series-parallel-like situation

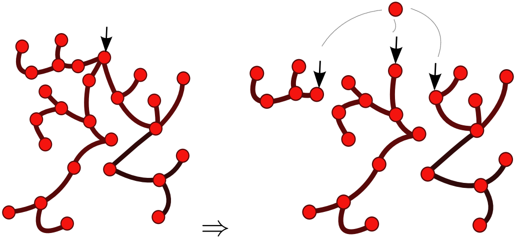


Planar-like situation

The bipartite framework

A key example: Trees

We count *rooted* trees



$$\mathcal{T} = \bullet \times \mathrm{Set}(\mathcal{T}) \to T(x) = xe^{T(x)}$$

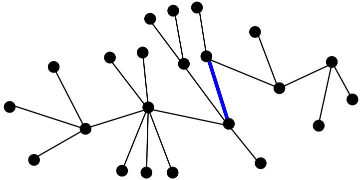
To forget the root, we just integrate: $(xU'(x) = T(x))$

$$\int_0^x \frac{T(s)}{s} ds = \left\{ \begin{array}{c} T(s) = u \\ T'(s)\, ds = du \end{array} \right\} = \int_{T(0)}^{T(x)} 1 - u \, du = T(x) - \frac{1}{2} T(x)^2$$

Question: can we interpret this formula **combinatorially**?

The dissymmetry theorem

Let $\mathcal{T}$ a class of unrooted trees $\Rightarrow$ canonical root (their centers).



Dissymmetry Theorem for trees:

$$\mathcal{T} \cup \mathcal{T}_{\bullet\to\bullet} \simeq \mathcal{T}_{\bullet-\bullet} \cup \mathcal{T}_{\bullet},$$

For trees:

$$\mathcal{T}_{\bullet\to\bullet} \to T(x)^2; \mathcal{T}_{\bullet-\bullet} \to \frac{1}{2}T(x)^2; \mathcal{T}_{\bullet} \to T(x).$$

Dissymmetry Theorem $\equiv$ Combinatorial **Integration**.

Returning to the equations

$$\frac{\partial B}{\partial y}(x,y) = \frac{x^2}{2}\left(\frac{1+D(x,y)}{1+y}\right) \leftrightarrow 2(1+y)\frac{\partial B}{\partial y}(x,y) = \frac{x^2}{2}\left(1+D(x,y)\right)$$

$$\Downarrow$$

$$B(x,y) = \frac{x^2}{2}\int_0^y \left(\frac{1+D(x,s)}{1+s}\right) ds$$

Amazingly, an **EXACT** formula exists!

$$\begin{aligned} B(x,y) &= T(x, D(x,y)) - \frac{1}{2}xD(x,y) + \frac{1}{2}\log(1+xD(x,y)) + \\ &\quad \frac{x^2}{2}\left(D(x,y) + \frac{1}{2}D(x,y)^2 + (1+D(x,y))\log\left(\frac{1+y}{1+D(x,y)}\right)\right). \end{aligned}$$

Is there a “tree-like” argument to explain this formula?

The complete grammar for graphs

A Grammar for Decomposing a Family of Graphs into 3-connected Components; (Chapuy, Fusy, Kang, Shoilekova)

(1). General from Connected (folklore).

$$\mathcal{G} = \mathbf{Set}(\mathcal{G}_1)$$

(2). Connected from 2-Connected (Bergeron, Labelle, Leroux).

$$\mathcal{G}_1 = \mathcal{C} = \mathcal{C}_v + \mathcal{C}_B - \mathcal{C}_{Bv} \quad \text{[dissymmetry theorem]}$$
$$\mathcal{C}_v = v * \mathcal{C}'$$
$$\mathcal{C}' = \mathbf{Set}(\mathcal{G}_2' \circ_v \mathcal{C}')$$
$$\mathcal{C}_B = \mathcal{G}_2 \circ_v \mathcal{C}'$$
$$\mathcal{C}_{Bv} = (v * \mathcal{G}_2') \circ_v \mathcal{C}'$$

(3). 2-Connected from 3-Connected.

