An algebraic Birkhoff decomposition for the continuous renormalization group

P. Martinetti

Università di Roma *Tor Vergata* and CMTP

Séminaire CALIN, LIPN Paris 13, 8^{th} February 2011

What is the algebraic (geometric) structure underlying renormalization?

- Perturbative renormalization in qft is a Birkhoff decomposition
  $\longrightarrow$ Hopf algebra of Feynman diagrams.(*Connes-Kreimer* 2000)

- Exact renormalization is an *algebraic* Birkhoff decomposition
  $\longrightarrow$ Hopf algebra of decorated rooted trees.

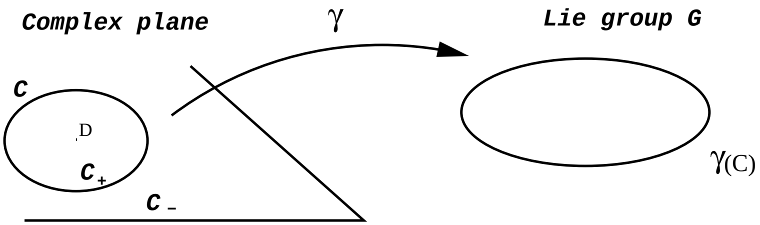
Program

- Birkhoff decomposition
- Exact Renormalization Group equations as fixed point equation
- Power series of trees
- Algebraic Birkhoff decomposition for the ERG

Algebraic Birkhoff decomposition for the continuous renormalization group, with F. Girelli and T. Krajewski, J. Math. Phys. **45** (2004) 4679-4697.

Wilsonian renormalization, differential equations and Hopf algebras, with T. Krajewski, to appear in Contemporary Mathematics Series of the AMS.

Birkhoff decomposition



$$\boxed{\gamma(z) = \gamma_-^{-1}(z)\gamma_+(z), \quad z \in \mathcal{C}}$$ where $\gamma_\pm : \mathcal{C}_\pm \to G$ are holomorphic.

$\to$ G nice enough: exists for any loop γ, unique assuming $\gamma_-(\infty) = 1$.

$\to$ γ defined on $\mathcal{C}_+$ with pole at D:

$$\gamma \to \gamma_+(D)$$

is a natural principle to extract finite value from singular expression $\gamma(D)$.

$\to$ dimensional regularization in QFT: D is the dimension of space time, G is the group of characters of the Hopf algebra of Feynman diagrams.

Birkhoff decomposition: Hopf algebra of Feynman diagrams

Coalgebra $\mathcal{C}_o$: reverse the arrow !

Coproduct $\Delta : \mathcal{C}_0 \mapsto \mathcal{C}_0 \otimes \mathcal{C}_0$, counity $\eta : \mathcal{C}_0 \mapsto \mathbb{C}$,

$$\begin{array}{ccc}
\mathcal{C}_o \otimes \mathcal{C}_o \otimes \mathcal{C}_o & \xleftarrow{\Delta \otimes \mathrm{id}_{\mathcal{C}}} & \mathcal{C}_o \otimes \mathcal{C}_o \\
\Big\uparrow {\scriptstyle \mathrm{id}_{\mathcal{C}} \otimes \Delta} & & \Big\uparrow {\scriptstyle \Delta} \\
\mathcal{C}_o \otimes \mathcal{C}_o & \xleftarrow{\Delta} & \mathcal{C}_o
\end{array}$$

$$\begin{array}{ccc}
\mathbb{C} \otimes \mathcal{C}_o & \xleftarrow{\eta \otimes \mathrm{id}_{\mathcal{C}}} & \mathcal{C}_o \otimes \mathcal{C}_o \\
\Big\updownarrow & & \Big\uparrow {\scriptstyle \Delta} \\
\mathcal{C}_o & \xleftarrow{\mathrm{id}_{\mathcal{C}}} & \mathcal{C}_o
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{C}_o \otimes \mathbb{C} & \xleftarrow{\mathrm{id}_{\mathcal{C}} \otimes \eta} & \mathcal{C}_o \otimes \mathcal{C}_o \\
\Big\updownarrow & & \Big\uparrow {\scriptstyle \Delta} \\
\mathcal{C}_o & \xleftarrow{\mathrm{id}_{\mathcal{C}}} & \mathcal{C}_o
\end{array}$$

Birkhoff decomposition: Hopf algebra of Feynman diagrams

Bialgebra $\mathcal{B}$: algebra + coalgebra.

