

Discovering hook length formulas by an expansion technique

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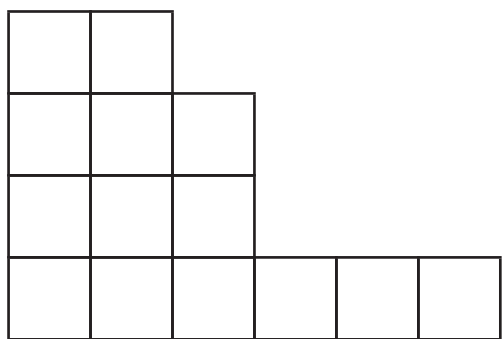
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Hook length formulas for partitions and trees

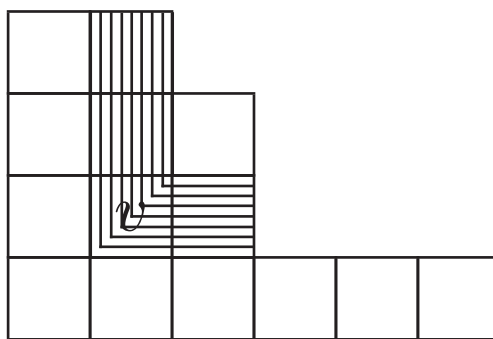
Summary:

- Some well-known examples
- How to discover new hook formulas ?
- The Main Theorem
- Specializations
- Number Theory
- Hook formulas for binary trees

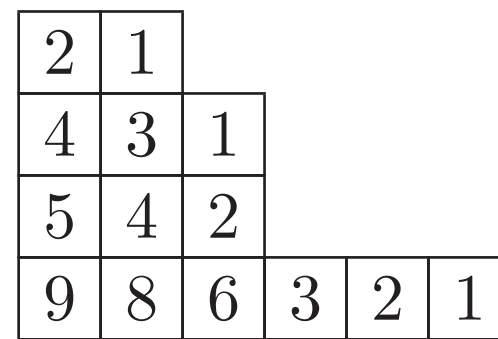
Some well-known examples: Hook length multi-set



Partition
 $\lambda = (6, 3, 3, 2)$



Hook length of v
 $h_v(\lambda) = 4$



Hook lengths
 $\mathcal{H}(\lambda)$

The hook length multi-set of λ is

$$\mathcal{H}(\lambda) = \{2, 1, 4, 3, 1, 5, 4, 2, 9, 8, 6, 3, 2, 1\}$$

Some well-known examples: permutations

f_λ : the number of standard Young tableaux of shape λ

Frame, Robinson and Thrall, 1954

$$f_\lambda = \frac{n!}{\prod_{h \in \mathcal{H}(\lambda)} h}$$

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$$f_\lambda = \frac{n!}{\prod_{h \in \mathcal{H}(\lambda)} h}$$

Robinson-Schensted correspondence: $\sum_{\lambda \vdash n} f_\lambda^2 = n!$

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} = e^x$$

Some well-known examples: involutions

Robinson-Schensted correspondence: The number of standard Young tableaux of $\{1, 2, \dots, n\}$ is equal to the number of *involutions* of order n .

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h} = e^{x+x^2/2}$$

Some well-known examples: partitions

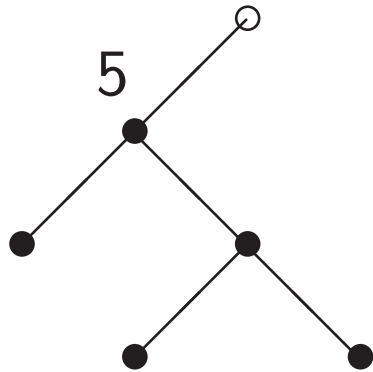
Euler: Generating function for partitions:

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} 1 = \prod_{k \geq 1} \frac{1}{1 - x^k}$$

Some well-known examples: binary trees

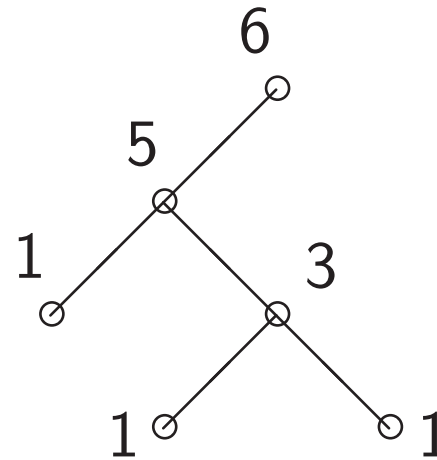
hook length for unlabeled binary trees

T



$$\mathcal{H}_v(T) = 5$$

T



$$\mathcal{H}(T) = \{1, 1, 1, 3, 5, 6\}$$

Some well-known examples: binary trees

f_T : the number of *increasing labeled binary trees*

$$f_T = \frac{n!}{\prod_{h \in \mathcal{H}(T)} h}$$

Some well-known examples: binary trees

Each labeled binary tree with n vertices is in bijection with a permutation of order n

$$\sum_{T \in \mathcal{B}(n)} n! \prod_{v \in T} \frac{1}{h_v} = n!$$

Some well-known examples: binary trees

Each labeled binary tree with n vertices is in bijection with a permutation of order n

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Generating function form:

$$\sum_{T \in \mathcal{B}} x^{|T|} \prod_{h \in \mathcal{H}(T)} \frac{1}{h} = \frac{1}{1-x}$$

Some well-known examples: binary trees, Catalan

The number of binary trees with n vertices is equal to the n -th Catalan number

$$\sum_{T \in \mathcal{B}(n)} 1 = \frac{1}{n+1} \binom{2n}{n}$$

Generating function form:

$$\sum_{T \in \mathcal{B}} x^{|T|} \prod_{h \in \mathcal{H}(T)} 1 = \frac{1 - \sqrt{1 - 4x}}{2x}$$

Some well-known examples: tangent numbers

The tangent number counts the *alternating permutations* (André, 1881), which are in bijection with the labeled complete binary trees.

