The Saddle point method in combinatorics asymptotic analysis: successes and failures (A personal view)

Guy Louchard

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Outline

The number of inversions in permutations

Let $a_1 \dots a_n$ be a permutation of the set $\{1, \dots, n\}$. If $a_i > a_k$ and $i < k$, the pair (a_i, a_k) is called an inversion; $I_n(j)$ is the number of permutations of length n with j inversions.

Here, we show how to extend previous results using the saddle point method. This leads, e. g., to asymptotics for $I_{\alpha n+\beta}(\gamma n + \delta)$, for integer constants α, β, γ, δ and more general ones as well.

With this technique, we will also show the known result that $I_n(j)$ is asymptotically normal, with additional corrections.

The generating function for the numbers $I_n(j)$ is given by

$$\Phi_n(z) = \sum_{j \geq 0} I_n(j) z^j = (1-z)^{-n} \prod_{i=1}^{n} (1 - z^i).$$

By Cauchy's theorem,

$$I_n(j) = \frac{1}{2\pi i} \int_{\mathcal{C}} \Phi_n(z) \frac{dz}{z^{j+1}},$$

where $\mathcal{C}$ is, say, a circle around the origin.

The Gaussian limit, $j = m + x\sigma$, $m = n(n-1)/4$

Actually, we obtain here local limit theorems with some corrections (=lower order terms).
The Gaussian limit of $I_n(j)$ is easily derived from the generating function $\Phi_n(z)$ (using the Lindeberg-Lévy conditions. Indeed, this generating function corresponds to a sum for $i = 1, \ldots, n$ of independent, uniform $[0..i-1]$ random variables. As an exercise, let us recover this result with the saddle point method, with an additional correction of order $1/n$. We have, for the random variable X_n characterized by

$$\mathbb{P}(X_n = j) = J_n(j),$$

with $J_n := I_n/n!$,

$$m := \mathbb{E}(X_n) = n(n-1)/4,$$
$$\sigma^2 := \mathbb{V}(X_n) = n(2n+5)(n-1)/72.$$

We know that

$$I_n(j) = \frac{1}{2\pi i} \int_\Omega e^{S(z)} dz$$

where Ω is inside the analyticity domain of the integrand, encircles the origin, passes through the saddle point $\tilde{z}$ and

$$S = \ln(\Phi_n(z)) - (j+1) \ln z,$$

$\tilde{z}$ is the solution of

$$S^{(1)}(\tilde{z}) = 0. \qquad (1)$$

Figure 1 shows the real part of $S(z)$ together with a path Ω through the saddle point.

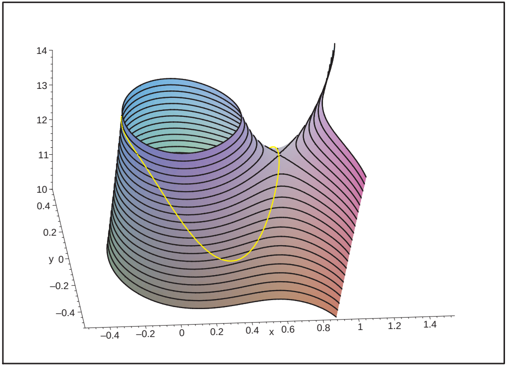

Figure 1: Real part of $S(z)$. Saddle-point and path, $n = 10$

We have

$$J_n(j) = \frac{1}{n!2\pi i} \int_\Omega \exp\left[S(\tilde{z}) + S^{(2)}(\tilde{z})(z-\tilde{z})^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde{z})(z-\tilde{z})^l/l!\right] dz$$

(note carefully that the linear term vanishes). Set $z = \tilde{z} + i\tau$. This gives

$$J_n(j) = \frac{1}{n!2\pi} \exp[S(\tilde{z})] \int_{-\infty}^{\infty} \exp\left[S^{(2)}(\tilde{z})(i\tau)^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde{z})(i\tau)^l/l!\right] d\tau. \quad (2)$$

We can now compute (2), for instance by using the classical trick of setting

$$S^{(2)}(\tilde{z})(i\tau)^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde{z})(i\tau)^l/l! = -u^2/2.$$

To justify this procedure, we proceed in three steps (to simplify, we use $\tilde{z} = 1$).

- setting $z = e^{\mathbf{i}\theta}$, we must show that the tail integral

$$\int_{\theta_0}^{2\pi-\theta_0} e^{S(z)} d\theta$$

 is negligible for some θ_0,

- we must insure that a central Gaussian approximation holds:

$$S(z) \sim S(\tilde{z}) + S^{(2)}(\tilde{z})(z-\tilde{z})^2/2!$$

 in the integration domain $|\theta| \leq \theta_0$, by chosing for instance $S^{(2)}(\tilde{z})\theta_0^2 \to \infty, S^{(3)}(\tilde{z})\theta_0^3 \to 0, n \to \infty$,

- me must have a tail completion: the incomplete Gaussian integral must be asymptotic to a complete one.

We split the exponent of the integrand as

$$\begin{aligned} S &:= S_1 + S_2, \quad (3)\\ S_1 &:= \sum_{i=1}^{n} \ln(1 - z^i),\\ S_2 &:= -n \ln(1 - z) - (j + 1) \ln z. \end{aligned}$$

Set

$$S^{(i)} := \frac{d^i S}{dz^i}.$$

Set $\tilde{z} := z^* - \varepsilon$, where $z^* = \lim_{n\to\infty} \tilde{z}$. Here, $z^* = 1$. (This notation always means that z^* is the approximate saddle point and $\tilde{z}$ is the exact saddle point; they differ by a quantity that has to be computed to some degree of accuracy.) This leads, to first order, to

$$\begin{aligned} &(n+1)^2/4 - 3n/4 - 5/4 - j]\\ &+ [-(n+1)^3/36 + 7(n+1)^2/24 - 49n/72 - 91/72 - j]\varepsilon = 0. \end{aligned} \quad (4)$$

Set $j = m + x\sigma$ in (4). This shows that, asymptotically, ε is given by a Puiseux series of powers of $n^{-1/2}$, starting with $-6x/n^{3/2}$. To obtain the next terms, we compute the next terms in the expansion of (1), i.e., we first obtain

$$\begin{aligned}
&[(n+1)^2/4 - 3n/4 - 5/4 - j] \\
&+ [-(n+1)^3/36 + 7(n+1)^2/24 - 49n/72 - 91/72 - j]\varepsilon \\
&+ [-j - 61/48 - (n+1)^3/24 + 5(n+1)^2/16 - 31n/48]\varepsilon^2 = 0.
\end{aligned} \tag{5}$$

More generally, even powers ε^{2k} lead to a $\mathcal{O}(n^{2k+1}) \cdot \varepsilon^{2k}$ term and odd powers ε^{2k+1} lead to a $\mathcal{O}(n^{2k+3}) \cdot \varepsilon^{2k+1}$ term. Now we set $j = m + x\sigma$, expand into powers of $n^{-1/2}$ and equate each coefficient with 0. This leads successively to a full expansion of ε. Note that to obtain a given precision of ε, it is enough to compute a given finite number of terms in the generalization of (5). We obtain

$$\varepsilon = -6x/n^{3/2} + (9x/2 - 54/25x^3)/n^{5/2} - (18x^2 + 36)/n^3 + x[-30942/30625x^4 + 27/10x^2 - 201/16]/n^{7/2} + \mathcal{O}(1/n^4). \quad (6)$$

Let us first analyze $S(\tilde{z})$. We obtain

$$
\begin{aligned}
S_1(\tilde{z}) &= \sum_{i=1}^{n} \ln(i) + [-3/2\ln(n) + \ln(6) + \ln(-x)]n \\
&\quad +3/2x\sqrt{n} + 43/50x^2 - 3/4 \\
&\quad + [3x/8 + 6/x + 27/50x^3]/\sqrt{n} \\
&\quad +[5679/12250x^4 - 9/50x^2 + 173/16]/n + \mathcal{O}(n^{-3/2}), \\
S_2(\tilde{z}) &= [3/2\ln(n) - \ln(6) - \ln(-x)]n - 3/2x\sqrt{n} - 34/25x^2 + 3/4 \\
&\quad - [3x/8 + 6/x + 27/50x^3]/\sqrt{n} \\
&\quad -[5679/12250x^4 - 9/50x^2 + 173/16]/n + \mathcal{O}(n^{-3/2}),
\end{aligned}
$$

and so

$$S(\tilde{z}) = -x^2/2 + \ln(n!) + \mathcal{O}(n^{-3/2}).$$

Also,

$$\begin{aligned} S^{(2)}(\tilde{z}) &= n^3/36 + (1/24 - 3/100x^2)n^2 + \mathcal{O}(n^{3/2}), \\ S^{(3)}(\tilde{z}) &= \mathcal{O}(n^{7/2}), \\ S^{(4)}(\tilde{z}) &= -n^5/600 + \mathcal{O}(n^4), \\ S^{(l)}(\tilde{z}) &= \mathcal{O}(n^{l+1}), \quad l \geq 5. \end{aligned}$$

We compute τ as a truncated series in u, setting $d\tau = \frac{d\tau}{du}du$, expanding w.r.t. n and integrating on $[u = -\infty..\infty]$. This amounts to the reversion of a series. Finally (2) leads to

$$J_n \sim e^{-x^2/2} \cdot \exp\left[\left(-51/50 + 27/50x^2\right)/n + \mathcal{O}(n^{-3/2})\right]/(2\pi n^3/36)^{1/2}. \tag{7}$$

Note that $S^{(3)}(\tilde{z})$ does not contribute to the $1/n$ correction.

To check the effect of the correction, we first give in Figure 2, for $n = 60$, the comparison between $J_n(j)$ and the asymptotics (7), without the $1/n$ term. Figure 3 gives the same comparison, with the constant term $-51/(50n)$ in the correction. Figure 4 shows the quotient of $J_n(j)$ and the asymptotics (7), with the constant term $-51/(50n)$. The "hat" behaviour, already noticed by Margolius, is apparent. Finally, Figure 5 shows the quotient of $J_n(j)$ and the asymptotics (7), with the full correction.

