Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

FWF Der Wissenschaftsfonds.

Extending algorithms for D-finite functions

D^n-finite functions

Antonio Jiménez-Pastor, Veronika Pillwein

LAND OBERÖSTERREICH

JOHANNES KEPLER UNIVERSITY LINZ

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Outline

1. D-finite functions
2. DD-finite functions
3. Implementation of closure properties
4. D^n-finite functions
5. Inclusion properties
6. Conclusions and future work

Extending D-finite to DD-finite

Notation

Through this talk we consider:

- K: a **computable** field
- $K[[x]]$: ring of formal power series over K.
- Given F a field:

$$V_F(f) = \langle f, f', f'', \ldots \rangle_F.$$

D-finite functions

Definition

Let $f \in K[[x]]$. We say that f is *D-finite* (or *holonomic*) if there exist $d \in \mathbb{N}$ and polynomials $p_0(x), \dots, p_d(x)$ such that:

$$p_d(x)f^{(d)}(x) + \dots + p_0(x)f(x) = 0.$$

We say that d is the *order* of f.

Examples

A lot of special functions are D-finite:

- Exponential function: e^x.
- Trigonometric functions: $\sin(x)$, $\cos(x)$.
- Logarithm function: $\log(x+1)$.
- Bessel functions: $J_n(x)$.
- Hypergeometric functions: ${}_pF_q\left(\begin{array}{c} a_1, \dots, a_p \\ b_1, \dots, b_q \end{array}; x\right)$.
- Airy functions: $Ai(x), Bi(x)$.
- Combinatorial generating functions: $F(x), C(x)$,,...

Non-D-finite examples

There are power series that are not D-finite:

- Double exponential: $f(x) = e^{e^x}$.
- Tangent: $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

- Gamma function: $f(x) = \Gamma(x+1)$.
- Partition Generating Function: $f(x) = \sum_{n\geq 0} p(n)x^n$.

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DD-finite Functions

Definition

Let $f \in K[[x]]$. We say that f is *DD-finite* if there exist $d \in \mathbb{N}$ and D-finite elements $r_0(x), ..., r_d(x)$ such that:

$$r_d(x)f^{(d)}(x) + ... + r_0(x)f(x) = 0.$$

Examples

The set is bigger than the D-finite functions:

$$
\begin{array}{ccc}
f \text{ is D-finite} & \Rightarrow & f \text{ is DD-finite,} \\
f(x) = e^{e^x} & \Rightarrow & f'(x) - e^x f(x) = 0, \\
f(x) = \tan(x) & \Rightarrow & \cos(x)^2 f''(x) - 2f(x) = 0, \\
f(x) = e^{\int_0^x J_n(t)dt} & \Rightarrow & f'(x) - J_n(x)f(x) = 0
\end{array}
$$

Differentially Definable Functions

Definition

Let $f \in K[[x]]$. We say that f is *DD-finite* if there exist $d \in \mathbb{N}$ and D-finite elements $r_0(x), \ldots, r_d(x)$ such that:

$$r_d(x) f^{(d)}(x) + \ldots + r_0(x) f(x) = 0.$$

Differentially Definable Functions

Definition

Let $f \in K[[x]]$ and $R \subset K[[x]]$ a ring. We say that f is differentially definable over R if there exist $d \in \mathbb{N}$ and elements in R $r_0(x), ..., r_d(x)$ such that:

$$r_d(x)f^{(d)}(x) + ... + r_0(x)f(x) = 0.$$

D(R): differentially definable functions over R.

Characterization Theorem

The following are equivalent:

$$f(x) \in D(R).$$

There are elements $r_0(x), \ldots, r_d(x) \in R$ and $g(x) \in D(R)$ such:

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Let F be the *field of fractions* of R. Then $V_F(f)$ has finite dimension.

Closure properties

$f(x), g(x) \in D(R)$ of order d_1, d_2.
F the field of fractions of R.
$a(x)$ algebraic over F of degree p.