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$$\sum_{T \in \mathcal{C}} x^{|T|} \prod_{h \in \mathcal{H}(T)} \frac{1}{h} = \tan(x) + \sec(x)$$

\mathcal{C} : complete binary trees

Discover new hook formulas

	<i>Partitions</i>	<i>Trees</i>
Discovering		
Proving		

Discover new hook formulas

	<i>Partitions</i>	<i>Trees</i>
Discovering	Hard	
Proving		

Discover new hook formulas

	<i>Partitions</i>	<i>Trees</i>
Discovering	Hard	Hard
Proving		

Discover new hook formulas

	<i>Partitions</i>	<i>Trees</i>
Discovering	Hard	Hard
Proving	Hard	

Discover new hook formulas

	<i>Partitions</i>	<i>Trees</i>
Discovering	Hard	Hard
Proving	Hard	Easy

Discover new hook formulas

We now introduce an efficient technique for discovering new hook length formulas:

Discover new hook formulas

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hook length expansion

Discover new hook formulas: expansion

$\rho(h)$: weight function

$f(x)$: formal power series

They are connected by the relation:

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \rho(h) = f(x)$$

Discover new hook formulas: expansion

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- *generating function* : $\rho \longrightarrow f$

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- *hook length expansion* : $\rho \longleftarrow f$

Discover new hook formulas: expansion

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- *generating function* : $\rho \longrightarrow f$
- *hook length expansion* : $\rho \longleftarrow f$
- *hook length formula* : when both ρ and f have “nice” forms

Discover new hook formulas: algorithm

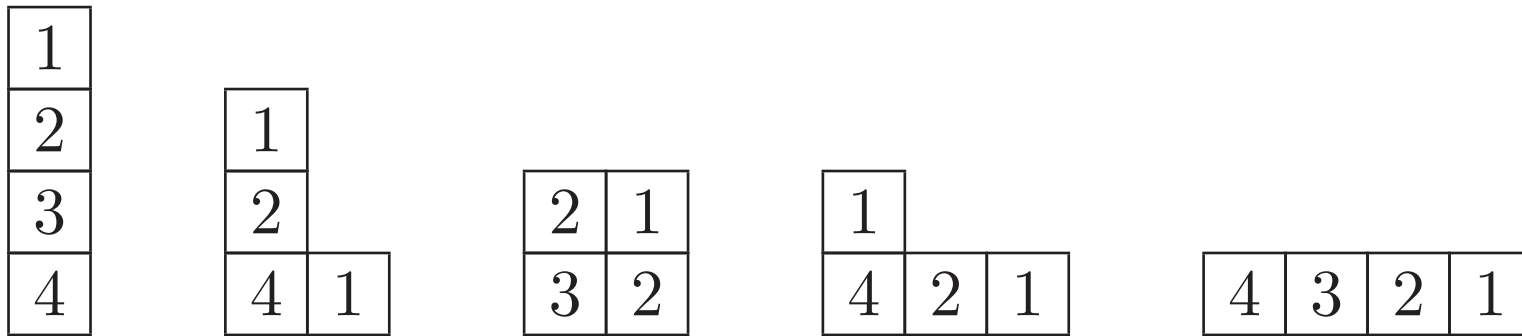
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Discover new hook formulas: algorithm

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- Is there an *algorithm* for computing the hook length expansion ? **Yes**.

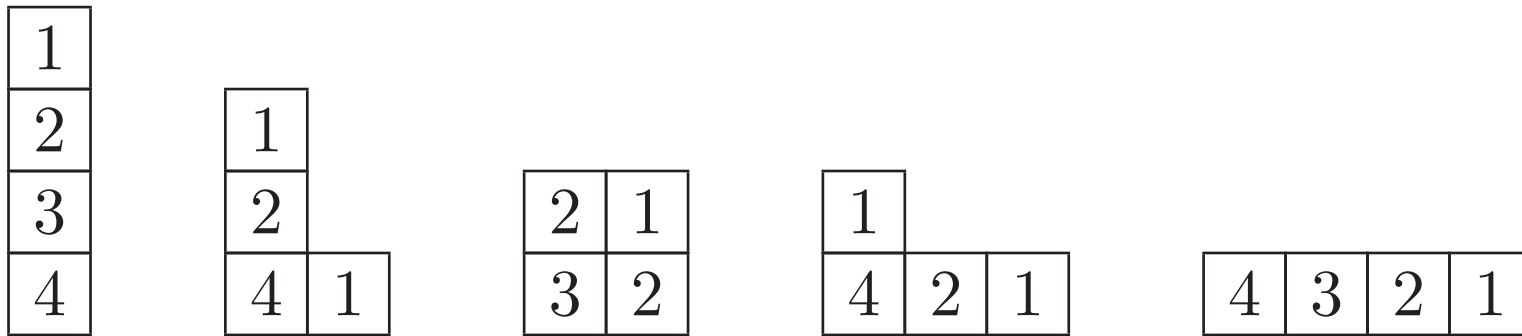
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Discover new hook formulas: algorithm