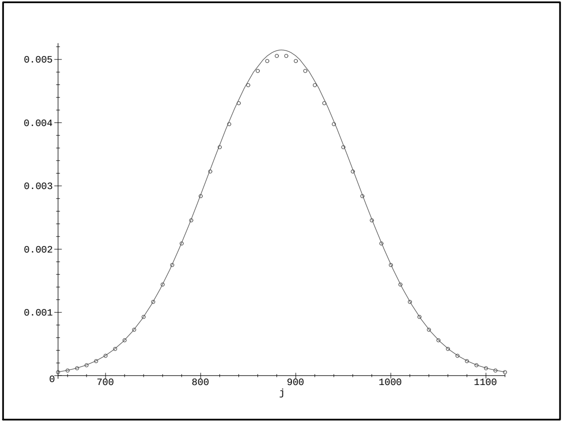

Figure 2: $J_n(j)$ (circle) and the asymptotics (7) (line), without the $1/n$ term, $n = 60$

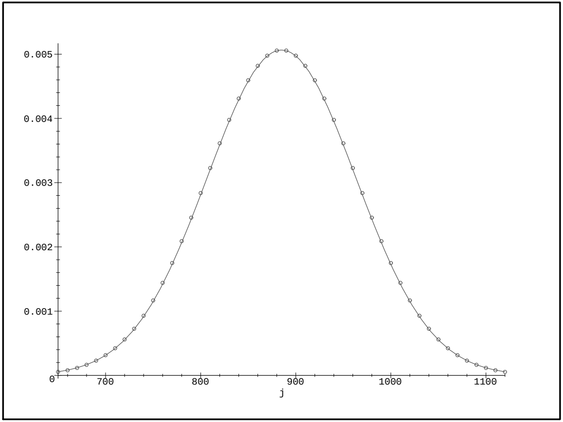

Figure 3: $J_n(j)$ (circle) and the asymptotics (7) (line), with the constant in the $1/n$ term, $n = 60$

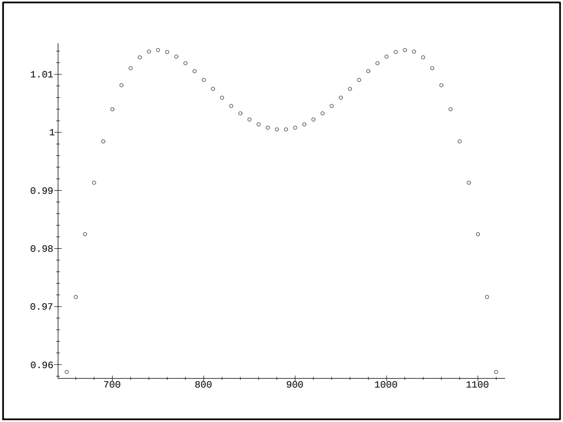

Figure 4: Quotient of $J_n(j)$ and the asymptotics (7), with the constant in the $1/n$ term, $n = 60$

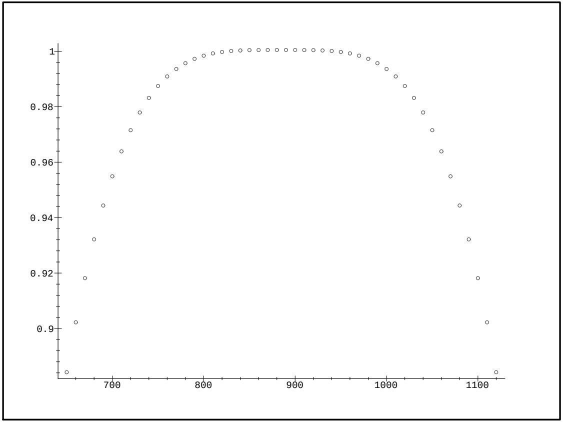

Figure 5: Quotient of $J_n(j)$ and the asymptotics (7), with the full $1/n$ term, $n = 60$

Case $j = n - k$

It is easy to see that here, we have $z^* = 1/2$. We obtain, to first order,

$$[C_{1,n} - 2j - 2 + 2n] + [C_{2,n} - 4j - 4 - 4n]\varepsilon = 0$$

with

$$\begin{aligned}
C_{1,n} &= C_1 + \mathcal{O}(2^{-n}),\\
C_1 &= \sum_{i=1}^{\infty} \frac{-2i}{2^i - 1} = -5.48806777751\ldots,\\
C_{2,n} &= C_2 + \mathcal{O}(2^{-n}),\\
C_2 &= \sum_{i=1}^{\infty} 4\frac{i(i2^i - 2^i + 1)}{(2^i - 1)^2} = 24.3761367267\ldots.
\end{aligned}$$

Set $j = n - k$. This shows that, asymptotically, ε is given by a Laurent series of powers of n^{-1}, starting with $(k - 1 + C_1/2)/(4n)$. We next obtain

$$[C_1 - 2j - 2 + 2n] + [C_2 - 4j - 4 - 4n]\varepsilon + [C_3 + 8n - 8j - 8]\varepsilon^2 = 0$$

for some constant C_3. More generally, powers ε^{2k} lead to a $\mathcal{O}(1) \cdot \varepsilon^{2k}$ term, powers ε^{2k+1} lead to a $\mathcal{O}(n) \cdot \varepsilon^{2k+1}$ term. This gives

$$\varepsilon = (k-1+C_1/2)/(4n)+(2k-2+C_1)(4k-4+C_2)/(64n^2)+\mathcal{O}(1/n^3).$$

Now we derive

$$S_1(\tilde{z}) = \ln(Q) - C_1(k - 1 + C_1/2)/(4n) + \mathcal{O}(1/n^2)$$

with $Q := \prod_{i=1}^{\infty}(1 - 1/2^i) = .288788095086\ldots$. Similarly

$$S_2(\tilde{z}) = 2\ln(2)n+(1-k)\ln(2)+(-k^2/2+k-1/2+C_1^2/8)/(2n)+\mathcal{O}(1/n^2)$$

and so

$$S(\tilde{z}) = \ln(Q) + 2\ln(2)n + (1-k)\ln(2) + (A_0 + A_1 k - k^2/4)/n + \mathcal{O}(1/n^2)$$

with

$$\begin{aligned} A_0 &:= -(C_1-2)^2/16, \\ A_1 &:= (-C_1/2+1)/2. \end{aligned}$$

Finally, we obtain

$$I_n(n-k) \sim e^{2\ln(2)n+(1-k)\ln(2)} \frac{Q}{(2\pi S^{(2,1)})^{1/2}} \times$$
$$\times \exp\left\{\left[(A_0 + 1/8 + C_2/16) + (A_1 + 1/4)k - k^2/4\right]/n + \mathcal{O}(1/n^2)\right\}. \tag{8}$$

Figure 6 shows, for $n = 300$, the quotient of $I_n(n-k)$ and the asymptotics (8).

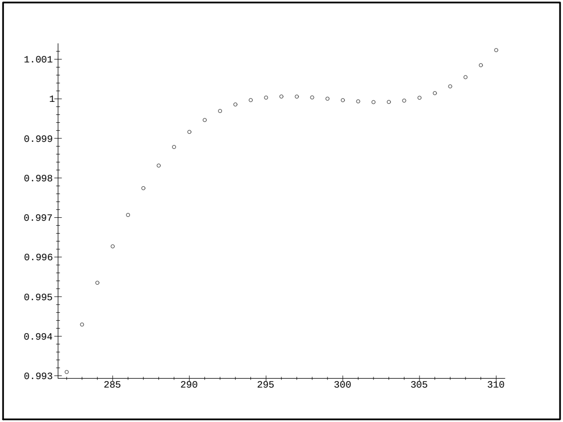

Figure 6: Quotient of $I_n(n-k)$ and the asymptotics (8), $n = 300$

Case $j = \alpha n - x$, $\alpha > 0$

Of course, we must have that $\alpha n - x$ is an integer. For instance, we can choose α, x integers. But this also covers more general cases, for instance $I_{\alpha n+\beta}(\gamma n + \delta)$, with α, β, γ, δ integers. We have here $z^* = \alpha/(1+\alpha)$. We derive, to first order,

$$\begin{aligned} & C_{1,n}(\alpha) - (j+1)(1+\alpha)/\alpha + (1+\alpha)n] \\ & + [C_{2,n}(\alpha) - (j+1)(1+\alpha)^2/\alpha^2 - (1+\alpha)^2 n]\varepsilon = 0 \end{aligned}$$

with, setting $\varphi(i,\alpha) := [\alpha/(1+\alpha)]^i$,

$$\begin{aligned} C_{1,n}(\alpha) &= C_1(\alpha) + \mathcal{O}([\alpha/(1+\alpha)]^{-n}), \\ C_1(\alpha) &= \sum_{i=1}^{\infty} \frac{i(1+\alpha)\varphi(i,\alpha)}{\alpha[\varphi(i,\alpha)-1]}, \\ C_{2,n}(\alpha) &= C_2(\alpha) + \mathcal{O}([\alpha/(1+\alpha)]^{-n}), \\ C_2(\alpha) &= \sum_{i=1}^{\infty} \varphi(i,\alpha) i (1+\alpha)^2 (i-1+\varphi(i,\alpha))/[(\varphi(i,\alpha)-1)^2\alpha^2]. \end{aligned}$$

$$\hat{Q}(\alpha) := \prod_{i=1}^{\infty}(1-\varphi(i,\alpha)) = \prod_{i=1}^{\infty}\left(1-\left(\frac{\alpha}{1+\alpha}\right)^{i}\right) = Q\left(\frac{\alpha}{1+\alpha}\right).$$

We finally derive

$$\begin{aligned}
&I_n(\alpha n - x) \sim e^{[-\ln(1/(1+\alpha))-\alpha\ln(\alpha/(1+\alpha))]n+(x-1)\ln(\alpha/(1+\alpha))} \times \\
&\frac{\hat{Q}(\alpha)}{(2\pi S^{(2,1)})^{1/2}} \exp\left[\left\{-(1+3\alpha+4\alpha^2-12\alpha^2 C_1+6C_1^2\alpha^2\right.\right. \\
&+\alpha^4+3\alpha^3-6C_2\alpha^2-12C_1^3\alpha)/[12\alpha(1+\alpha)^3] \\
&+x(2\alpha^2-2C_1\alpha+3\alpha+1)/[2\alpha(1+\alpha)^2] \\
&\left.\left.-x^2/[2\alpha(1+\alpha)]\right\}/n+\mathcal{O}(1/n^2)\right]. \qquad (9)
\end{aligned}$$

Figure 7 shows, for $\alpha = 1/2$, $n = 300$, the quotient of $I_n(\alpha n - x)$ and the asymptotics (9).

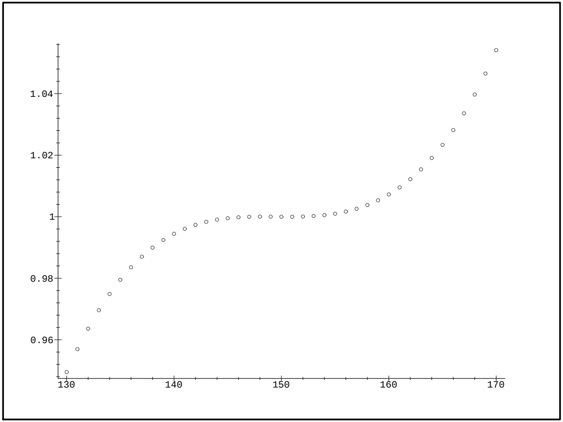

Figure 7: Quotient of $I_n(\alpha n - x)$ and the asymptotics (9), $\alpha = 1/2$, $n = 300$

The moderate Large deviation, $j = m + xn^{7/4}$

Now we consider the case $j = m + xn^{7/4}$. We have here $z^* = 1$.
We observe the same behaviour as in in the Gaussian limit for the coefficients of ε in the generalization of (5).
Finally we obtain

$$\begin{aligned} J_n &\sim e^{-18x^2\sqrt{n}-2916/25x^4} \times \\ &\times \exp\Big[x^2(-1889568/625x^4 + 1161/25)/\sqrt{n} \\ &+ (-51/50 - 1836660096/15625x^8 + 17637426/30625x^4)/n \\ &+ \mathcal{O}(n^{-5/4})\Big]/(2\pi n^3/36)^{1/2}. \end{aligned} \tag{10}$$

Note that $S^{(3)}(\tilde{z})$ does not contribute to the correction and that this correction is equivalent to the Gaussian case when $x = 0$. Of course, the dominant term is null for $x = 0$. The exponent $7/4$ that we have chosen is of course not sacred; any fixed number below 2 could also have been considered.