Antipode $S : \mathcal{B} \mapsto \mathcal{B}$,

$$\mathrm{id}_{\mathcal{B}} * S \doteq m(\mathrm{id}_{\mathcal{B}} \otimes S)\Delta = \eta 1, \quad S * \mathrm{id}_{\mathcal{B}} \doteq m(S * \mathrm{id}_{\mathcal{B}})\Delta = \eta 1.$$

Bialgebra with antipode = Hopf algebra $\mathcal{H}$.

$\rightarrow$ 1PI-Feynman diagrams form an Hopf algebra,

$\rightarrow$ Combinatorics of perturbative renormalization is encoded within the coproduct Δ.

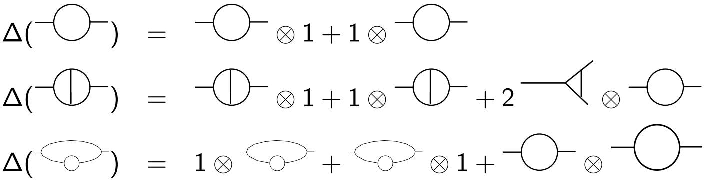
Birkhoff decomposition: Hopf algebra of Feynman diagrams

<u>The Hopf algebra H_F of Feynman diagrams:</u>

Algebra structure:
-product: disjoint union of graphs,
-unity: the empty set.

Hopf algebra structure:
-counity: $\eta(\emptyset) = 1$, $\eta(\Gamma) = 0$ otherwise,
-coproduct:

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subsetneq \Gamma} \gamma \otimes \Gamma/\gamma$$



-antipode: built by induction.

Birkhoff decomposition: perturbative renormalization

$\mathcal{A}$: complex functions in $\mathbb{C}$, pole in D (=4).
$\mathcal{A}_+$: holomorphic functions in $\mathbb{C}$.
$\mathcal{A}_-$: polynômial in $\frac{1}{z-D}$ without constant term.

$$\begin{cases} \text{Feynman rules} : H_F \overset{U}{\Longrightarrow} \mathcal{A} \\ \text{Conterterms} : H_F \overset{C}{\Longrightarrow} \mathcal{A}_- \\ \text{Renormalized theory} : H_F \overset{R}{\Longrightarrow} \mathcal{A}_+ \end{cases}$$

$$\boxed{C * U = R}$$

Compose with character χ_z of $\mathcal{A}$,

$$\gamma(z) \doteq \chi_z \circ U, \quad \gamma_-(z) \doteq \chi_z \circ C, \quad \gamma_+(z) \doteq \chi_z \circ R,$$

$\gamma(z)$, $z \in \mathcal{C}$ is a loop within the group G of characters of H_F,

$$\gamma(z) = \gamma_-^{-1}(z)\,\gamma_+(z).$$

The renormalized theory is the evaluation at D of the positive part of the Birkhoff decomposition of the bare theory.

Birkhoff decomposition: algebraic formulation

The Exact Renormalization Group equations govern the evolution of the parameters of the theory with respect to the scale of observation (e.g. energie Λ),

$$\Lambda \frac{\partial}{\partial \Lambda} S = \beta(\Lambda, S)$$

where $S(\Lambda) \in \mathcal{E}$, vector space of "actions".

- no analogous to the dimension D where to localize the pole
- analogous to $C * U = R$.