(i) **Networks**

$$\mathcal{D} = e + \mathcal{S} + \mathcal{P} + \mathcal{H}$$
$$\mathcal{S} = (\mathcal{D} - \mathcal{S}) * v * \mathcal{D}$$
$$\mathcal{P} = \begin{cases} \mathbf{Set}_{\geq 2}(\mathcal{D} - \mathcal{P}), & \text{[Multi-edges allowed]} \\ e * \mathbf{Set}_{\geq 1}(\mathcal{D} - \mathcal{P} - e) + \mathbf{Set}_{\geq 2}(\mathcal{D} - \mathcal{P} - e), & \text{[No Multi-edges]} \end{cases}$$
$$\mathcal{H} = \overrightarrow{\mathcal{G}_3} \circ_e \mathcal{D}$$

(ii) **Unrooted 2-Connected**

$$\mathcal{G}_2 = \epsilon + \mathcal{B}, \quad \epsilon = \begin{cases} \ell_1 + \ell_2 & \text{[Multi-edges allowed]} \\ \ell_1 & \text{[No multi-edges]} \end{cases}$$
$$\mathcal{B} = \mathcal{B}_R + \mathcal{B}_M + \mathcal{B}_T - \mathcal{B}_{R-M} - \mathcal{B}_{R-T} - \mathcal{B}_{M-T} - \mathcal{B}_{T\to T} + \mathcal{B}_{T-T}$$
$$\mathcal{B}_R = \mathcal{R} \circ_e (\mathcal{D} - \mathcal{S})$$
$$\mathcal{B}_M = \begin{cases} \mathcal{M} \circ_e (\mathcal{D} - \mathcal{P}) = (v^2 * \mathrm{Set}_{\geq 3}(\mathcal{D} - \mathcal{P}))/\bullet \leftrightarrows \bullet, & \text{[Multi-edges allowed]} \\ (v^2 * e * \mathrm{Set}_{\geq 2}(\mathcal{D} - \mathcal{P} - e) + v^2 * \mathrm{Set}_{\geq 3}(\mathcal{D} - \mathcal{P} - e))/\bullet \leftrightarrows \bullet, & \text{[No multi-edges]} \end{cases}$$
$$\mathcal{B}_T = \mathcal{G}_3 \circ_e \mathcal{D}$$
$$\mathcal{B}_{R-M} = (v^2 * \mathcal{S} * \mathcal{P})/\bullet \leftrightarrows \bullet, \quad \text{[Up to pole exchange, denoted by } /\bullet \leftrightarrows \bullet\text{]}$$
$$\mathcal{B}_{R-T} = (v^2 * \mathcal{S} * \mathcal{H})/\bullet \leftrightarrows \bullet$$
$$\mathcal{B}_{M-T} = (v^2 * \mathcal{P} * \mathcal{H})/\bullet \leftrightarrows \bullet$$
$$\mathcal{B}_{T\to T} = (v^2 * \mathcal{H} * \mathcal{H})/\bullet \leftrightarrows \bullet$$
$$\mathcal{B}_{T-T} = (v^2 * \mathcal{H} * \mathcal{H})/(\bullet \leftrightarrows \bullet, H \leftrightarrows H), \quad \text{[Up to pole and component exchange]}$$

(iii) **Vertex-pointed 2-Connected**

$$\mathcal{G}_2' = \epsilon' + \mathcal{B}', \quad \epsilon' = \begin{cases} \ell_1' + \ell_2' = v * (e + \mathrm{Set}_2(e)) & \text{[Multi-edges allowed]} \\ \ell_1' = v * e & \text{[No multi-edges]} \end{cases}$$
$$\mathcal{B}' = \mathcal{V} = \mathcal{V}_R + \mathcal{V}_M + \mathcal{V}_T - \mathcal{V}_{R-M} - \mathcal{V}_{R-T} - \mathcal{V}_{M-T} - \mathcal{V}_{T\to T} + \mathcal{V}_{T-T}$$
$$\mathcal{V}_R = \mathcal{R}' \circ_e (\mathcal{D} - \mathcal{S})$$
$$\mathcal{V}_M = \begin{cases} v * \mathbf{Set}_{\geq 3}(\mathcal{D} - \mathcal{P}), & \text{[Multi-edges allowed]} \\ v * e * \mathbf{Set}_{\geq 2}(\mathcal{D} - \mathcal{P} - e) + v * \mathbf{Set}_{\geq 3}(\mathcal{D} - \mathcal{P} - e), & \text{[No multi-edges]} \end{cases}$$
$$\mathcal{V}_T = \mathcal{G}_3' \circ_e \mathcal{D}$$
$$\mathcal{V}_{R-M} = v * \mathcal{S} * \mathcal{P}$$
$$\mathcal{V}_{R-T} = v * \mathcal{S} * \mathcal{H}$$
$$\mathcal{V}_{M-T} = v * \mathcal{P} * \mathcal{H}$$
$$\mathcal{V}_{T\to T} = v * \mathcal{H} * \mathcal{H}$$
$$\mathcal{V}_{T-T} = (v * \mathcal{H} * \mathcal{H})/H \leftrightarrows H$$