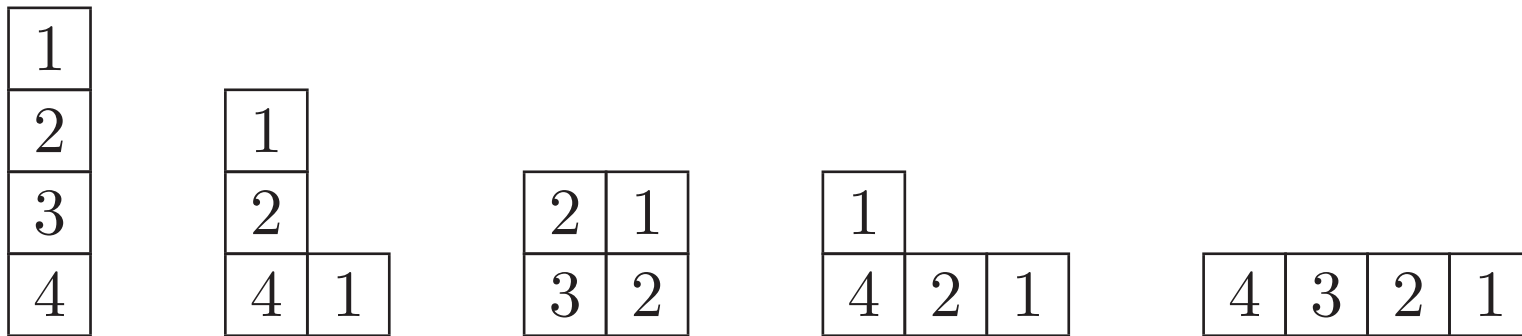
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$$\rho_4 \rho_3 \rho_2 \rho_1 + \rho_4 \rho_2 \rho_1 \rho_1 + \rho_3 \rho_2 \rho_2 \rho_1 + \rho_4 \rho_2 \rho_1 \rho_1 + \rho_4 \rho_3 \rho_2 \rho_1 = f_4$$

Discover new hook formulas: algorithm

- Does the *hook length expansion* exist ? **Yes.**
- Is there an *algorithm* for computing the hook length expansion ? **Yes.**



$$\rho_4 \rho_3 \rho_2 \rho_1 + \rho_4 \rho_2 \rho_1 \rho_1 + \rho_3 \rho_2 \rho_2 \rho_1 + \rho_4 \rho_2 \rho_1 \rho_1 + \rho_4 \rho_3 \rho_2 \rho_1 = f_4$$

We can solve ρ_4 when knowing $\rho_1, \rho_2, \rho_3, f_4$, because there is at most one “4” in each partition (linear equation with one variable)

Discover new hook formulas: maple package

Maple package for the hook length expansion

HookExp

Two procedures

hookgen: $\rho \longrightarrow f$

hookexp: $\rho \longleftarrow f$

Discover new hook formulas: permutation

Example : permutations

```
> read("HookExp.mp1") :  
> hookexp(exp(x), 8);
```

$$\left[1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \frac{1}{49}, \frac{1}{64}\right]$$

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} = e^x$$

Discover new hook formulas: involution

Example: involutions

```
> hookexp(exp(x+x^2/2), 8);
```

$$\left[1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}\right]$$

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h} = e^{x+x^2/2}$$

Discover new hook formulas: interpolation

permutations :
$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} = e^x$$

involutions :
$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h} = e^{x+x^2/2}$$

♥♥♥ What about the interpolation

$$e^{x+zx^2/2} \quad ?$$

Discover new hook formulas: interpolation

Try

```
> hookexp(exp(x+z*x^2/2), 8);
```

$$\left[1, \frac{1+z}{4}, \frac{3z+1}{9+3z}, \frac{z^2+6z+1}{16+16z}, \frac{5z^2+10z+1}{5z^2+50z+25}, \frac{z^3+15z^2+15z+1}{120z+36z^2+36}, \frac{7z^3+35z^2+21z+1}{7z^3+147z^2+245z+49} \right]$$

Many binomial coefficients, so that ...

Discover new hook formulas: interpolation

Interpolation between permutations and involutions:

First Conjecture (H., 2008)

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \rho(z; h) = e^{x + zx^2/2}$$

where

$$\rho(z; n) = \frac{\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} z^k}{n \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} z^k}$$

Latest news on the subject

The First Conjecture has been proved by:

Kevin Carde, Joe Loubert, Aaron Potechin, Adrian Sanborn

under the guidance of Dennis Stanton and Vic Reiner

(the Minnesota school)

Discover new hook formulas: partition

Another example. Euler: generating function for partitions

```
> hookexp(product(1/(1-x^k), k=1..9), 9);
```

```
[1, 1, 1, 1, 1, 1, 1, 1, 1]
```

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} 1 = \prod_{k \geq 1} \frac{1}{1 - x^k}$$

Discover new hook formulas: partition

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♥♥♥ What about $\prod_k (1 - x^k)$?