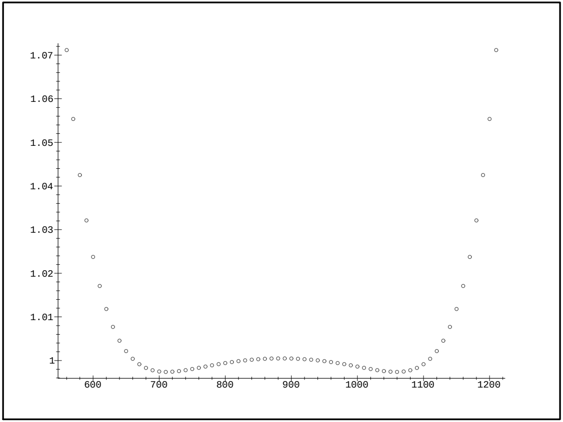

Figure 8: Quotient of $J_n(j)$ and the asymptotics (10), with the $1/\sqrt{n}$ and $1/n$ term, $n = 60$

Large deviations, $j = \alpha n(n-1)$, $0 < \alpha < 1/2$

Here, again, $z^* = 1$. Asymptotically, ε is given by a Laurent series of powers of n^{-1}, but here the behaviour is quite different: all terms of the series generalizing (5) contribute to the computation of the coefficients. It is convenient to analyze separately $S_1^{(1)}$ and $S_2^{(1)}$. This gives, by substituting

$$\tilde{z} := 1 - \varepsilon,\ j = \alpha n(n-1),\ \varepsilon = a_1/n + a_2/n^2 + a_3/n^3 + \mathcal{O}(1/n^4),$$

and expanding w.r.t. n,

$$\begin{aligned} S_2^{(1)}(\tilde{z}) &\sim (1/a_1 - \alpha)n^2 + (\alpha - \alpha a_1 - a_2/a_1^2)n + \mathcal{O}(1), \\ S_1^{(1)}(\tilde{z}) &\sim \sum_{k=0}^{n-1} f(k), \end{aligned}$$

$$\begin{aligned} f(k) &:= -(k+1)(1-\varepsilon)^k/[1-(1-\varepsilon)^{k+1}] \\ &= -(k+1)(1-[a_1/n + a_2/n^2 + a_3/n^3 + \mathcal{O}(1/n^4)])^k \\ &\quad /\{1-(1-[a_1/n + a_2/n^2 + a_3/n^3 + \mathcal{O}(1/n^4)])^{k+1}\}. \end{aligned}$$

This immediately suggests to apply the Euler-Mac Laurin summation formula, which gives, to first order,

$$S_1^{(1)}(\tilde{z}) \sim \int_0^n f(k)dk - \frac{1}{2}(f(n) - f(0)),$$

so we set $k = -un/a_1$ and expand $-f(k)n/a_1$. This leads to

$$\begin{aligned}
& \int_0^n f(k)dk \\
\sim \quad & \int_0^{-a_1} \left[-\frac{ue^u}{a_1^2(1-e^u)} n^2 \right. \\
+ \quad & \left. \frac{e^u[2a_1^2 - 2e^u a_1^2 - 2u^2 a_2 - u^2 a_1^2 + 2e^u u a_1^2]}{2a_1^3(1-e^u)^2} n \right] du \\
+ \quad & \mathcal{O}(1) - \frac{1}{2}(f(n) - f(0)).
\end{aligned}$$

This readily gives

$$\int_0^n f(k)dk \sim -\mathrm{dilog}(e^{-a_1})/a_1^2 n^2$$
$$+ \quad [2a_1^3 e^{-a_1} + a_1^4 e^{-a_1} - 4a_2 \mathrm{dilog}(e^{-a_1}) + 4a_2 \mathrm{dilog}(e^{-a_1}) e^{-a_1}$$
$$+ \quad 2a_2 a_1^2 e^{-a_1} - 2a_1^2 + 2a_1^2 e^{-a_1}]/[2a_1^3(e^{-a_1} - 1)]n + \mathcal{O}(1).$$

Combining $S_1^{(1)}(\tilde{z}) + S_2^{(1)}(\tilde{z}) = 0$, we see that $a_1 = a_1(\alpha)$ is the solution of

$$-\mathrm{dilog}(e^{-a_1})/a_1^2 + 1/a_1 - \alpha = 0.$$

We check that $\lim_{\alpha\to 0} a_1(\alpha) = \infty$, $\lim_{\alpha\to 1/2} a_1(\alpha) = -\infty$. Similarly, $a_2(\alpha)$ is the solution of the linear equation

$$\alpha - \alpha a_1 - a_2/a_1^2 + e^{-a_1}/[2(1 - e^{-a_1})] - 1/(2a_1)$$
$$+ \quad [2a_1^3 e^{-a_1} + a_1^4 e^{-a_1} + 4a_2 \mathrm{dilog}(e^{-a_1})(e^{-a_1} - 1)$$
$$+ \quad 2a_2 a_1^2 e^{-a_1} - 2a_1^2 + 2a_1^2 e^{-a_1}]/[2a_1^3(e^{-a_1} - 1)]$$
$$= \quad 0$$

and $\lim_{\alpha\to 0} a_2(\alpha) = -\infty$, $\lim_{\alpha\to 1/2} a_2(\alpha) = \infty$.

We could proceed in the same manner to derive $a_3(\alpha)$ but the computation becomes quite heavy. So we have computed an approximate solution $\tilde{a}_3(\alpha)$.

Finally,

$$
\begin{aligned}
&J_n(\alpha n(n-1)) \\
&\sim e^{[1/72 a_1(a_1-18+72\alpha)]n+C_0} \frac{1}{(2\pi n^3/36)^{1/2}} \times \\
&\times \exp\left[\left(1/72 a_2^2 + 1/36 a_1 a_3 - 1/4 a_3 + 1/4 a_2 - 1/2 a_1 + a_3\alpha + a_1 a_2 \alpha \right.\right. \\
&+ 1/3 a_1^3 \alpha - a_2\alpha - 1/2 a_1^2 \alpha - 5/24 a_1 a_2 + 1/24 a_1^2 a_2 + 1139/18000 a_1^2 \\
&\left.\left. -1/16 a_1^3 + 87/25 - 18\alpha\right)/n + \mathcal{O}(1/n^2)\right]. \qquad (11) \\
&C_0 := 1/72 a_1^3 - 1/4 a_2 + 1/4 a_1 - a_1\alpha - 5/48 a_1^2 \\
&+ 1/36 a_1 a_2 + a_2\alpha + 1/2 a_1^2 \alpha
\end{aligned}
$$

Note that, for $\alpha = 1/4$, the $1/n$ term gives $-51/50$, again as expected.

Figure 9 shows, for $n = 80$ and $\alpha \in [0.15..0.35]$, the quotient of $J_n(\alpha n(n-1))$ and the asymptotics (11).

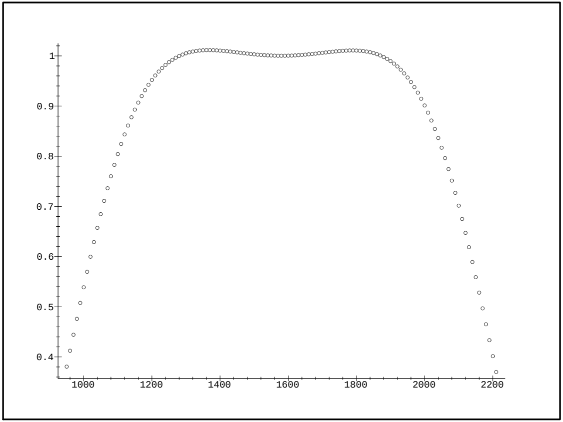

Figure 9: Quotient of $J_n(\alpha n(n-1))$ and the asymptotics (11), $n = 80$

Median versus A (A large) for a Luria-Delbruck-like distribution, with parameter A

The distribution studied by Zheng, $p(n)$, depending on two parameters, A, k, is defined as follows. Set $\theta := 1/A$. We have

$$\xi(n) = \frac{1}{1+\theta}\frac{1}{n}\left[-\frac{1}{n+1} + \sum_{1}^{n-1} j\xi(j)\frac{1}{(n-j)(n-j+1)}\right], n \geq 1,$$
$$\xi(0) = \ln(1+1/\theta). \qquad (12)$$

Note that

$$\frac{1}{n}\sum_{1}^{n-1} j\frac{1}{(n-j)(n-j+1)} \sim 1 - \frac{\ln n}{n}.$$

Also, with some extra integer parameter k, we set

$$p(n) = -\frac{k}{n}\sum_{1}^{n} j\xi(j)p(n-j), n \geq 1, \quad p(0) = \frac{1}{(1+1/\theta)^k}.$$

A picture of $p(n)$ ($A = 20, k = 2$) is given in Figure 10.

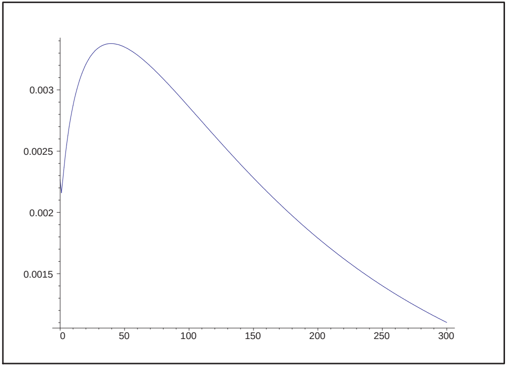

Figure 10: $p(n)$

The analysis

We know that the GF of $\xi(n)$ is given by

$$G(z) = \sum_{0}^{\infty} \xi(n) z^n = \ln \left[1 - \frac{1}{\theta} (-1 + \varphi(z)) \right], G(1) = 0, \quad (13)$$

with

$$\varphi(z) = \sum_{1}^{\infty} \frac{z^n}{n(n+1)} = 1 + \left(\frac{1}{z} - 1 \right) \ln(1 - z).$$

Set

$$F(z) := \sum_{1}^{\infty} z^n p(n).$$

Now

$$zF'z) = \sum_{1}^{\infty} z^n np(n) = -k \sum_{n=1}^{\infty} \sum_{j=1}^{n} z^j j\xi(j) z^{n-j} p(n-j)$$

$$= -k \sum_{1}^{\infty} z^n n\xi(n)p(0) - k \sum_{j=1}^{\infty} z^j j\xi(j) \sum_{n=j+1}^{\infty} z^{n-j} p(n-j)$$

$$= -kzG'(z)p(0) - kzG'(z)F(z).$$

Solving, this gives

$$F(z) = \left[\int_0^z -u^{-1-k}(\ln(1-u)+u)k[\theta u - \ln(1-u) + u\ln(1-u)]^{k-1} \left(\frac{\theta+1}{\theta}\right)^{-k} du + C\right] z^k \Big/ (\theta z - \ln(1-z) + z\ln(1-z))^k .$$

For instance, for $k=2$, we have

$$F(z)=$$
$$-\left[2z^2\ln(1-z)\theta-2\theta z^2-2z\ln(1-z)\theta+\ln(1-z)^2-z^2-2z\ln(1-z\right.$$
$$\left.+z^2\ln(1-z)^2\right]\theta^2\Big/\left[(1+\theta)^2[\theta z-\ln(1-z)+z\ln(1-z)]^2\right].$$

Of course

$$p(0)+F(1)=1.$$

The GF of $\sum_n^\infty p(j)$ is given by

$$FD(z)=\sum_1^\infty z^n\sum_n^\infty p(j)=\frac{z}{1-z}[1-p(0)-F(z)].$$

How to find n^* such that

$$[z^{n^*}]FD(z) = 1/2.$$

Of course, we can't use the singularity of $FD(z)$ at $z = 1$, as it gives

$$[z^n]FD(z)$$

for large n, but FIXED θ.