Property	**Is in** $D(R)$	**Order bound**
Addition	$(f+g)$	$d_1 + d_2$
Product	(fg)	$d_1 d_2$
Differentiation	f'	d_1
Integration	$\int f$	$d_1 + 1$
Be Algebraic	$a(x)$	p

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$\longrightarrow$ Proof by direct formula
$\longrightarrow$ Proof by linear algebra

Implementation of closure properties

Vector spaces

Let $R \subset K[[x]]$, F its field of fractions and $V_F(f)$ the F-vector space spanned by f and its derivatives.

The Characterization theorem provides

$$f(x) \in D(R) \quad \Leftrightarrow \quad \dim(V_F(f)) < \infty$$

The ansatz method

Specifications

Input: A power series $h(x)$ $(f(x) + g(x)$, $f(x)g(x)$ or $a(x))$
Output: An operator $\mathcal{A} \in R[\partial]$ such that $\mathcal{A}h = 0$

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Method

1. Compute $W \subset K[[x]]$ such that $\dim(W) < \infty$ and $V_F(h) \subset W$.

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3. For $i = 0, ..., \dim(W)$, compute vectors $v_i \in F^n$ such that:
$$h^{(i)}(x) = \sum_{j=0}^{n} v_{ij}\phi_j.$$

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Method

4. Set up the ansatz:

$$\alpha_0 h(x) + ... + \alpha_n h^{(n)} = 0.$$

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Method

4. Set up the ansatz:

$$\alpha_0 h(x) + \ldots + \alpha_n h^{(n)} = 0.$$

5. Solve the induced F-linear system for the variables α_k.

The ansatz method

Specifications

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Method

4. Set up the ansatz:

$$\alpha_0 h(x) + \ldots + \alpha_n h^{(n)} = 0.$$

5. Solve the induced F-linear system for the variables α_k.
6. Return $\mathcal{A} = \alpha_n \partial^n + \ldots + \alpha_1 \partial + \alpha_0$.

The ansatz method

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Input: A power series $h(x)$ ($f(x) + g(x)$, $f(x)g(x)$ or $a(x)$)
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1. Compute $W \subset K[[x]]$ such that $\dim(W) < \infty$ and $V_F(h) \subset W$.
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Mathieu: definition

Mathieu functions

Mathieu functions are solutions of the differential equation:

$$w''(x) + (a - 2q\cos(2x))w(x) = 0$$

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The sine and cosine

- Cos: $w_1(x)$ with $w_1(0) = 1$ and $w_1'(0) = 0$.
- Sin: $w_2(x)$ with $w_2(0) = 0$ and $w_2'(0) = 1$.

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- Sin: $w_2(x)$ with $w_2(0) = 0$ and $w_2'(0) = 1$.

$$\mathcal{W} = \begin{vmatrix} w_1 & w_2 \\ w_1' & w_2' \end{vmatrix} = w_1(x)w_2'(x) - w_1'(x)w_2(x) = 1.$$

Mathieu: derivative

Equation for $w_1'(x)$ and $w_2'(x)$

$$\begin{aligned} (a - 2q\cos(2x)) \quad & y'' \\ -(4q\sin(2x)) \quad & y' \\ +\left(a^2 - 4aq\cos(2x) + 4q^2\cos(2x)^2\right) \quad & y \quad = \quad 0 \end{aligned}$$

Mathieu: product

Equation for $w_1(x)w_2'(x)$ and $w_2(x)w_1'(x)$

$$\beta_4 y^{(4)} + \beta_3 y^{(3)} + \beta_2 y'' + \beta_1 y' = 0,$$

$$\begin{aligned}
\beta_4 &= q\sin(2x)^2 - a\cos(2x) + 2q \\
\beta_3 &= -2\sin(2x)\left(2q\cos(2x) + a\right) \\
\beta_2 &= -4\left(2q\sin(2x)^2\cos(2x) - q(a+1)\cos(2x)^2 + \right. \\
&\quad \left. +(4q^2 + a^2)\cos(2x) - 3q(a+1)\right) \\
\beta_1 &= 8\sin(2x)\left(q^2\sin(2x)^2 - 5aq\cos(2x) + 14q^2 - a^2\right)
\end{aligned}$$

D^n-finite functions: iterating the process

D^n-finite functions

Remark

Given a differential ring $R \subset K[[x]]$, the closure properties show that $D(R)$ is again a ring. Hence we can iterate the construction with the same algorithms.