Definition(*Connes, Kreimer, Kastler*): H commutative Hopf algebra, $\mathcal{A}$ commutative algebra. p_- projection onto a subalgebra $\mathcal{A}_-$.
An algebra morphism $\gamma : H \to \mathcal{A}$ has a unique algebraic Birkhoff decomposition if there exist two algebra morphisms γ_+, γ_- from H to $\mathcal{A}$ such that

$$\gamma_+ = \gamma_- * \gamma$$

$$p_+ \gamma_+ = \gamma_+, \quad p_- \gamma_- = \gamma_-$$

with p_+ the projection on

$$\mathcal{A}_+ = \text{Ker } p_-.$$

ERG as fixed point equation

Dimensional analysis : $\Lambda \to t$, $S \to x$, $\beta \mapsto X$,

$$\frac{\partial x}{\partial t} = Dx + X(x)$$

$x(t) \in \mathcal{E}$, D diagonal matrix of dimensions, X smooth operator $\mathcal{E} \to \mathcal{E}$,

$$X(x+y) = X(x) + X'_x(y) + X''_x(y,y) + \ldots + \frac{1}{n!} X_x^{[n]}(y, \ldots, y) + \mathcal{O}(\|y\|^{n+1})$$

where $X_x^{[n]}$ is a linear symmetric application from $\mathcal{E}^{[n]}$ to $\mathcal{E}$.

$$x(t) = e^{(t-t_0)D} x_0 + \int_{t_0}^{t} e^{(t-u)D} X(x(u)) du.$$

x belongs to the space $\tilde{\mathcal{E}}$ of smooth maps from $\mathbb{R}^{*+}$ to $\mathcal{E}$, as well as

$$\tilde{x}_0 : \; t \mapsto e^{(t-t_0)D} x_0.$$

Define χ_0, smooth map from $\tilde{\mathcal{E}}$ to $\tilde{\mathcal{E}}$,

$$\chi_0(x) : t \mapsto \int_{t_0}^{t} e^{(t-u)D} X(x(u)) du.$$

ERG as fixed point equation

Fixed point equation

$$\boxed{x = \tilde{x}_0 + \chi_0(x)}$$

- $x(t)$ represents the parameters at a scale t.
- $\tilde{x}_0$ encodes the initial conditions at a fixed scale t_0.

Wilson's ERG context: t_0 is an UV cutoff. One interested in $t_0 \to +\infty$.

ERG as fixed point equation: mixed initial conditions

$$\tilde{x}_0(t) = e^{(t-t_0)D} x_0 \begin{cases} \text{converges on } \mathcal{E}^+ \\ \text{is constantly zero on } \mathcal{E}^0 \quad \text{as } t_0 \to +\infty \\ \text{diverges on } \mathcal{E}^- \end{cases}$$

where $\mathcal{E}^+$, $\mathcal{E}^0$, $\mathcal{E}^-$ are proper subspaces of D corresponding to positive, zero and negative eigenvalues (*irrelevant*, *marginal*, *relevant*).

- Finiteness of $x(t)$ at high scale by imposing initial conditions for relevant sector at scale $t_1 \neq t_0$.
- P orthogonal projection $\mathcal{E} \mapsto \mathcal{E}^-$ allows mixed initial conditions

$$x_R \doteq P\tilde{x}_1 + (\mathbb{I} - P)\tilde{x}_0 :$$

- $\chi_R \doteq P\chi_1 + (\mathbb{I} - P)\chi_0$ with $\chi_i(x) : t \mapsto \int_{t_i}^{t} e^{(t-u)D} X(x(u)) du$

$$\boxed{x(t) = x_R + \chi_R(x)}$$

Renormalization deals with change of initial condition in fixed point equation.

Power series of trees: smooth non linear operators

χ is a smooth operator from $\tilde{\mathcal{E}}$ to $\tilde{\mathcal{E}}$:

$$\chi(x+y) = \chi(x) + \chi'_x(y) + \chi''_x(y,y) + ... + \frac{1}{n!}\chi_x^{[n]}(y,...,y) + \mathcal{O}(\|y\|^{n+1})$$

where $\chi_x^{[n]}$ is a linear symmetric application from $\tilde{\mathcal{E}}^{[n]}$ to $\tilde{\mathcal{E}}$.