The Saddle point

We have

$$[z^j]FD(z) = \frac{1}{2\pi\mathbf{i}}\int_\Omega \frac{FD(z)}{z^{j+1}}dz = \frac{1}{2\pi\mathbf{i}}\int_\Omega \frac{e^{\ln(FD(z))}}{z^{j+1}}dz. \qquad (14)$$

It appears, after some experiments, that the values of j such that $[z^j]FD(z) = 1/2$ are such that

$$j \sim \alpha^* A \ln(A) = -\frac{\alpha^* \ln(\theta)}{\theta},$$

for some constant α^*. Note that this does not confirm Zheng's conjecture. For instance, $k = 1$ leads to $\alpha^* = 0.92\ldots$, $k = 2$ to $\alpha^* = 0.2.52\ldots$, $k = 3$ to $\alpha^* = 4.25\ldots$. For now on, we set for instance $k = 2$.

Set

$$G(z) := \ln(FD(z)) - \left(-\frac{\alpha^* \ln(\theta)}{\theta} + 1\right) \ln(z).$$

Set z to C and expand $G'(z)$ into θ. This gives, to first order,

$$\frac{\alpha \ln(\theta)}{C\theta} + \frac{1}{1-C} = 0,$$

or

$$C = 1 + \frac{\theta}{\alpha \ln(\theta)} + \mathcal{O}(\theta^2).$$

So we try

$$\tilde{z} = 1 + \frac{\beta\theta}{\ln(\theta)}. \tag{15}$$

This gives, after some complicated manipulations (done by Maple)

$$G'(\tilde{z}) \sim \frac{[-\beta + \alpha\beta^2 + 2\alpha + 3\beta\alpha - 3]\ln(\theta)}{(1+\beta)(2+\beta)\theta}.$$

Solving $G'(\tilde{z}) = 0$ gives

$$\beta = \frac{1 - 3\alpha + \sqrt{1 + 6\alpha + \alpha^2}}{2\alpha}.$$

But this is impossible: β is decreasing as expected, but PROBLEM 1: $\beta(3/2) = 0!$. The same problem appears for all k. To be convinced that (15) is the correct asymptotics, we have computed numerically the solution zn of $G'(zn) = 0$ for different values of θ and α and we have extracted the corresponding values of β from (15). For instance, for $\theta = 1/1000$, we obtain in Figure 11 the graph of β vs α. And the values of θ: $1/100, 1/500, 1/1500$ give the same function, with some remarkable fit.

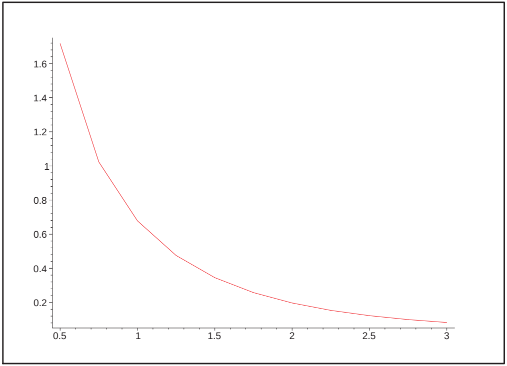

Figure 11: β vs α, $\theta = 1/1000, k = 2$

Independently of PROBLEM 1, it appears, after some experiments, that the next term would be

$$\tilde{z} = 1 + \frac{\beta\theta}{\ln(\theta)} + \frac{\gamma\theta \ln(-\ln(\theta))}{\ln(\theta)^2}.$$

This leads to some equation

$$\varphi(\gamma, \alpha) = 0.$$

Also, we obtain, to first order,

$$G''(\tilde{z}) \sim f(\beta)\frac{\ln(\theta)^2}{\theta^2}, \quad f(\beta) = \frac{7 + 6\beta + \beta^2}{(2+\beta)^2(1+\beta)^2},$$

and

$$G(\tilde{z}) \sim -\ln(\theta) + g(\beta) + \ln(-\ln(\theta)) + \alpha\beta, \quad g(\beta) = \ln\left(\frac{2+\beta}{(1+\beta)^2}\right).$$

Hence, by standard manipulations,

$$[z^j]FD(z) \sim \frac{e^{\alpha\beta+g(\beta)}}{\sqrt{2\pi}\sqrt{f(\beta)}} = \phi(\alpha) \text{ say },$$

independent of θ.
If we knew $\beta(\alpha)$, this would give α^* from $\phi(\alpha^*) = 1/2$.

Sum of positions of records in random permutations

The statistic *srec* is defined as the sum of positions of records in random permutations. The generating function (GF) of *srec* is given by

$$G(z) = \prod_{i=1}^{n} (z^i + i - 1), \tag{16}$$

and the probability generating function (PGF) is given by

$$Z(z) = \frac{\prod_{i=1}^{n} (z^i + i - 1)}{n!}. \tag{17}$$

This statistic has been the object of recent interest in the litterature. Kortchemski obtains the the GF (16) and also proves that

$$J_n(j) := z^j[G(z)] \sim e^{n \ln(n) y}, \text{ where } j = \frac{n(n+1)}{2} x, \quad x = 1 - y^2, \tag{18}$$

with an error $\mathcal{O}(1/\ln(n))$.

The large deviation $j = \frac{n(n+1)}{2}(1 - y^2)$

By Cauchy's theorem,

$$J_n(j) = \frac{1}{2\pi\mathbf{i}} \int_\Omega \frac{G(z)}{z^{j+1}} dz = \frac{1}{2\pi\mathbf{i}} \int_\Omega e^{S(z)} dz, \quad (19)$$

where Ω is inside the analyticity domain of the integrand and encircles the origin and

$$S(z) = \sum_{i=1}^{n} \ln(z^i + i - 1) - \left(\frac{n(n+1)}{2}(1 - y^2) + 1 \right) \ln(z).$$

Set

$$S^{(i)} := \frac{d^i S}{dz^i}.$$

First we must find the solution of

$$S^{(1)}(\tilde{z}) = 0$$

with smallest module. This leads to

$$\sum_{i=1}^{n} \frac{i\tilde{z}^i}{\tilde{z}^i + i - 1} - \left(\frac{n(n+1)}{2}(1-y^2)+1\right) = 0. \qquad (20)$$

In some previous sections , we simply tried $\tilde{z} = z^* + \varepsilon$, plug into (20), and expanded into ε. Here it appears that we cannot get this expansion. So we expand first (20) itself. But we must be careful. There exists some $\tilde{i}$ such that $\tilde{z}^{\tilde{i}} = \tilde{i}$. Some numerical experiments suggest that $\tilde{i} = \mathcal{O}(n)$. So we set $\tilde{i} = \alpha n, 0 < \alpha < 1$ and we must now compute α.

We obtain $\tilde{z} = e^{\xi}$, with

$$\xi = \frac{L + \ln(\alpha)}{\alpha n},$$

where here and in the sequel, $L := \ln(n)$. Note that this leads to

$$\tilde{z}^n = \exp\left(\frac{L + \ln(\alpha)}{\alpha}\right) = n^{1/\alpha}\alpha^{1/\alpha}.$$

Using the classical splitting of the sum technique (20) leads to (we provide in the sequel only a few terms in the expansions, but Maple knows and uses more)

$$\Sigma_1 := \sum_{i=1}^{\tilde{i}} \frac{i\tilde{z}^i}{\tilde{z}^i + i - 1},$$

$$\Sigma_2 := \sum_{i=\tilde{i}+1}^{n} \frac{i\tilde{z}^i}{\tilde{z}^i + i - 1} - \left(\frac{n(n+1)}{2}(1 - y^2) + 1\right).$$

As

$$\frac{\tilde{z}^i - 1}{i} < \frac{\tilde{z}^{\tilde{i}} - 1}{\tilde{i}} < \frac{\tilde{z}^{\tilde{i}}}{\tilde{i}} = 1, i < \tilde{i},$$

we have

$$\Sigma_1 = \sum_{i=1}^{\tilde{i}} \frac{\tilde{z}^i}{1 + \frac{\tilde{z}^i - 1}{i}} = \sum_{i=1}^{\tilde{i}} \tilde{z}^i \left[1 - \frac{\tilde{z}^i - 1}{i} + \left(\frac{\tilde{z}^i - 1}{i} \right)^2 + \dots \right] \tag{21}$$

The first summation is immediate

$$\sum_{i=1}^{\tilde{i}} \tilde{z}^i = \frac{\tilde{z}^{\tilde{i}+1} - 1}{\tilde{z} - 1} - \frac{\tilde{z}}{\tilde{z} - 1}.$$

For the next summations, we again use Euler-Mac Laurin summation formula. First of all, the correction (to first order) is given by

$$\frac{1}{2}+\frac{1}{2}\frac{\tilde{i}}{1+\frac{\tilde{i}-1}{\tilde{i}}}=\frac{1}{2}+\frac{1}{2}\frac{\alpha n}{2-1/(\alpha n)}\sim\frac{1}{4}\alpha n. \tag{22}$$

Next, we must compute integrals such as

$$\int_1^{\tilde{i}}\tilde{z}^i\left(\frac{\tilde{z}^i-1}{i}\right)^k di. \tag{23}$$

But we know that

$$\int_1^{\tilde{i}}\tilde{z}^i\left(\frac{\tilde{z}^i-1}{i}\right)di=\int_1^{\tilde{i}}e^{\xi i}\left(\frac{e^{\xi i}-1}{i}\right)di=Ei(1,-2\xi)-Ei(1,-\xi)$$
$$+Ei(1,-\tilde{i}\xi)-Ei(1,-2\tilde{i}\xi)$$
$$=Ei(1,-2\xi)-Ei(1,-\xi)+Ei(1,-(L+\ln(\alpha)))-Ei(1,-2(L+\ln(\alpha))),$$

where $Ei(x)$ is the exponential integral.

Setting $L_1 := L + \ln(\alpha)$, we have

$$\Re(Ei(1,-\xi)) = -\gamma - \ln(\xi) - \xi - \frac{\xi^2}{4} + \dots,$$
$$\Re(Ei(1,-L_1)) = e^{L_1}\left[-\frac{1}{L_1} - \frac{1}{L_1^2} + \dots\right] = \alpha n\left[-\frac{1}{L_1} - \frac{1}{L_1^2} + \dots\right].$$

We use similar expansions for terms like (23). This finally leads, with (22), to

$$\Sigma_1 = n^2\left[\frac{7\alpha^2}{12L} + \frac{\alpha^2(-84\ln(\alpha) - 31)}{144L^2} + \dots\right] + \mathcal{O}(n).$$

Now we turn to Σ_2. As

$$\frac{i-1}{\tilde{z}^i} < \frac{\tilde{i}+1-1}{\tilde{z}^{\tilde{i}+1}} < \frac{\tilde{i}}{\tilde{z}^i} = 1, i > \tilde{i},$$

we have

$$\Sigma_2 = \sum_{i=\tilde{i}+1}^{n} \frac{i}{1+\frac{i-1}{\tilde{z}^i}} - \left(\frac{n(n+1)}{2}(1-y^2)+1\right)$$

$$= \sum_{i=\tilde{i}+1}^{n} i\left[1 - \frac{i-1}{\tilde{z}^i} + \left(\frac{i-1}{\tilde{z}^i}\right)^2 + \dots\right] - \left(\frac{n(n+1)}{2}(1-y^2)+1\right)$$

$$= n^2\left[\frac{1}{2} - \frac{\alpha^2}{2} - \frac{47\alpha^2}{60L} + \dots\right] + \mathcal{O}(n)$$

$$+ \frac{n^3}{n^{1/\alpha}}\left[\frac{\alpha}{\alpha^{1/\alpha}L} + \dots\right] - \left(\frac{n(n+1)}{2}(1-y^2)+1\right) + \dots$$

So

$$S'(\tilde{z}) = \Sigma_1 + \Sigma_2 = n^2 \left[\frac{1}{2} - \frac{\alpha^2}{2} - \frac{\alpha^2}{5L} + \ldots - (1 - y^2)/2 \right] + \mathcal{O}(n)$$
$$+ \frac{n^3}{n^{1/\alpha}} \left[\frac{\alpha}{\alpha^{1/\alpha} L} + \ldots \right] + \ldots = 0 \qquad (24)$$

Putting the coefficient of n^2 to 0, and solving wrt α gives

$$\alpha^* = y - \frac{y}{5L} + \frac{-3199y/1800 + \ln(y)y/5}{L^2} + \ldots \qquad (25)$$

Now we must consider the other terms of (24). First we must compare n with $\frac{n^3}{n^{1/\alpha}}$.
If $\alpha > \frac{1}{2}$, $\quad n^{3-1/\alpha} > n$ and vice-versa. The most interesting case is the case $\alpha > \frac{1}{2}$ (the other one can be treated by similar method). Note hat there are also other terms in (24) of order $n^{k-(k-2)/\alpha}$, $k \geq 4$. This is greater than n if $\alpha > (k-2)/(k-1)$.