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D^n-finite functions

We define the D^n-finite functions as the nth iteration over the polynomials, i.e., $D^n(K[x])$.

$$K[x] \subset D(K[x]) \subset D^2(K[x]) \subset \ldots \subset D^n(K[x]) \subset \ldots$$

New Properties

$f(x) \in D^n(K[x])$ of order d_1.
$g(x) \in D^m(K[x])$ of order d_2.
$a(x)$ algebraic over $D^m(K[x])$ of degree p.

Property	Function	Is in	Order bound
Composition	$f \circ g$	$D^{n+m}(K[x])$	d_1
Alg. subs.	$f \circ a$	$D^{n+m}(K[x])$	pd_1

$D^n \subsetneq D^{n+1}$: Iterated exponentials

Iterated exponentials

$$K[x] \subsetneq \mathrm{D}(K[x]) \subset \mathrm{D}^2(K[x]) \subset \ldots \subset \mathrm{D}^n(K[x]) \subset \ldots$$

Iterated exponentials

$$K[x] \subsetneq D(K[x]) \subsetneq D^2(K[x]) \subset \ldots \subset D^n(K[x]) \subset \ldots$$

$$e^x \in D(K[x]), \qquad e^{e^x - 1} \in D^2(K[x])$$

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$$e^x \in D(K[x]), \qquad e^{e^x-1} \in D^2(K[x])$$

Iterated Exponentials

- $e_0(x) = 1$,
- $\hat{e}_n(x) = \int_0^x e_n(t)dt$,
- $e_{n+1}(x) = \exp(\hat{e}_n(x))$.

Increasing chain

Proposition

- $e_n(x) \in D^n(K[x])$.
- $e_n(x) \notin D^{n-1}(K[x])$.

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First is trivial: $e_n'(x) = e_{n-1}(x)e_n(x)$.

Increasing chain

Proposition

- $e_n(x) \in D^n(K[x])$.
- $e_n(x) \notin D^{n-1}(K[x])$.

Second: proof using Differential Galois Theory (M. F. Singer)

Picard-Vessiot

Picard-Vessiot closure

Let (K, ∂) be a differential field with constants C. The *Picard-Vessiot* closure is the *smallest* field with same constants such that **all** linear differential equation with coefficients in K have all the C-linearly independent solutions.

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$$
\begin{array}{ccccccccccc}
C[x] & \subset & D(C[x]) & \subset & \dots & \subset & D^n(C[x]) & \subset & \dots & \subset & C[[x]] \\
\cap & & \cap & & \ddots & & \cap & & \ddots & & \\
F_0 & \subset & F_1 & \subset & \dots & \subset & F_n & \subset & \dots & \subset & C((x)) \\
\cap & & \cap & & \ddots & & \cap & & \ddots & & \\
K_0 & \subset & K_1 & \subset & \dots & \subset & K_n & \subset & \dots & \subset & K_{PV}
\end{array}
$$

Main result

Proposition

Let (K, ∂) be a differential field with algebraically closed field of constants C. Let E be a PV-extension of K. Let $u, v \in E \setminus \{0\}$ such that:

$$\frac{u'}{u} = a \in K, \qquad \frac{v'}{v} = u,$$

then u is algebraic over K.

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$$\frac{u'}{u} = a \in K, \qquad \frac{v'}{v} = u,$$

then u is algebraic over K.

Corollary

Let $c \in C^*$ and $n \in \mathbb{N} \setminus \{0\}$. Then $e_n^c = \exp(c\hat{e}_{n-1}) \notin K_{n-1}$.

Main result

$$\begin{array}{ccccccccccc}
C[x] & \subset & D(C[x]) & \subset & \dots & \subset & D^{n-1}(C[x]) & \subset & \dots & \subset & C[[x]] \\
\cap & & \cap & & \ddots & & \cap & & \ddots & & \\
F_0 & \subset & F_1 & \subset & \dots & \subset & F_{n-1} & \subset & \dots & \subset & C((x)) \\
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\end{array}$$

$e_n(x)$

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$e_n(x) \notin K_{n-1}$, and...