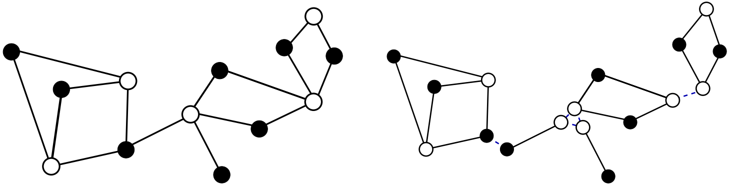
This system is obtained applying the **dissymmetry theorem for trees** in an ingenious way.

The key step is the one which translates combinatorially the integration!

Bipartite Graphs: the strategy (I)

Can we apply the same decomposition for bipartite graphs?

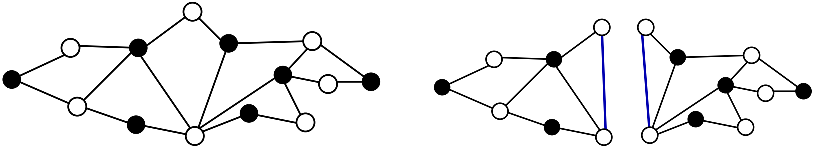
1-sums are easy!



The 2-connected components are also bipartite

Bipartite graphs: the strategy (II)

For 2-sums we have problems



We need to study something more general **Ising Model.**

Bipartite graphs: the strategy (III)

PROBLEM: going from 3-connected level to 2-connected level.

Networks in the general case:

$$\begin{cases} D(x,y) = y + S(x,y) + P(x,y) + H(x,y) \\ S(x,y) = D(x,y)x\,(D(x,y) - S(x,y)) \\ P(x,y) = (1+y)\,(\exp(S(x,y)+H(x,y)) - 1 - S(x,y) - H(x,y)) \\ H(x,y) = \frac{2}{x^2} T_y(x, D(x,y)). \end{cases}$$

Networks in the Ising model:

$$\begin{cases} S_{\circ-\bullet} = xD_{\circ-\bullet}\frac{x\left(D_{\circ-\circ}^2 - D_{\circ-\bullet}^2\right)+2D_{\circ-\circ}}{(1+x(D_{\circ-\circ}+D_{\circ-\bullet}))(1+x(D_{\circ-\circ}-D_{\circ-\bullet}))} \\ S_{\circ-\circ} = x\frac{D_{\circ-\circ}^2+D_{\circ-\bullet}^2+D_{\circ-\circ}^3-xD_{\circ-\circ}D_{\circ-\bullet}^2}{(1+x(D_{\circ-\circ}+D_{\circ-\bullet}))(1+x(D_{\circ-\circ}-D_{\circ-\bullet}))} \end{cases}$$

and

$$2(1+y_{\circ-\bullet})\frac{\partial}{\partial y_{\circ-\bullet}}B(x,y_{\circ-\bullet},y_{\circ-\circ}) + 2(1+y_{\circ-\circ})\frac{\partial}{\partial y_{\circ-\circ}}B(x,y_{\circ-\bullet},y_{\circ-\circ}) = x^2(1+D_{\circ-\circ}+D_{\circ-\bullet})$$

We do not have any choice: Combinatorial Integration!

The Program (Coming soon!)

One needs to rephrase the grammar for graphs including the colours.

Once we have this (+ Singularity analysis), we can:

i.- Study SP-graphs.

ii.- Study families of graphs defined by "easy" 3-connected components.

ii.- Study limit laws for several parameters

What we **CANNOT** do (for the moment!):

STUDY GENERAL
PLANAR BIPARTITE PLANAR GRAPHS

$\Updownarrow$

OBTAIN GF FOR 3-CONNECTED MAPS
(BERNARDI & BOUSQUET-MÉLOU)

Merci

Bipartite subfamilies of planar graphs

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