Returning to (24), we first compute $n^{1/\alpha} = n^{1/y}\varphi(y, L)$, with

$$\varphi(y, L) = e^{L(1/\alpha - 1/y)} = e^{1/(5y)} - \frac{e^{1/(5y)}(-3271 + 360\ln(y))}{1800yL} + \dots \tag{26}$$

So we obtain from (24) the term

$$\frac{n^3}{n^{1/y}\varphi(y, L)}\left[\frac{\alpha}{\alpha^{1/\alpha}L} + \dots\right],$$

and with (25),

$$\frac{n^3}{n^{1/y}}\left[\frac{y}{y^{1/y}e^{1/(5y)}L} + \dots\right].$$

Now we set $\alpha = \alpha^* + \frac{Cn}{n^{1/y}}$, plug into (24) (ignoring the $\mathcal{O}(n)$ term), and expand. The n^2 term must theoretically be 0. Actually, it is given by a series of large powers of $1/L$ as we only use a finite number of terms in (25). Solving the coefficient of $\frac{n^3}{n^{1/y}}$ wrt C, we obtain

$$C = \frac{e^{-1/(5y)} y^{(3y-1)/y}}{Ly^3} + \dots$$

and

$$\alpha = \alpha^* + \frac{Cn}{n^{1/y}} + \dots \qquad (27)$$

$$\tilde{J}_n(j) = \frac{e^{S(\tilde{z})}}{\sqrt{2\pi S''(\tilde{z})}} \qquad (28)$$

In Figure 12, we give, for $n = 150$, a comparison between $\ln(J_n(j))$ (circle) and $\ln(\tilde{J}_n(j))$ (line). The fit is quite good, but when y is near 1. But j is then small and our symptotics are no more very efficient. We also show the first approximation (18): nLy (line blue) which is only efficient for very large n.

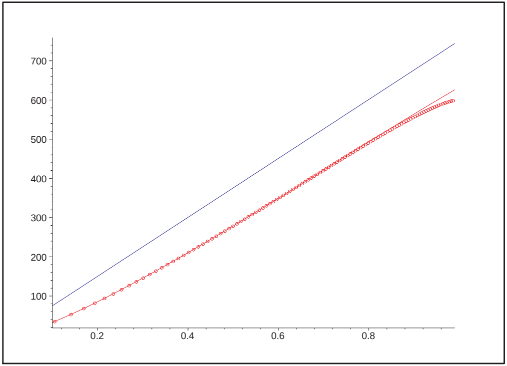

Figure 12: Comparison between between $\ln(J_n(j))$ (circle) and $\ln(\tilde{J}_n(j))$ (line), $n = 150$. Also it shows the first approximation (1): nLy (line blue)

The central region $j = yn$

From (17), we see that, as expected, $Z(1) = 1$. Moreover

$$Z'(z) = \sum_{i=1}^{n} \frac{iz^{i-1}}{z^i + i - 1} Z(z),$$

and

$$Z''(z) = \sum_{i=1}^{n} \frac{i[z^{i-2}i^2 - 2z^{i-2} - z^{2i-2} + z^{i-2}]}{(z^i + i - 1)^2} Z(z)$$
$$+ \left(\sum_{i=1}^{n} \frac{iz^{i-1}}{z^i + i - 1} \right)^2 Z(z).$$

So

$$\mathbb{E}(srec) = Z'(1) = n,$$

and the variance is given by

$$\mathbb{V}(srec) = Z''(1) + n - n^2 = \sum_{i=1}^{n}(i-2) + n^2 + n - n^2 = \frac{n(n-1)}{2}.$$

Of course $Z(z)$ corresponds to a sum of independent *non* identically distributed random variables, but it is clear that the Lindeberg-Lévy conditions are not satified here. The distribution is *not* asymptotically Gaussian. As will be clear later on, it is convenient to separate the cases $j \geq 1$ and $j < 1$.

The case $y \geq 1$

Again, by Cauchy's theorem,

$$\mathbb{P}(srec = j) = \frac{1}{2\pi\mathbf{i}} \int_\Omega \frac{Z(z)}{z^{j+1}} dz = \frac{1}{2\pi\mathbf{i}} \int_\Omega e^{S(z)} dz, \tag{29}$$

where Ω is inside the analyticity domain of the integrand and encircles the origin and

$$S(z) = \sum_{i=1}^{n} \ln(z^i + i - 1) - (yn+1)\ln(z) - \ln(n!).$$

First we must find the solution of

$$S^{(1)}(\tilde{z}) = 0$$

with smallest module.

This leads to

$$\sum_{i=1}^{n} \frac{i\tilde{z}^i}{\tilde{z}^i + i - 1} - (yn+1) = 0. \qquad (30)$$

Set $\tilde{z} := z^* + \varepsilon$, where, here, it is easy to check that $z^* = 1$. Set $j = yn$.

This leads, to first order (keeping only the ε term in (30)), to

$$\varepsilon := \frac{2(y-1)}{n} + \frac{-4+10y-4y^2}{n^2} + \mathcal{O}(1/n^3).$$

This shows that, asymptotically, ε is given by a Laurent series of powers of n^{-1}. To obtain more precision, we set again $j = yn$, expand (30) into powers of ε (we use 9 terms), set

$$\varepsilon = \frac{a_1}{n} + \sum_{i=2}^{4} \frac{a_i}{n^i}$$

expand in powers of n^{-1}, and equate each coefficient to 0.

This gives, for the coefficient of n

$$1 - y + 1/120a_1^4 + 1/2a_1 + 1/720a_1^5 + 1/362880a_1^8 + 1/6a_1^2 + 1/5040a_1^6 + 1/3628800a_1^9 + 1/24a_1^3 + 1/40320a_1^7 = 0.$$

We observe that *all* terms of the expansion of (30) into ε contribute to the computation of the coefficients. We have already encountered this sitution in analyzing the number in inversions in permutations. So we must turn to another approach. Setting $i = k + 1$, (30) becomes

$$\sum_{k=0}^{n-1} f(k) - (yn + 1) = 0, \tag{31}$$

where

$$f(k) = \frac{(k+1)\tilde{z}^{k+1}}{\tilde{z}^{k+1} + k}.$$

This gives, always by Euler-Mac Laurin summation formula, to first order,

$$\int_0^n f(k)dk - \frac{1}{2}(f(n) - f(0)) - (yn + 1) = 0,$$

so we set $k = un/a_1, \tilde{z} = 1 + a_1/n$ and expand $f(k)n/a_1$. This leads to

$$\int_0^{a_1} e^u \frac{ndu}{a_1} - yn + \mathcal{O}(1) = 0,$$

or

$$e^{a_1} = ya_1 + 1. \tag{32}$$

Note that

$$a_1(y) \sim \ln(y), y \to \infty,$$
$$a_1(1) = 0.$$

The explicit solution of (5) is easily found, we have

$$\begin{aligned} e^{a_1} &= ya_1 + 1 = y[a_1 - 1/y] = e^{a_1+1/y}e^{-1/y}, \\ -e^{-[a_1+1/y]}[a_1 + 1/y] &= -e^{-1/y}/y, \\ -[a_1 + 1/y] &= W(-1, -e^{-1/y}/y), \\ a_1 &= -1/y - W(-1, -e^{-1/y}/y), \end{aligned}$$

where $W(-1, x)$ is the suitable branch of the Lambert equation: $W(x)e^{W(x)} = x$.
PROBLEM 2: THE THIRD DERIVATIVE IS OF ORDER n^3 ALTHOUGH THE SECOND DERIVATIVE IS OF ORDER n^2.

Finally, by standard technique, (29) should lead to

$$\mathbb{P}(srec = j) \sim \frac{e^{F(y)}}{\sqrt{2\pi n^2 F_2(y)}}, \tag{33}$$

and

$$\mathbb{P}(srec = n) \sim \frac{1}{\sqrt{2\pi n^2/2}}. \tag{34}$$

Also

$$\mathbb{P}(srec = j) \sim \frac{e^{-y\ln(y)}}{\sqrt{2\pi n^2 y}}, y \to \infty.$$

The case $y < 1$

We have

$$\mathbb{P}(srec = j) = \frac{1}{n!}[z^j]\prod_{i=1}^{j}[z^i + i - 1]\prod_{u=j}^{n-1} u,$$

and, if j is large, by (34),

$$[z^j]\prod_{i=1}^{j}[z^i + i - 1] \sim j!\frac{1}{\sqrt{\pi j}}.$$

So

$$\mathbb{P}(srec = j) \sim \frac{1}{n!}\frac{(j-1)!}{\sqrt{\pi}}\prod_{u=y}^{n-1} u = \frac{1}{n\sqrt{\pi}}.$$

For j large enough, $\mathbb{P}(srec = j)$ is constant and given by (34).
We have made a numerical comparison of
$\mathbb{P}(srec = j), n = 200, j = 1..3n$ with (34) and (33). This is given in Figure 13. This is quite satisfactory.

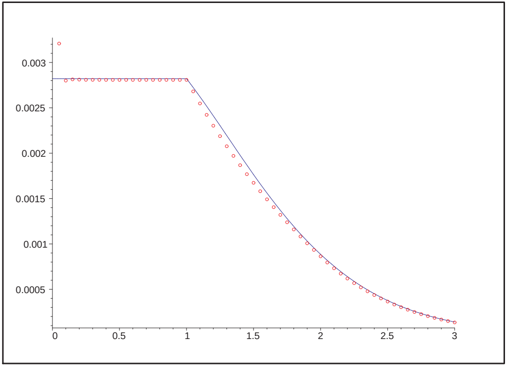

Figure 13: Comparison between $\mathbb{P}(srec = j), n = 200, j = 1..3n$ (circle) and the asymptotics (34) and (33) (line)

Merten's theorem for toral automorphisms

Let

$$\varphi_n(z) := \prod_1^n (1 - z^k)(1 - z^{-k})$$

What is the symptotic behaviour of

$$[z^j]\varphi_n(z),$$

in particular the asymptotic value of

$$[z^0]\varphi_n(z).$$

A plot of $[z^j]\varphi_n(z)$, $n = 15$ is given in Figure 14. This seems to have a Gaussian envelope.

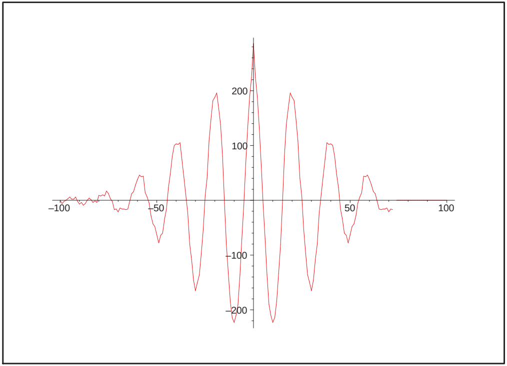

Figure 14: $[z^j]\varphi_n(z)$, $n = 15$

But

$$S(z) := \ln(\varphi_n(z)) - \ln(z) = \sum_1^n (\ln(1 - z^k) + \ln(1 - z^{-k}) - \ln(z)$$

doesn't appear to posses zeroes, two plots of

$$|S(z)|^2$$

given in Figures 15, 16, reveal a quite irregular behaviour.