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$e_n(x) \notin F_{n-1}$, and...

Main result

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K_0 & \subset & K_1 & \subset & \dots & \subset & K_{n-1} & \subset & \dots & \subset & K_{PV}
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$$

$e_n(x) \notin D^{n-1}(K[x])$, finishing the proof.

Non linear differential equations

- Diff. definable over $R \longrightarrow$ linear differential equation.
- Diff. algebraic over $R \longrightarrow$ non-linear differential equation.

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Theorem

Let $f \in K[[x]]$. If there is $n \in \mathbb{N}$ with $f \in D^n(R)$, then f is differentially algebraic over R.

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Theorem

Let $f \in K[[x]]$. If there is $n \in \mathbb{N}$ with $f \in D^n(R)$, then f is differentially algebraic over R.

The proof is constructive and it is implemented.

Non linear differential equations

- Double exponential $(\exp(\exp(x) - 1))$:

$$f'(x) - \exp(x)f(x) = 0 \rightarrow f''(x)f(x) - f'(x)^2 - f'(x)f(x) = 0$$

Non linear differential equations

- Double exponential $(\exp(\exp(x) - 1))$:

$$f'(x) - \exp(x)f(x) = 0 \rightarrow f''(x)f(x) - f'(x)^2 - f'(x)f(x) = 0$$

- Mathieu functions:

$$f''(x) - (a - 2q\cos(2x))f(x) = 0$$
$$\downarrow$$
$$f^{(5)}(x)f(x)^3 - 3f^{(4)}(x)f'(x)f(x)^2 - 4f'''(x)f''(x)f(x)^2 + 6f'''(x)f'(x)^2f(x) + 4f'''(x)f(x)^3 + 6f''(x)^2f'(x)f(x) - 6f''(x)f'(x)^3 - 4f''(x)f'(x)f(x)^2 = 0$$

The reverse is not true

Remark

Not all Diff. algebraic functions are D^n-finite (M. Van der Put)

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Key property

Let $P(x, y, y', \dots, y^{(n)})$ be a differential polynomial and $\mathcal{A} = \{f_1, \dots, f_n\}$ be a finite set of solutions, i.e.,

$$P(x, f_i(x), \dots, f_i^{(n)}(x)) = 0$$

Then $\mathcal{A}$ is a **algebraically independent set**.

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Let $P(x, y, y', \dots, y^{(n)})$ be a differential polynomial and $\mathcal{A} = \{f_1, \dots, f_n\}$ be a finite set of solutions, i.e.,

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Then $\mathcal{A}$ is a **algebraically independent set**.

Example

The equation $y' = y^2(y-1)$ has that property.

The SAGE package

SAGE system

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- Open Source computer system based on Python
- Interfaces to many mathematical tools.

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Public repository

https://www.dk-compmath.jku.at/Members/antonio/sage-package-dd_functions

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Based on package *ore_algebra* by M. Kauers and M. Mezzarobba

SAGE system

Features

- Implementation of $D(R)$ for any ring R.
- Computation of initial values for elements of $D(R)$.
- Implementation of closure properties $(+, -, *, /, \circ)$.
- Possibility to have constant parameters.
- Computation of non-linear differential equations.
- Library of examples (extracted from DLMF)

Conclusions and Future work

Conclusions

Achievements

- Extended the framework of D-finite to wider class of computable functions
- Implemented closure properties for DD-finite
- Implemented composition of D^n-finite functions
- Detected limits of the class of differentially definable
- Code available for SAGE

Conclusions

Future work

- Improve performance of the current code
- Study analytic properties of DD-finite functions
- Study combinatorial properties of DD-finite functions
- Study the annalog of DD-finite functions in sequences
- Multivariate case
- Generalize for other type of operators (*q-shift*)

Thank you!

Contact webpage:

- https://www.dk-compmath.jku.at/people/antonio
- https://www.risc.jku.at/home/ajpastor

SAGE code:

- https://www.dk-compmath.jku.at/Members/antonio/sage-package-dd_functions