Discover new hook formulas: partition

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♥♥♥ What about $\prod_k (1 - x^k)$?

♥♥♥ or more generally $\prod_k (1 - x^k)^z$?

Discover new hook formulas: partition

Try it by using HookExp

Discover new hook formulas: partition

Try it by using HookExp

```
> hookexp(product((1-x^k)^z, k=1..7), 7);
```

$$\left[-z, \frac{3-z}{4}, \frac{8-z}{9}, \frac{15-z}{16}, \frac{24-z}{25}, \frac{35-z}{36}, \frac{48-z}{49}\right]$$

Discover new hook formulas: partition

Try it by using HookExp

```
> hookexp(product((1-x^k)^z, k=1..7), 7);  
[-z,  $\frac{3-z}{4}$ ,  $\frac{8-z}{9}$ ,  $\frac{15-z}{16}$ ,  $\frac{24-z}{25}$ ,  $\frac{35-z}{36}$ ,  $\frac{48-z}{49}$ ]
```

We see that the ρ has a very simple expression:

$$\rho(h) = \frac{h^2 - 1 - z}{h^2} = 1 - \frac{z + 1}{h^2}$$

Discover new hook formulas: partition

The previous hook length expansion suggests:

Theorem

$$\sum_{\lambda \in \mathcal{P}} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z+1}{h^2}\right) x = \prod_{k \geq 1} (1 - x^k)^z$$

The Russian-Physics Proof

Nekrasov, Okounkov (2003): arXiv: hep-th/0306238, 90 pages

(The last formula is deeply hidden in N-O's paper. See formula (6.12) on page 55)

Discover new hook formulas: proofs

The Russian-Physics Proof

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The “Elementary” Proof

H. (2008): arXiv:0805.1398 [math.CO]

Discover new hook formulas

“This is great !

But can we do more ?”

Main Theorem: $1 + x^k$

Partition with distinct parts - shift Young tableaux

$$\prod_k (1 + x^k)$$

Thrall, 1952

The number of standard shifted Young tableaux is given by

$$\frac{n!}{\prod_{h \in \mathcal{H}(\lambda)} h}$$

Main Theorem: $1 + x^k$

```
> hooktype := "PAD";  
> hookexp(product( 1+x^k, k=1..9), 9);
```

```
[1,  $\frac{1}{2}$ ,  $\frac{2}{3}$ ,  $\frac{3}{8}$ ,  $\frac{18}{29}$ ,  $\frac{52}{113}$ ,  $\frac{43539}{71974}$ ,  $\frac{50712791}{136184240}$ ,  $\frac{224560049745548}{376968863190753}$ ]
```

No formula !

Main Theorem: $1 + x^k$

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> hooktype := "PAD";  
> hookexp(product( 1+x^k, k=1..9), 9);
```

$$\left[1, \frac{1}{2}, \frac{2}{3}, \frac{3}{8}, \frac{18}{29}, \frac{52}{113}, \frac{43539}{71974}, \frac{50712791}{136184240}, \frac{224560049745548}{376968863190753}\right]$$

No formula !

```
> hooktype := "PA";  
> hookexp(product( 1+x^k, k=1..14), 14);
```

$$\left[1, \frac{1}{2}, 1, \frac{7}{8}, 1, \frac{17}{18}, 1, \frac{31}{32}, 1, \frac{49}{50}, 1, \frac{71}{72}, 1, \frac{97}{98}\right]$$

Main Theorem: $1 + x^k$

We have

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda), h \text{ even}} \left(1 - \frac{2}{h^2}\right) = \prod_{k \geq 1} (1 + x^k).$$

Main Theorem: $1 + x^k$

We have

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Compare with ($z = 1$ in N-O formula):

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{2}{h^2}\right) = \prod_{k \geq 1} (1 - x^k).$$

Main Theorem: $1 + x^k$

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 - No, with the right-hand side by `hookexp(←—)`, because no “nice” expansion for

$$\prod \frac{1}{1 + x^k} \quad \text{or} \quad \prod (1 + x^k)^z.$$

Main Theorem: $1 + x^k$

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 - No, with the right-hand side by `hookexp(←—)`, because no “nice” expansion for

$$\prod \frac{1}{1 + x^k} \quad \text{or} \quad \prod (1 + x^k)^z.$$

- Yes, with the left-hand side by `hookgen(→—)`.

Main Theorem: $1 + x^k$ variation

We have just seen:

$$\rho = \left[1, \frac{1}{2}, 1, \frac{7}{8}, 1, \frac{17}{18}, 1, \frac{31}{32}, 1, \frac{49}{50}, 1, \frac{71}{72}, 1, \frac{97}{98}\right] \longrightarrow \prod_{k \geq 1} (1 + x^k).$$

Try the following variations of ρ with [hookgen](#):

$$\left[1, 1 - \frac{z}{2}, 1, 1 - \frac{z}{8}, 1, 1 - \frac{z}{18}, 1, 1 - \frac{z}{32}, 1, 1 - \frac{z}{50}, 1\right]$$

$$[1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1]$$

$$[1, 1, z, 1, 1, z, 1, 1, z, 1, 1, z, 1, 1, z]$$

Main Theorem: $1 + x^k$ variation

> ...

$$\left[1, 1 - \frac{z}{2}, 1, 1 - \frac{z}{8}, 1, 1 - \frac{z}{18}, 1, 1 - \frac{z}{32}, 1, 1 - \frac{z}{50}, 1\right]$$

> hookgen(%): etamake(%, x, 10): simplify(%);

$$\prod_{k \geq 1} \frac{(1 - x^{2k})^z}{1 - x^k}$$

When $z = 1$

$$\prod_{k \geq 1} \frac{(1 - x^{2k})^z}{1 - x^k} = \prod_{k \geq 1} \frac{1 - x^{2k}}{1 - x^k} = \prod_{k \geq 1} (1 + x^k)$$

Main Theorem: $1 + x^k$ variation