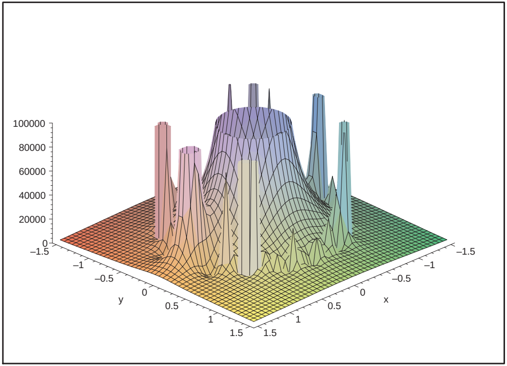

Figure 15: $|S(z)|^2$, $n = 15$

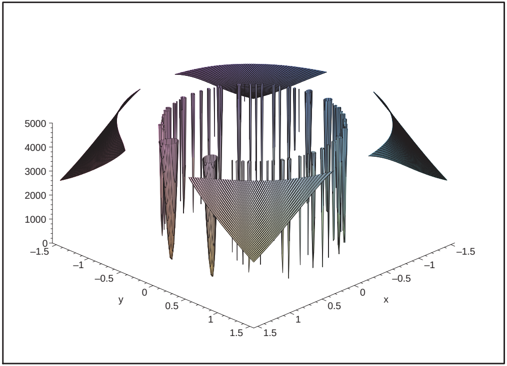

Figure 16: $|S(z)|^2$, $n = 15$

Representations of numbers as $\sum_{k=-n}^{n} \varepsilon_k k$

We consider the number of representations of m as $\sum_{k=-n}^{n} \varepsilon_k k$, where $\varepsilon_k \in \{0,1\}$. For $m=0$, this is sequence A000980 in Sloane's encyclopedia. This problem has a long history. Here, we extend previous ranges a bit, to $\mathcal{O}(n^{3/2})$. But we improve at the same time the quality of the approximation

The generating function of the number of representations for fixed n is given by

$$C_n(z) = 2\prod_{k=1}^{n}(1+z^k)(1+z^{-k}), C_n(1) = 2\cdot 4^n.$$

By normalisation, we get the probability generating function of a random variable X_n:

$$F_n(z) = 4^{-n}\prod_{k=1}^{n}(1+z^k)(1+z^{-k}),$$

The Gaussian limit

We obtain mean $\mathbb{M}$ and variance $\mathbb{V}$:

$$\mathbb{M}(n) = 0, \quad \sigma^2 := \mathbb{V}(n) = \frac{n(n+1)(2n+1)}{12}.$$

We consider values $j = x\sigma$, for $x = \mathcal{O}(1)$ in a neighbourhood of the mean 0.

The Gaussian limit of X_n can be obtained by using the Lindeberg-Lévy conditions, but we want more precision.

We know that

$$P_n(j) = \frac{1}{2\pi i} \int_\Omega e^{S(z)} dz$$

where

$$S(z) := \ln(F_n(z)) - (j+1) \ln z$$

with

$$\ln(F_n(z)) = \sum_{i=1}^{n} [\ln(1+z^i) + \ln(1+z^{-i}) - \ln 4].$$

Set $\tilde{z} := z^* - \varepsilon$, where, here, $z^* = 1$.
Finally this leads to

$$P_n(j) \sim e^{-x^2/2} \cdot \exp\left([-39/40 + 9/20 x^2 - 3x^4/40]/n + \mathcal{O}(n^{-3/2}) \right) / (2\pi n^3/6) \tag{35}$$

Note again that $S^{(3)}(\tilde{z})$ does not contribute to the $1/n$ correction. Note also that, unlike in the instance of the number of inversions in permutations, we have an x^4 term in the first order correction. Figure 17 shows, for $n = 60$, Q_3: the quotient of $P_n(j)$ and the asymptotics (35), with the constant term $-39/(40n)$ and the x^2 term $9x^2/(20n)$, and Q_4: the quotient of $P_n(j)$ and the full asymptotics (35). Q_4 gives indeed a good precision on a larger range.

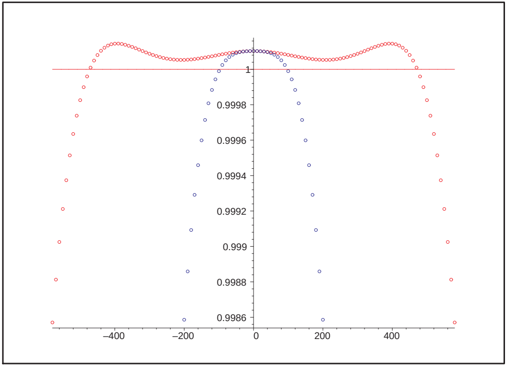

Figure 17: Q_3 (blue) and Q_4 (red)

The case $j = n^{7/4} - x$

Now we show that our methods are strong enough to deal with probabilities that are far away from the average; viz. $n^{7/4} - x$, for fixed x. Of course, they are very small, but nevertheless we find asymptotic formulæ for them. Later on, $n^{7/4} - x$ will be used as an integer

This finally gives

$$P_n(j) \sim e^{-3n^{1/2}-27/10}\Big[1 + 369/175/n^{1/2} + 931359/245000/n + 1256042967/471625000/n^{3/2} + 4104x/175/n^{7/4} - 9521495156603/2145893750000/n^2 + 7561917x/122500/n^{9/4} + (-2359744124681290 59/341392187500000 + 18x^2)/n^{5/2} + \cdots\Big] \Big/ (2\pi n^3/6)^{1/2}. \tag{36}$$

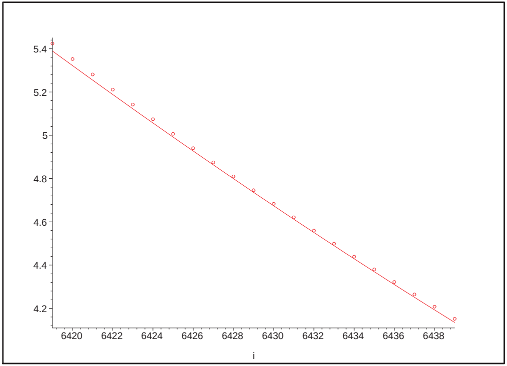

Figure 18: $P_n(j)$ (circle) and the full asymptotics (36) up to the $n^{-5/2}$ term,(line), $n = 150$, scaling$= 10^{21}$

The q-Catalan numbers

The q-Catalan numbers $C_n(q)$ are defined as

$$C_n(q) = \frac{1-q}{1-q^{n+1}} \frac{(q;q)_{2n}}{(q;q)_n(q;q)_{2n}},$$

with $(q;q)_n = (1-q)(1-q^2)\dots(1-q^n)$. Note that $C_n = C_n(1)$, a Catalan number, and the polynomial $F_n(q) = C_n(q)/C_n$ is the probability generating function of a distribution X_n that we call the Catalan distribution. Is has been shown recently that this distribution is asymptotically normal.

Since we attack $F_n(q)$ and $C_n(q)$ mostly with analytic methods, we find it more appropriate to replace the letter q by z; note that

$$C_n(z) = \prod_{i=1}^{n-1} \frac{1-z^{n+i+1}}{1-z^{i+1}}.$$

It is easy to get the mean m and variance σ^2 of the random variable X_n characterized by

$$P_n(i) := \mathbb{P}(X_n = i) := \frac{[z^i]C_n(z)}{C_n},$$

namely

$$m = \frac{n(n-1)}{2},$$
$$\sigma^2 = \frac{n(n^2-1)}{6}.$$

(In the full paper we sketch how these moments and higher ones can be computed quite easily.)

The Gaussian limit

Set $j = m + x\sigma$, with $m = n(n-1)/2$ and $\sigma = \sqrt{n(n^2-1)/6}$. Then, for fixed x, the following approximation holds:

$$P_n(j) \sim e^{-x^2/2} \cdot \exp\Big((-9/40 + 9/20x^2 - 3x^4/40)/n\Big) \Big/ (2\pi n^3/6)^{1/2}. \tag{37}$$

Note that, like in the Representations of numbers as $\sum_{k=-n}^{n} \varepsilon_k k$, we have a x^4 term in the first order correction.
Figure 19 shows, for $n = 60$, Q_3: the quotient of $P_n(j)$ and the asymptotics (37), with the constant term $-9/(40n)$ and the x^2 term $9x^2/(20n)$ and Q_4: the quotient of $P_n(j)$ and the full asymptotics (37). Q_4 gives indeed a good precision on a larger range.

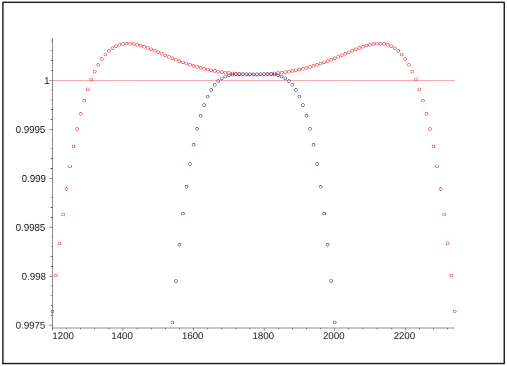

Figure 19: Q_3 (blue) and Q_4 (red)

The case $j = n - k$

We deal with probabilities that are far away from the average; viz. $j = n - k$, for fixed k. Again, they are very small, but nevertheless we find asymptotic formulæ for them.

We have $z^* = 1$ and, as we will see, ε is again given by a Puiseux series of powers of $n^{-1/2}$. But the approach of previous Sections is doomed to failure: all terms of the generalization of (5) contribute to the computation of the coefficients. So we have to turn to another technique. We have

$$P_n(n-k) = [z^{n-k}] \prod_{i=1}^{n-1} \frac{1 - z^{n+i+1}}{1 - z^{i+1}} \Big/ C_n.$$

So we set

$$S := S_1 + S_2 + S_3 + S_4,$$

with

$$S_1 := \sum_{i=1}^{n-1} \ln(1 - z^{n+i+1}),$$

$$S_2 := -\sum_{i=1}^{n-1} \ln(1 - z^{i+1}) = -\sum_{i=2}^{n} \ln(1 - z^i),$$

$$S_3 := -(n+1-k)\ln z, \quad S_4 := -\ln(C_n).$$

Set $\tilde{z} = z^* - \varepsilon = 1 - \varepsilon$. We must have $S'(\tilde{z}) = 0$, with

$$S'(z) = S_1'(z) + S_2'(z) + S_3'(z),$$

$$S_1'(z) = \sum_{i=1}^{n-1} \frac{-(n+i+1)z^{n+i}}{1 - z^{n+i+1}},$$

$$S_2'(z) = \sum_{i=1}^{n-1} \frac{(i+1)z^i}{1 - z^{i+1}},$$

$$S_3'(z) = -\frac{n+1-k}{z}.$$

It is clear that we must have either $S_1'(\tilde{z}) = \mathcal{O}(n)$ or $S_2'(\tilde{z}) = \mathcal{O}(n)$. Set

$$\varepsilon = \frac{f(n)}{n},$$

for a function still to be determined. Let us first (roughly) solve

$$S_1'(\tilde{z}) = n.$$

We have

$$S_1'(\tilde{z}) \sim n\frac{\tilde{z}^n}{1-\tilde{z}} \sim \frac{n^2 e^{-f(n)}}{f(n)}.$$

This leads to the equation

$$\frac{n^2 e^{-f(n)}}{f(n)} = n, \text{ i.e. } e^{f(n)} f(n) = n,$$

which is solved by

$$f(n) = W(n),$$

where W is the Lambert function. But we know that

$$W(n) \sim \ln(n) - \ln\ln(n) + \mathcal{O}\left(\frac{\ln\ln(n)}{\ln(n)}\right).$$

So

$$\varepsilon \sim \frac{\ln(n)}{n}.$$

But then

$$S_2'(\tilde{z}) \sim \frac{1}{(1-\tilde{z})^2} \sim \frac{n^2}{\ln(n)^2} = \Omega(n).$$

So we must turn to the other choice, i.e.

$$S_2'(\tilde{z}) \sim \mathcal{O}\left(\frac{1}{(1-\tilde{z})^2}\right) \sim \mathcal{O}\left(\frac{n^2}{f(n)^2}\right).$$

The equation

$$\frac{n^2}{f(n)^2} = n$$

leads now to

$$f(n) = \sqrt{n},$$

and

$$S_1'(\tilde{z}) \sim \frac{n^2 e^{-\sqrt{n}}}{\sqrt{n}} = n^{3/2} e^{-\sqrt{n}},$$

which is exponentially negligible.