- Physicists' notations: $x = \{x^\mu\}$, $\chi(x) = \{\chi^\mu(x)\}$,

$$\chi'_x(y) = \partial_\nu \chi^\mu_{/x}\, y^\nu, \quad \chi''_x(y_1,y_2) = \partial_{\nu\rho}\chi^\mu_{/x}\, y_1^\nu y_2^\rho.$$

- Coordinate free notations: $\chi'(\chi)$ is the map $\tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$

$$y \mapsto \chi'_y(\chi(y)).$$

Power series of trees: smooth non linear operators

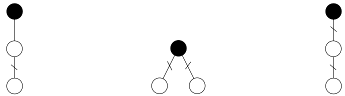
$$\chi^{\emptyset} \doteq \mathbb{I}, \quad \chi^{\bullet} \doteq \chi, \quad \chi^{\overset{\bullet}{\circ}} \doteq \chi'(\chi), \quad \chi^{\overset{\bullet}{\circ\ \circ}} \doteq \frac{1}{2}\chi''(\chi,\chi)\dots$$

<u>Taylor expansion</u>:

$$\begin{aligned} \chi(\mathbb{I}+\chi) &= \chi^{\bullet} + \chi^{\overset{\bullet}{\circ}} + \chi^{\overset{\bullet}{\circ\ \circ}} + \dots \\ &= \sum_T \phi(T)\chi^T \\ &= f_\phi[\chi] \end{aligned}$$

where $\phi(T) = 1$ for any rooted tree T, except $\phi(\emptyset) = 0$.

Power series of trees: characters of the Hopf algebra



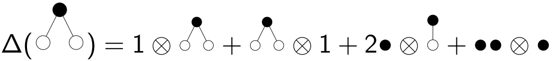
simple cut   simple cut   non simple cut

H_T is a Hopf algebra with counit $\epsilon = 0$ except $\epsilon(1) = 1$, the antipode

$$S : \quad \bullet \mapsto \quad -\bullet$$
$$T \mapsto \quad -T - \sum_{c \in C(T)} S(P_c(T)) R_c(T)$$

and the coproduct

$$\Delta(T) = T \otimes 1 + 1 \otimes T + \sum_{c \in C(T)} P_c(T) \otimes R_c(T), \quad \Delta(1) = 1 \otimes 1.$$



Proposition: *Butcher group, B-series; T.K, P.M.*:

The group of formal power series starting with $\mathbb{I}$ (i.e. $\phi(\emptyset) = 1$) is isomorphic to the opposite group of characters of H_T.

$$\begin{aligned} f_\phi[\chi] \circ f_\psi[\chi] &= \sum_T \phi(T)\chi^T \left(\sum_{T'} \psi(T')\chi^{T'} \hbar^{|T'|} \right) \hbar^{|T|} \\ &= \sum_T (\psi * \phi)(T)\chi^T \hbar^{|T|} \\ &= f_{\psi * \phi}[\chi]. \end{aligned}$$

Power series of trees: solution of fixed point equation

- $x = x_0 + \chi_0(x) \iff x_0 = (\mathbb{I} - \chi_0)(x)$.

$$\boxed{x = (\mathbb{I} - \chi_0)^{-1}(x_0) = f_\varphi[\chi_0]^{-1}(x_0) = f_{\phi_1}[\chi_0](x_0)}$$

where $\varphi = 0$ except $\varphi(\emptyset) = 1$, $\varphi(\bullet) = -1$ and $\phi_1 = \varphi^{-1} = 1$.

- $x = x_R + \chi_R(x) \Longrightarrow x = f_{\phi_1}[\chi_R](x_R)$
- $\xi \doteq \mathbb{I} - (\mathbb{I} - \chi_R) \circ (\mathbb{I} - \chi_0)^{-1} \Longrightarrow (\mathbb{I} - \chi_R)^{-1} = (\mathbb{I} - \chi_0)^{-1} \circ (\mathbb{I} - \xi)^{-1}$