```
> r:=n-> if n mod 3=0 then -1 else 1 fi:  
> [seq(r(i), i=1..17)];
```

[1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1]

```
> hookgen(%): etamake(%, x, 17): simplify(%) ;
```

$$\prod_{k \geq 1} \frac{(1 - x^{12k})^3 (1 - x^{3k})^6}{(1 - x^{6k})^9 (1 - x^k)}$$

Main Theorem

The previous and many other experimentations suggest:

Main Theorem (H. 2008)

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}_t(\lambda)} \left(y - \frac{tyz}{h^2} \right) = \prod_{k \geq 1} \frac{(1 - x^{tk})^t}{(1 - (yx^t)^k)^{t-z} (1 - x^k)}$$

$$\mathcal{H}_t(\lambda) = \{h \mid h \in \mathcal{H}(\lambda), h \equiv 0(\text{mod } t)\}.$$

Main Theorem: fields of interest

This work has some links with the following fields:

- General Mathematical Community: Euler, Jacobi, Gauss
- High Energy Physics Theory: Nekrasov, Okounkov
- Lie Algebra and Representation Theory: Macdonald, Dyson, Kostant, Milne, Schlosser, Bessenrodt
- Modular Forms and Number Theory: Ramanujan, Lehmer, Ono
- q-Series, Combinatorics: Andrews, Stanton, Stanley
- Symmetric Functions: Cauchy, Schur, Lascoux
- Algorithm, Computer Algebra: RSK, Krattenthaler (rate), Garvan (qseries), Rubey, Sloane
- Plane Trees: Viennot, Foata, Schützenberger, Strehl, Gessel, Postnikov

Main Theorem: Specializations

The Main Theorem has so many specializations:

- the Jacobi triple product identity \rightarrow
- the Gauss identity \rightarrow
- the Nekrasov-Okounkov formula
- the generating function for partitions
- the Macdonald identity for $A_\ell^{(a)}$
- the classical hook length formula
- the marked hook formula \rightarrow
- the generating function for t -cores
- the t -core analogues of the hook formula
- the t -core analogues of the marked hook formula
- ...

Main Theorem: Specializations

Why it has so many specializations?

- It contains 3 variables t, y, z
- We can give special values to t, y, z
- Compare the coefficients of the minimal terms
- Compare the coefficients of the maximal terms

Specializations, Jacobi + Gauss

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}_t(\lambda)} \left(y - \frac{tyz}{h^2} \right) = \prod_{k \geq 1} \frac{(1 - x^{tk})^t}{(1 - (yx^t)^k)^{t-z} (1 - x^k)}$$

$t = 1, y = 1, z = 4$:

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}_1(\lambda)} \left(1 - \frac{4}{h^2} \right) = \prod_{k \geq 1} (1 - x^k)^3$$

$t = 2, y = 1, z = 2$:

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}_2(\lambda)} \left(1 - \frac{4}{h^2} \right) = \prod_{k \geq 1} \frac{(1 - x^{2k})^2}{1 - x^k}$$

Specializations, Jacobi + Gauss

Simplify

$$J := \prod_{h \in \mathcal{H}_1(\lambda)} \left(1 - \frac{4}{h^2}\right) \quad \text{and} \quad G := \prod_{h \in \mathcal{H}_2(\lambda)} \left(1 - \frac{4}{h^2}\right)$$

- If a partition λ contains one box v whose hook length is $h_v = 2$, then

$$J = G = 0.$$

- Otherwise λ must be a *staircase partition*

1			
3	1		
5	3	1	
7	5	3	1

Specializations, Jacobi + Gauss

-3			
$\frac{5}{9}$	-3		
$\frac{21}{25}$	$\frac{5}{9}$	-3	
$\frac{45}{49}$	$\frac{21}{25}$	$\frac{5}{9}$	-3

J

1			
1	1		
1	1	1	
1	1	1	1

G

$$J = \left(\frac{(2m-1)^2 - 4}{(2m-1)^2} \right)^1 \cdots \left(\frac{5}{9} \right)^{m-1} \left(\frac{-3}{1} \right)^m = (-1)^m (2m+1)$$

$$G = 1$$

Specializations, Jacobi + Gauss

The Main Theorem unifies Jacobi and Gauss identities.

$t = 1, y = 1, z = 4:$

Jacobi

$$\prod_{m \geq 1} (1 - x^m)^3 = \sum_{m \geq 0} (-1)^m (2m + 1) x^{m(m+1)/2}$$

$t = 2, y = 1, z = 2:$

Gauss

$$\prod_{m \geq 1} \frac{(1 - x^{2m})^2}{1 - x^m} = \sum_{m \geq 0} x^{m(m+1)/2}$$

Specializations, t -cores

Let $\{z = t \text{ or } y = 0\}$, we get the well known formula:

$$\sum_{\lambda: t\text{-cores}} x^{|\lambda|} = \prod_{k \geq 1} \frac{(1 - x^{tk})^t}{1 - x^k}$$

Specializations, t -cores

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♥♥♥ What about

$$\prod_{k \geq 1} \frac{(1 + x^{tk})^t}{1 - x^k} ?$$

Specializations, t -cores

Let $\{z = t \text{ or } y = 0\}$, we get the well known formula:

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♥♥♥ What about

$$\prod_{k \geq 1} \frac{(1 + x^{tk})^t}{1 - x^k} ?$$

♥♥♥ How to generalize it ?

Specializations, t -cores

First, try `hookexp` (\leftarrow):