This rough analysis leads to the more precise asymptotics

$$\varepsilon = \frac{a_1}{n^{1/2}} + \frac{a_2}{n} + \frac{a_3}{n^{3/2}} + \cdots \qquad (38)$$

and we must solve

$$S_2'(\tilde{z}) + S_3'(\tilde{z}) = 0,$$

i.e.

$$\sum_{j=1}^{n-1} \frac{(j+1)\tilde{z}^j}{1-\tilde{z}^{j+1}} - \frac{n+1-k}{\tilde{z}} = 0.$$

We rewrite this equation as

$$\sum_{j=2}^{n} \frac{j\tilde{z}^j}{1-\tilde{z}^j} = M,$$

and will later replace M by $n+1-k$. It is not hard to see that we can solve

$$\sum_{j=2}^{\infty} \frac{j\tilde{z}^j}{1-\tilde{z}^j} = M,$$

since the extra terms introduce an exponentially small error.

Now we replace $\tilde{z}$ by e^{-t}:

$$g(t) = \sum_{j=2}^{\infty} \frac{je^{-jt}}{1 - e^{-jt}},$$

and compute its Mellin transform:

$$g^*(s) = (\zeta(s-1) - 1)\zeta(s)\Gamma(s). \tag{39}$$

The original function can be recovered by a contour integral:

$$g(t) = \frac{1}{2\pi i} \int_{3-i\infty}^{3+i\infty} (\zeta(s-1) - 1)\zeta(s)\Gamma(s)t^{-s} ds.$$

This integral can be approximately (in a neighbourhood of $t = 0$) solved by taking residues into account:

$$g(t) = \frac{\pi^2}{6t^2} - \frac{3}{2t} + \frac{13}{24} - \frac{t}{12} + \frac{t^3}{720} - \frac{t^5}{30240} + \mathcal{O}(t^7).$$

Now we solve

$$\frac{\pi^2}{6t^2} - \frac{3}{2t} + \frac{13}{24} - \frac{t}{12} + \frac{t^3}{720} - \frac{t^5}{30240} \sim M,$$

and find

$$t(M) \sim \frac{\pi}{\sqrt{6}} M^{-1/2} - \frac{3}{4} M^{-1} + \frac{(81 + 13\pi^2)\sqrt{6}}{288\pi} M^{-3/2} - \left(\frac{13}{32} + \frac{\pi^2}{144}\right) M^{-2}$$

To simplify the next expressions, we will set $n = w^2$.

Now, since $1 - \varepsilon = e^{-t}$, we set $M = n + 1 - k$, expand and reorganize:

$$\varepsilon \sim \frac{\pi}{\sqrt{6}} w^{-1} - \left(\frac{\pi^2}{12} + \frac{3}{4}\right) w^{-2} + \frac{\sqrt{6}(4\pi^4 + 3\pi^2 + 72\pi^2 k + 243)}{864\pi} w^{-3}. \tag{40}$$

Note that k appears (linearly) only in the coefficient of w^{-3}. The quadratic term in k appears in the coefficient of w^{-5}.

This finally leads to

$$P_n(n-k) \sim e^{T_4}\frac{2^{-1/2}\pi^{3/2}}{12}\left(1 - 2^{1/2}3^{1/2}\frac{216+13\pi^2+24\pi^2 k}{144\pi w}\right.$$

$$+\frac{19008\pi^2+20736\pi^2 k+31104+217\pi^4+624\pi^4 k+576\pi^4 k^2}{6912\pi^2 w^2}$$

$$-3^{1/2}2^{1/2}[-229635+11104128\pi^2+1907712\pi^4 k+11197440\pi^2 k$$

$$+4069\pi^6+771984\pi^4+1244160\pi^4 k^2+15624\pi^6 k$$

$$\left.+13824\pi^6 k^3+22464\pi^6 k^2]/[2985984\pi^3 w^3]\right)$$

$$T_4 := -2\ln(2)w^2 + \frac{\sqrt{6}\pi w}{3}.$$

The quality of the approximation is given in Figure 20, with the w^{-2} term (and k^2 contribution) and in Figure 21, with the w^{-2} and w^{-3} terms (and k^3 contribution). The fit is rather good: the curves cover the exact graph above and below.

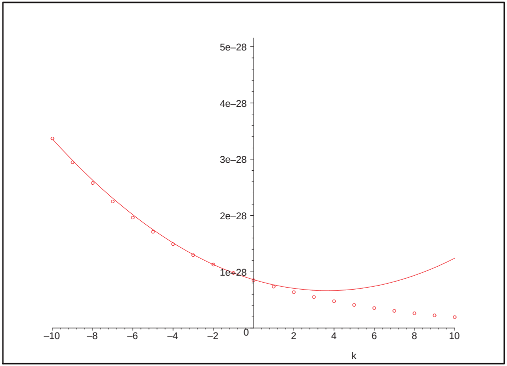

Figure 20: $P_n(n-k), n = 60$ exact (circle), asymptotics (line), with w^{-2} term

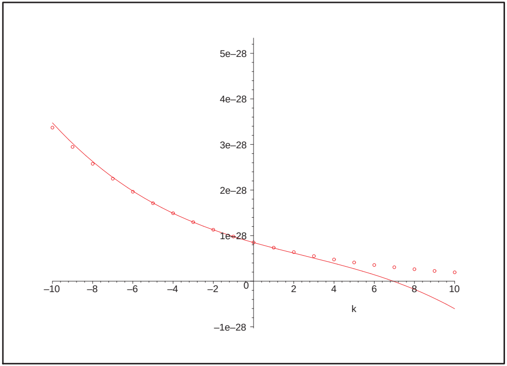

Figure 21: $P_n(n-k), n = 60$ exact (circle), asymptotics (line), with w^{-2} and w^{-3} terms

A simple case of the Mahonian statistic

Canfield, Janson and Zeilberger have analyzed the Mahonian distribution on multiset permutations: classic permutations on m objects can be viewed as words in the alphabet $\{1, \dots, m\}$. If we allow repetitions, we can consider all words with a_1 occurences of 1, a_2 occurences of $2, \dots$, a_m occurences of m. Let J_m denote the number of inversions. Assuming that all words are equally likely, the probability generating function of J_m is given, setting $N = a_1 + \cdots + a_m$, by

$$\phi_{a_1,\dots,a_m}(z) = \frac{\prod_{i=1}^{m} a_i! \prod_{i=1}^{N}(1 - z^i)}{N! \prod_{j=1}^{m} \prod_{i=1}^{a_j}(1 - z^i)}.$$

The mean μ and variance σ^2 are given by

$$\mu(J_m) = e_2(a_1, \dots, a_m)/2, \quad \sigma^2(J_m) \quad = \frac{(e_1 + 1)e_2 - e_3}{12},$$

where $e_k(a_1, \dots, a_m)$ is the degree k elementary symmetric function.

Let $a^* = \max_j a_j$ and $N^* = N - a^*$. Recently it was proved that, if $N^* \to \infty$ then the sequence of normalized random variables

$$\frac{J_m - \mu(J_m)}{\sigma(J_m)}$$

tends to the standard normal distribution. The authors also conjecture a local limit theorem and prove it under additional hypotheses.

In this talk, we analyze simple examples of the Mahonian statistic, for instance, we consider the case

$$m = 2, a_1 = an, a_2 = bn, n \to \infty.$$

We analyze the central region $j=\mu+x\sigma$ and one large deviation $j=\mu+xn^{7/4}$. The exponent $7/4$ that we have chosen is of course again not sacred, any fixed number below 2 and above $3/2$ could also have been considered.

We have here

$$\phi(z)=\frac{(an)!(bn)!\prod_{i=1}^{(a+b)n}(1-z^i)}{((a+b)n)!\prod_{i=1}^{an}(1-z^i)\prod_{i=1}^{bn}(1-z^i)},$$

$$\mu=\frac{abn^2}{2},$$

$$\sigma^2=\frac{ab(a+b+1/n)n^3}{12}.$$

The Gaussian limit

To compute $S_1(\tilde{z})$, we first compute the asymptotics of the i term, this leads to a $\ln(i)$ contribution, which will be cancelled by the factorials. We obtain

$$Z_2(j) \sim e^{-x^2/2}.$$

$$\exp\left[\left[-\frac{3a^2+13ab+3b^2}{20ab(a+b)} + \frac{3(a^2+b^2+ab)x^2}{10ab(a+b)}\right] \Big/ n + \mathcal{O}(n^{-3/2})\right]$$

$$\Big/ (\pi ab(a+b)n^3/6)^{1/2}\ . \tag{41}$$

The Large deviation, $j = \mu + xn^{7/4}$

We derive

$$Z_2(j) \sim \exp\left[-\frac{6x^2}{ab(a+b)}n^{1/2} - \frac{36(a^2+ab+b^2)x^4}{5[ab(a+b)]^3} + C_9(x,a,b)/n^{1/2} + C_{10}(x,a,b)/n + \dots\right] \Big/ (\pi ab(a+b)n^3/6)^{1/2}\ . \tag{42}$$

To check the effect of the correction, we first give in Figure 22, for $n = 50, a = b = 1/2$ and $x \in [0..0.2]$, the comparison between $Z_2(j)$ and the asymptotics (42). Figure 23 shows the quotient of $Z_2(j)$ and the asymptotics (42)

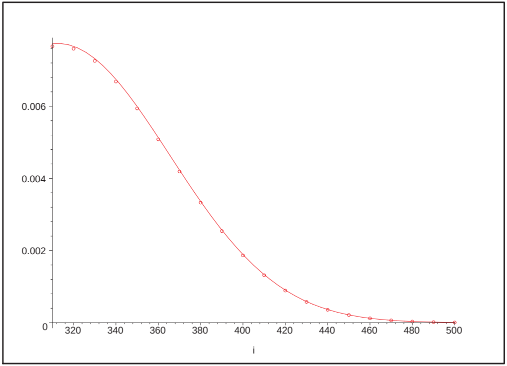

Figure 22: The comparison between $Z_2(j)$ and the asymptotics (42).