Power series of trees: rooted trees with two decorations

$$f_{\phi_1}[\chi_R] = f_{\phi_1}[\chi_0] \circ f_{\phi_1}[\xi]$$

1 character, 2 operators $\Longleftrightarrow$ 1 operator, 2 characters :

$$f_{\phi_+}[Y] = f_{\phi}[Y] \circ f_{\phi_-}[Y]$$

$$Y^{\blacksquare} = \chi_R, \quad Y^{\bullet} = \xi, \quad Y^{\blacksquare_{\circ\circ}} = \chi_R''(\xi, \xi),$$

$$\phi \doteq \phi_-^{-1} * \phi_+$$

$$\phi_-(\mathcal{T}) \doteq \begin{cases} \phi_1(\mathcal{T}) \text{ if } \mathcal{T} \in H_\bullet \\ 0 \text{ if } \mathcal{T} \notin H_\bullet, \end{cases} \quad , \quad \phi_+(\mathcal{T}) \doteq \begin{cases} \phi_1(\mathcal{T}) \text{ if } \mathcal{T} \in H_\blacksquare \\ 0 \text{ if } \mathcal{T} \notin H_\blacksquare \end{cases}$$

where $H_\bullet$, $H_\blacksquare$ are the set of trees decorated with one decoration only so that

$$f_{\phi_1}[\chi_R] = f_{\phi_+}[Y], \quad f_{\phi_1}[\xi] = f_{\phi_-}[Y]$$

Proposition: $\lim_{t_0 \to +\infty} f_{\phi_1}[\chi_R](x_R)$ is finite order by order and does not depend on x_0.

Algebraic Birkhoff decomposition for the ERG

Perturbative renormalization: $H_F \xrightarrow{\text{Feyman rules}} \mathcal{A} \xrightarrow{\text{evaluation at } z} G$.
Exact renormalization: $H_{\mathcal{T}} \xrightarrow{\text{evaluation on decorations}} G$.

→ No Birkhoff decomposition since no loop in G.
→ Algebraic Birkhoff decomposition on which algebra ?

As U, C, R map a Feynman diagram to a meromorphic funtion, characters map a decorated rooted tree to a monomial in $Y^{\mathcal{T}}$,

$$\gamma(\mathcal{T}) \doteq \phi(\mathcal{T}) Y^{\mathcal{T}}, \; \gamma_{\pm}(\mathcal{T}) \doteq \phi_{\pm}(\mathcal{T}) Y^{\mathcal{T}}.$$

Unfortunately γ, $\gamma_{\pm}$ do not define an algebraic Birkhoff decomposition.

$$\gamma_{+}(\overset{\blacksquare}{\circ}) = 0$$

$$(\gamma_{-} * \gamma)(\overset{\blacksquare}{\circ}) = \langle \gamma_{-} \otimes \gamma, 1 \otimes \overset{\blacksquare}{\circ} + \overset{\blacksquare}{\circ} \otimes 1 + \bullet \otimes \blacksquare \rangle$$

essaibirk $Y^{\overset{\blacksquare}{\circ}} + Y^{\circ} Y^{\blacksquare}$.

Algebraic Birkhoff decomposition for the ERG

→ Algebraic Birkhoff decomposition with

- target

$$\mathcal{A} = \overline{\{1, \bullet, \blacksquare\}}, \quad \mathcal{A}_- = \overline{\{1, \bullet\}}.$$

- projection $p_- : \mathcal{A} \to \mathcal{A}_-$

$$p_-(1) = 1, \quad p_-(\bullet) = \bullet, \quad p_-(\blacksquare) = 0.$$

- Algebra homorphism $H_{\mathcal{T}} \to \mathcal{A}$

$$\gamma(\mathcal{T}) = \phi(\mathcal{T})\Gamma(\mathcal{T}),\ \gamma_\pm(\mathcal{T}) = \phi_\pm(\mathcal{T})\Gamma(\mathcal{T}).$$

where $\phi = \phi_-^{-1} * \phi_+$ and Γ counts the decoration

$\Gamma($  $) = \bullet^3\blacksquare$

$$\boxed{\gamma_+ = \gamma_- * \gamma}$$

Conclusion

Perturbative renormalization with dimensional regularization has a nice description in terms of Birkhoff decomposition of a loop around the dimension D of space time

- geometrical interpretation (bundles on the Riemann sphere),
- Galois theory for the renormalization group (Connes, Marcolli).

Analogous formulation for ERG, only at the algebraic level

- Is the algebra of decorations an artificial tool ?
- Deeper structure (Rota-Baxter operator, cf Ebrahimi-Fard) ?
- Signification of the characters ?