```
> hookexp( product( (1+x^k)/(1-x^k), k=1..9), 9);  
[2, 1, 1, 1, 1, 1, 1, 1, 1]
```

Specializations, t -cores

First, try `hookexp` (\leftarrow):

```
> hookexp( product( (1+x^k)/(1-x^k), k=1..9), 9);  
[2, 1, 1, 1, 1, 1, 1, 1, 1]
```

We have

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} 2^{\#\{h \in \mathcal{H}(\lambda), h=t\}} = \prod_{k \geq 1} \frac{(1 + x^{tk})^t}{1 - x^k}.$$

Specializations, t -cores

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Specializations, t -cores

First, try `hookexp` (\longleftarrow):

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Then, try `hookgen` (\longrightarrow):

Theorem (H. 2008)

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} y^{\#\{h \in \mathcal{H}(\lambda), h=t\}} = \prod_{k \geq 1} \frac{(1 + (y - 1)x^{tk})^t}{1 - x^k}$$

Special cases: $y = 0, 1, 2$

Latest news on the subject

A. Velingker, E. Clader, Y. Kemper, M. Wage, D. Collins, S. Wolfe
were working on these new hook length formulas and found
interesting applications on Modular Forms and Number Theory
under the guidance of Ken Ono

(the Wisconsin school)

Specializations, marked hook formula

- $\{z = -b/y, y \rightarrow 0\}$ in Main Theorem:

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{tb}{h^2} = e^{bx^t} \prod_{k \geq 1} \frac{(1 - x^{tk})^t}{1 - x^k}$$

- Compare the coefficients of $b^n x^{tn}$:

$$\sum_{\lambda \vdash tn, \# \mathcal{H}_t(\lambda) = n} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h^2} = \frac{1}{t^n n!}$$

- $t = 1$:

$$\sum_{\lambda \vdash n} f_{\lambda}^2 = n!$$

Specializations, marked hook formula

- Compare the coefficients of $(-z)^{n-1} x^{nt} y^n$

$$\sum_{\lambda \vdash nt, \# \mathcal{H}_t(\lambda) = n} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h^2} \sum_{h \in \mathcal{H}_t(\lambda)} h^2 = \frac{3n - 3 + 2t}{2(n-1)!}$$

- $t = 1$:

Marked hook formula

$$\sum_{\lambda \vdash n} f_{\lambda}^2 \sum_{h \in \mathcal{H}(\lambda)} h^2 = \frac{n(3n-1)}{2} n!$$

Specializations, marked hook formula

- Direct marked-RSK proof ? Not yet
- Generalizations ? Yes

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Specializations, marked hook formula

$$\sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h \in \mathcal{H}(\lambda)} h^2 = \frac{3n - 1}{2(n - 1)!}$$

$$\sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h \in \mathcal{H}(\lambda)} h^4 = \frac{40n^2 - 75n + 41}{6(n - 1)!}$$

$$\sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h \in \mathcal{H}(\lambda)} h^6 = \frac{1050n^3 - 4060n^2 + 5586n - 2552}{24(n - 1)!}$$

Second Conjecture (H. 2008)

$$P_k(n) = (n - 1)! \sum_{\lambda \vdash n} \left(\prod_{v \in \lambda} \frac{1}{h_v^2} \right) \left(\sum_{u \in \lambda} h_u^{2k} \right)$$

is a polynomial in n of degree k .

Latest news on the subject

The Second Conjecture has been proved by
Richard Stanley

Tewodros Amdeberhan slightly simplified Stanley's proof

(the MIT school)

Number Theory, Corollary

Corollary [$y = 1$]. *We have*

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}_t(\lambda)} \left(1 - \frac{tz}{h^2}\right) = \prod_{k \geq 1} \frac{(1 - x^{tk})^z}{1 - x^k}.$$

Number Theory, Discrete interpolation

Discrete interpolation :

$$\sum_{\lambda} x^{|\lambda|} \prod_{h \in \mathcal{H}_1(\lambda)} \left(1 - \frac{36}{h^2}\right) = \prod_{k \geq 1} \frac{(1 - x^k)^{36}}{1 - x^k};$$

$$\sum_{\lambda} x^{|\lambda|} \prod_{h \in \mathcal{H}_2(\lambda)} \left(1 - \frac{36}{h^2}\right) = \prod_{k \geq 1} \frac{(1 - x^{2k})^{18}}{1 - x^k};$$