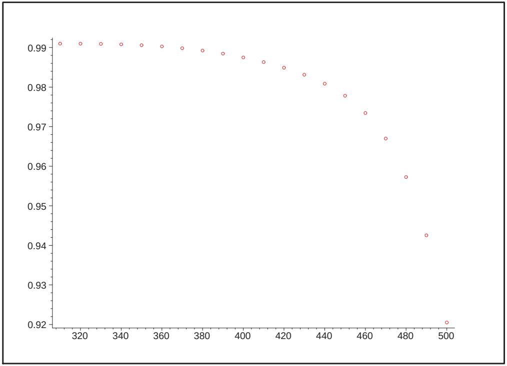

Figure 23: The quotient of $Z_2(j)$ and the asymptotics (42)

Asymptotics of the Stirling numbers of the first kind revisited

Let $\left[\begin{array}{c} n \\ j \end{array}\right]$ be the Stirling number of the first kind (unsigned version). Their generating function is given by

$$\phi_n(z) = \prod_0^{n-1}(z+i) = \frac{\Gamma(z+n)}{\Gamma(z)}, \quad \phi_n(1) = n!.$$

Consider the random variable J_n, with probability distribution

$$\mathbb{P}[J_n = j] = Z_n(j),$$

$$Z_n(j) := \frac{\left[\begin{array}{c} n \\ j \end{array}\right]}{n!}.$$

$$M := \mathbb{E}(J_n) = \sum_{0}^{n-1} \frac{1}{1+i} = H_n = \psi(n+1) + \gamma,$$

$$\sigma^2 := \mathbb{V}(J_n) = \sum_{0}^{n-1} \frac{i}{(1+i)^2} = \psi(1, n+1) + \psi(n+1) - \frac{\pi^2}{6} + \gamma,$$

where $\psi(k, x)$ is the kth polygamma function, and

$$M \sim \ln(n) + \gamma + \frac{1}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right),$$

$$\sigma^2 \sim \ln(n) - \frac{\pi^2}{6} + \gamma + \frac{3}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

It is convenient to set

$$A_n := \ln(n) - \frac{\pi^2}{6} + \gamma = \ln\left(ne^{\gamma - \pi^2/6}\right),$$

and to consider all our next asymptotics ($n \to \infty$) as functions of A_n. All asymptotics can be reformulated in terms of $\ln(n)$.

We have

$$M \sim A_n + \frac{\pi^2}{6} + \mathcal{O}\left(\frac{1}{n}\right),$$
$$\sigma^2 \sim A_n + \mathcal{O}\left(\frac{1}{n}\right).$$

A celebretated central limit theorem of Goncharov says that

$$J_n \sim \mathcal{N}\left(M, \sigma\right),$$

where $\mathcal{N}$ is the Gaussian distribution, with a rate of convergence $\mathcal{O}(1/\sqrt{\ln(n)})$.

In this Section, we want to obtain a more precise local limit theorem for J_n in terms of $x := \frac{J_n - M}{\sigma}$ and A_n. Actually, we obtain the following result , where we use $B_n := \sqrt{A_n}$ to simplify the expressions.

$$
\begin{aligned}
& Z_n(j) \sim R_1, \\
& \quad R_1 := \frac{1}{\sqrt{2\pi} B_n} e^{-x^2/2}. \\
& \quad \cdot \left[1 + \frac{x^3/6 - x/2}{B_n} + \frac{3x^2/8 - x^4/6 - 1/12 + x^6/72}{B_n^2} \right. \\
& \quad + \frac{1}{B_n^3} [-\pi^2 x^3/18 + 37x^5/240 - 355x^3/144 + x/8 - x^7/48 \\
& \quad \left. + x^9/1296 + \pi^2 x/6 - \zeta(3)x + \zeta(3)x^3/3] + \dots \right].
\end{aligned}
$$

A comparison of $Z_n(j) \Big/ \left[\frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\left(\frac{j-M}{\sigma} \right)^2 \Big/ 2 \right] \right]$ with $Z_n(j)/R_1$, with 2 terms in R_1, is given in Figure 24.

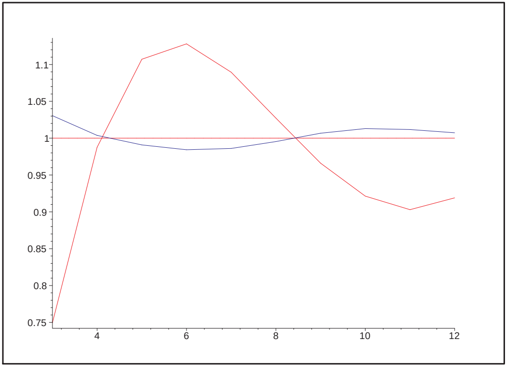

Figure 24: $Z_n(j) \Big/ \left[\frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\left(\frac{j-M}{\sigma}\right)^2 \Big/ 2\right]\right]$, color=red, $Z_n(j)/R_1$, color=blue, $n = 3000$

The precision of R_1 is of order 10^{-2}. Using 3 terms in R_1 leads to a less good result: A_n is not large enough to take advantage of the $A_n^{-3/2}$ term: $A_n = 6.94$ here, we deal with asymptotic series, not necessarily convergent ones. More terms can be computed in R_1 (which is almost automatic with Maple).

The justification of the integration procedure goes as folows. We proceed as in Flajolet and Sedgewick (Analytic combinatorics,FS), ch. *VIII*. We can choose here $\tilde{z} = 1$. This leads, with $z = e^{\mathbf{i}\theta}$, to

$$S(z) \sim S_0(z) + \mathcal{O}\left(\sqrt{\ln(n)}\theta\right) + \text{ constant term},$$

with

$$S_0(z) = \sum_{k=0}^{n-1} \ln[e^{\mathbf{i}\theta} + k] - H_n \mathbf{i}\theta$$
$$\sim \sum_{k=0}^{n-1} \frac{1}{1+k}[e^{\mathbf{i}\theta} - 1] - \frac{1}{2}\sum_{k=0}^{n-1}\left[\frac{1}{1+k}[e^{\mathbf{i}\theta} - 1\right]^2 - H_n\mathbf{i}\theta + \mathcal{O}(\theta^3)$$
$$\sim H_n[e^{\mathbf{i}\theta} - 1 - \mathbf{i}\theta] + \mathcal{O}(\theta^2).$$

Set

$$h(\theta) := e^{\mathbf{i}\theta} - 1 - \mathbf{i}\theta.$$

We have

$$h(\theta) \sim -\frac{\theta^2}{2},$$

The function $h(\theta)$ is the same as in FS, Ex. $VIII$.3, which proves the validity of our integration procedure: we use here $H_n \sim \ln(n)$ instead of n. The complete asymptotic expansion is justified as in FS, Ex. $VIII$.4.

Large deviation, $j = n - n^\alpha, \quad 1 > \alpha > 1/2$

We have

$$G_n(z) := \frac{\Gamma(z+n)}{\Gamma(z)z^{j+1}} = \exp[S(z)],$$

with

$$S(z) = S_1(z) + S_2(z), S_1(z) = \sum_0^{n-1} \ln(z+i), S_2(z) = -(j+1)\ln(z).$$

Some experiments with some values for α ($\alpha = 5/8$ is a good choice) show that $\tilde{z}$ must be a combination of $x = n^\alpha$ and $y = n^{1-\alpha}$ and $x \gg y \gg 1$. Note that both x and y are large. We will derive series of powers of x^{-1}, where each coefficient is a series of powers of y^{-1}.

First, by bootstrapping, we obtain (we give the first terms)

$$\tilde{z} = \frac{ny}{2}\left[1 - \frac{4}{3y} + \frac{2}{9y^2} + \frac{8}{135y^3} + \frac{8}{405y^4} + \frac{16}{1701y^5} + \frac{232}{45525y^6} + \frac{64}{18225y^7} + \dots + \frac{1}{x}\left[1 - \frac{1}{y} + \frac{4}{9y^2} - \frac{16}{135y^3} + \dots\right] + \frac{1}{x^2}\left[1 - \frac{1}{y} + \frac{0}{y^2} + \dots\right] + \frac{1}{x^3}\left[1 + \dots\right] + \mathcal{O}\left(\frac{1}{x^4}\right)\right]. \quad (43)$$

Note that the choice of dominant terms in the bracket of (43) depends on α. For instance, for $\alpha = 3/4$, the dominant terms (in decreasing order) are

$$1, \frac{1}{y}, \frac{1}{y^2}, \left\{\frac{1}{x}, \frac{1}{y^3}\right\}, \left\{\frac{1}{xy}, \frac{1}{y^4}\right\}, \left\{\frac{1}{xy^2}, \frac{1}{y^5}\right\}, \left\{\frac{1}{x^2}, \frac{1}{xy^3}, \frac{1}{y^6}\right\}, \dots$$

We obtain the following result

$$[z^j]\phi_n(z) \sim \frac{1}{\sqrt{2\pi}} \frac{y^2\sqrt{x}}{2} e^{S(\tilde{z})} T_4$$

$$T_4 = 1 - \frac{2}{3y} - \frac{2}{9y^2} + \ldots + \frac{1}{x}\left(\frac{5}{12} - \frac{11}{18y} + \ldots\right)$$

$$+ \frac{1}{x^2}\left(\frac{73}{288} - \frac{133}{432y} + \ldots\right) + \frac{1}{x^3}\left(\frac{721}{576} + \ldots\right) + \mathcal{O}\left(\frac{1}{x^4}\right) \quad (44)$$

We have made several experiments with (44), with n up to 500. The result is unsatisfactory, only values of x of order $\sqrt{n}$ give reasonable results. Actually, only very large values of n lead to good precision. So we turn to another formulation: instead of using an expansion for $e^{S(\tilde{z})}$, we plug directly $\tilde{z}$ into $G_n(z)$, ie we set

$$T_7 = G_n(\tilde{z}),$$

leading to

$$[z^j]\phi_n(z) \sim \frac{1}{\sqrt{2\pi}} \frac{y^2\sqrt{x}}{2} T_7 T_4 =: T_8 \text{ say } .$$

For $n = 500$, using two and three terms in $T4$, we give in Figures 25 and 26, the quotient $[z^j]\phi_n(z)/T_8$. The precision is now of order 10^{-5}.

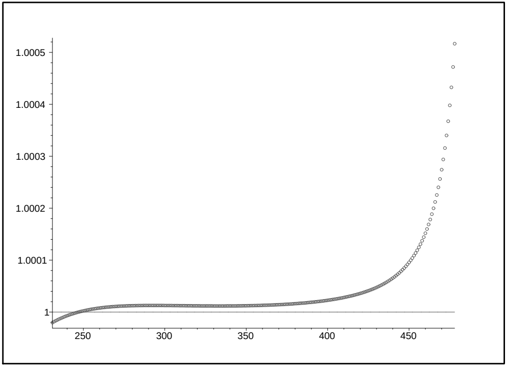

Figure 25: The quotient $[z^j]\phi_n(z)/T_8$, two terms in T_4, as function of j, $n = 500$

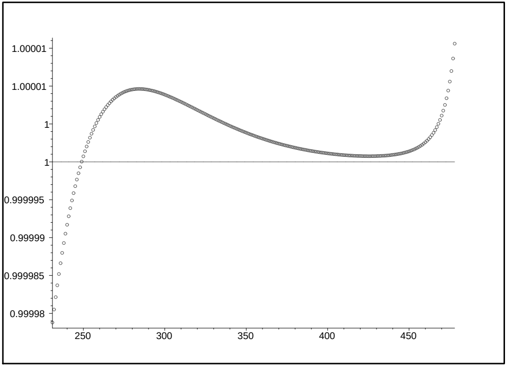

Figure 26: The quotient $[z^j]\phi_n(z)/T_8$, three terms in T_4, as function of j, $n = 500$