$$\sum_{\lambda} x^{|\lambda|} \prod_{h \in \mathcal{H}_3(\lambda)} \left(1 - \frac{36}{h^2}\right) = \prod_{k \geq 1} \frac{(1 - x^{3k})^{12}}{1 - x^k};$$

$$\sum_{\lambda} x^{|\lambda|} \prod_{h \in \mathcal{H}_6(\lambda)} \left(1 - \frac{36}{h^2}\right) = \prod_{k \geq 1} \frac{(1 - x^{6k})^6}{1 - x^k},$$

where each sum is over all 6-cores λ .

Number Theory, Conjecture

Notation: Let $f(x) = \sum f_n x^n$ and $g(x) = \sum g_n x^n$ be two power series. We write

$$f(x) \equiv g(x)$$

if

$$f_n = 0 \iff g_n = 0.$$

Third Conjecture (H. 2008)

Let n, s, t be positive integers such that $t \neq 4, 10$ and $s \mid t$.
Then

$$\prod_{k \geq 1} \frac{(1 - x^{sk})^{t^2/s}}{1 - x^k} \equiv \prod_{k \geq 1} \frac{(1 - x^{tk})^t}{1 - x^k}.$$

Number Theory, $t = 2$

$t = 2$: Third conjecture is true.

Jacobi

$$\prod_{m \geq 1} \frac{(1 - x^m)^4}{(1 - x^m)} = \sum_{m \geq 0} (-1)^m (2m + 1) x^{m(m+1)/2}$$

Gauss

$$\prod_{m \geq 1} \frac{(1 - x^{2m})^2}{1 - x^m} = \sum_{m \geq 0} x^{m(m+1)/2}$$

Number Theory, Ramanujan and Lehmer

Ramanujan τ -function is defined by

$$\begin{aligned} x \prod_{m \geq 1} (1 - x^m)^{24} &= \sum_{n \geq 1} \tau(n) x^n \\ &= x - 24x^2 + 252x^3 - 1472x^4 + 4830x^5 - 6048x^6 + \dots \end{aligned}$$

Conjecture (Lehmer)

For each n we have $\tau(n) \neq 0$.

$$\prod_{m \geq 1} (1 - x^m)^{24} \equiv \frac{1}{1 - x}$$

Specializations, $t = 5$

$t = 5$: Third conjecture becomes

Third Conjecture ($t = 5$)

$$\prod_{k \geq 1} \frac{(1 - x^k)^{25}}{1 - x^k} = \prod_{k \geq 1} (1 - x^k)^{24} \equiv \prod_{k \geq 1} \frac{(1 - x^{5k})^5}{1 - x^k}$$

Granville and Ono ($t = 5$)

$$\prod_{k \geq 1} \frac{(1 - x^{5k})^5}{1 - x^k} \equiv \frac{1}{1 - x}$$

Specializations, $t = 5$

$t = 5$: Third conjecture becomes Lehmer conjecture.

Number Theory, $t = 3$

$t = 3$:

$$\prod_{m \geq 1} (1 - x^m)^8 = 1 - 8x + 20x^2 - 70x^4 + 64x^5 + 56x^6 - 125x^8 + \dots - 20482x^{220} + 24050x^{224} - 21624x^{225} + \dots$$

$$\prod_{m \geq 1} \frac{(1 - x^{3m})^3}{1 - x^m} = 1 + x + 2x^2 + 2x^4 + x^5 + 2x^6 + x^8 + \dots + 2x^{220} + 2x^{224} + 3x^{225} + \dots$$

Number Theory, $t = 3$, Theorem

$t = 3$: Third conjecture is true.

Theorem (H., Ono, 2008)

$$\prod_{m \geq 1} (1 - x^m)^8 \equiv \prod_{m \geq 1} \frac{(1 - x^{3m})^3}{1 - x^m}$$

Binary trees: tangent numbers

The tangent number counts the *alternating permutations* (André, 1881), or the labeled complete binary trees.

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```
> hooktype:="CBT": # Complete Binary Trees  
> hookexp(tan(x)+sec(x), 9);
```

$$\left[1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}\right]$$

Binary trees: tangent numbers

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$$\sum_{T \in \mathcal{C}} x^{|T|} \prod_{h \in \mathcal{H}(T)} \frac{1}{h} = \tan(x) + \sec(x)$$

\mathcal{C} : complete binary trees

tangent numbers

```
> hooktype:="BT": # Binary Trees  
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```

$$\left[1, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}, \frac{1}{14}, \frac{1}{16}\right]$$

tangent numbers

```
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```

$$\left[1, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}, \frac{1}{14}, \frac{1}{16}\right]$$

$$\sum_{T \in \mathcal{B}} x^{|T|} \prod_{h \in \mathcal{H}(T), h \geq 2} \frac{1}{2h} = \tan(x) + \sec(x)$$

tangent numbers

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$$\sum_{T \in \mathcal{B}} x^{|T|} \prod_{h \in \mathcal{H}(T), h \geq 2} \frac{1}{2h} = \tan(x) + \sec(x)$$

The tangent number counts *André permutations* (Foata, Schützenberger, Strehl, 1973).

Hook length formulas for plane trees, T1

Theorem T1 [$a = 0, z = 1$, all permutations]. *We have*

$$\sum_{T \in \mathcal{B}} x^{|T|} \prod_{h \in \mathcal{H}(T)} \frac{1}{h} = \frac{1}{1-x}.$$

Hook length formulas for plane trees, T2

Theorem T2 [$a \rightarrow \infty, z = 1$, Catalan number]. *We have*

$$\sum_{T \in \mathcal{B}} x^{|T|} \prod_{h \in \mathcal{H}(T)} 1 = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

Hook length formulas for plane trees, T3

Theorem T3 [$a = 1, z = 1$, Postnikov]. *We have*

$$\sum_{T \in \mathcal{B}} x^{|T|} \prod_{h \in \mathcal{H}(T)} \left(1 + \frac{1}{h}\right) = \sum_{n \geq 0} (n+1)^{n-1} \frac{(2x)^n}{n!}.$$

Hook length formulas for plane trees, T4

Theorem T4 [$z = 1$, left-hand side extension of Postnikov identity (Du-Lu)]. *We have*

$$\sum_{T \in \mathcal{B}(n)} \prod_{v \in T} \left(a + \frac{1}{h_v} \right) = \frac{1}{(n+1)!} \prod_{k=0}^{n-1} ((n+1+k)a + n+1-k).$$

Hook length formulas for plane trees, T5

Theorem T5 [$a = 1$, right-hand side extension of Postnikov identity (Han, 2008)]. *We have*

$$\sum_{T \in \mathcal{B}} x^{|T|} \prod_{v \in T} \frac{(z + h)^{h-1}}{h(2z + h - 1)^{h-2}} = \sum_{n \geq 0} z(z + n)^{n-1} \frac{(2x)^n}{n!}.$$

Hook length formulas for plane trees

Theorem TX(H., 2008)

We have

$$\begin{aligned} \sum_{T \in \mathcal{B}(n)} \prod_{h \in \mathcal{H}(T)} \frac{\prod_{i=1}^{h-1} (za + z + (2h - i)a + i)}{2h \prod_{i=1}^{h-2} (2za + 2z + (2h - 2 - i)a + i)} \\ = \frac{z(a+1)}{n!} \prod_{i=1}^{n-1} (za + z + (2n - i)a + i). \end{aligned}$$

Hook length formulas for plane trees

Theorem TX(H., 2008)

We have

$$\sum_{T \in \mathcal{B}(n)} \prod_{h \in \mathcal{H}(T)} \frac{\prod_{i=1}^{h-1} (za + z + (2h - i)a + i)}{2h \prod_{i=1}^{h-2} (2za + 2z + (2h - 2 - i)a + i)}$$
$$= \frac{z(a+1)}{n!} \prod_{i=1}^{n-1} (za + z + (2n - i)a + i).$$

♥♥♥ Theorem TX unifies a lot of well known hook formulas by taking special values of a and z , including Theorems T1, T2, T3, T4 and T5.

Latest news on the subject

L. Yang, B. Sagan, W. Chen, O. Gao, P. Guo, N. Eriksen, M. Kuba
have found generalizations and other proofs of certain
hook length formulas for plane trees

- Discovering hook length formulas by an expansion technique
- New hook length formulas for binary trees
- Yet another generalization of Postnikov's hook length formula for binary trees
- Some conjectures and open problems on partition hook lengths
- An explicit expansion formula for the powers of the Euler product in terms of partition hook lengths (arXiv exclusive)
- The Nekrasov-Okounkov hook length formula: refinement, elementary proof, extension and applications
- (with K. Ono) Hook lengths and 3-cores
- Hook lengths and shifted parts of partitions
- (with K. Ji) Combining hook length formulas and BG-ranks for partitions via the Littlewood decomposition
- (with Ch. Bessenrodt) Symmetry distribution between hook length and part length for partitions

References

- L. Yang, Generalizations of Han's hook length identities
- B. Sagan, Probabilistic proofs of hook length formulas involving trees
- R. Stanley, Some combinatorial properties of hook lengths, contents, and parts of partitions
- T. Amdeberhan, Differential operators, shifted parts, and hook lengths
- G. Kalai, Powers of Euler products and Han's marked hook formula (blog)
- E. Clader, Y. Kemper, M. Wage, Lacunarity of certain partition theoretic generating functions
- A. Velingker, An exact formula for the coefficients of Han's generating function
- D. Collins, S. Wolfe, Congruences for Han's generating function
- K. Carde, J. Loubert, A. Potechin, A. Sanborn, Proof of Han's hook expansion conjecture
- G. Panova, Proof of a conjecture of Okada
- W. Chen, O. Gao, P. Guo, Hook length formulas for trees by Han's expansion
- H. Shin, J. Zeng, An involution for symmetry of hook length and part length of partitions
- N. Eriksen, Combinatorial proofs for some forest hook length identities
- M. Kuba, A note on hook length formulas for trees

Thank you!

`www-irma.u-strasbg.fr/~guoniu/hook`