About the arrow **can**: Tensor product of series $\longrightarrow$ Double series and its usage in computer science

(MO: 200442 & 201753), Schützenberger's calculus and continuation of Li.

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C. Lavault, C. Tollu, N. Behr, N. Gargava.

Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
C. Bui, Q.H. Ngô, Vu Dinh, S. Goodenough.

CIP seminar, 12 January 2021

Plan

Aim of the talk

The main objective of this talk is to introduce concepts, objects and results concerning the following ladder

$$
\begin{array}{ccc}
(\mathbb{C}\langle X\rangle, ⧢, 1_{X^*}) & \xhookrightarrow{\mathrm{Li}_\bullet} & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\
\big\downarrow & & \big\downarrow \\
(\mathbb{C}\langle X\rangle, ⧢, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] & \xrightarrow{\mathrm{Li}^{(1)}_\bullet} & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\big\downarrow & & \big\downarrow \\
\mathbb{C}\langle X\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle & \xrightarrow{\mathrm{Li}^{(2)}_\bullet} & \mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\big\uparrow & \nearrow & \\
\mathbb{C}\langle X\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle & &
\end{array}
$$

Also, general tools for exploring enlarged indexation (and identities) are provided. In passing, we will introduce tools about rational series and rational expressions.

$$\mathrm{Li}_\bullet \ : \ \mathbb{C}\langle X\rangle \longrightarrow\!\!\!\!\rightarrow \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}$$

Polylogarithms

- With $(s_1, \ldots s_r) \in \mathbb{C}^r$ (and $|z| < 1$)

$$\mathrm{Li}(s_1, \ldots s_r) = \sum_{n_1 > n_2 > \ldots n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}}$$

  They are a priori coded by lists $(s_1, \ldots s_r)$ but, when $s_i \in \mathbb{N}_+$, admit an iterated integral representation and are better coded by words with letters in $X = \{x_0, x_1\}$. We will use the one-to-one correspondence (see CAP 17).

$$(s_1, \ldots, s_r) \in \mathbb{N}_+^r \leftrightarrow x_0^{s_1 - 1} x_1 \ldots x_0^{s_r - 1} x_1 \in X^* x_1 \qquad (1)$$

- $\mathrm{Li}(s)[z]$ is Jonquière and, for $\Re(s) > 1$, one has $\mathrm{Li}(s)[1] = \zeta(s)$
- Completed by $Li(x_0^n) = \frac{log^n(z)}{n!}$ this provides a family of independant functions admitting an analytic continuation on the cleft plane $\mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$ or $\widetilde{\mathbb{C} \setminus \{0, 1\}}$.

Li From Noncommutative Diff Eq (CAP 17)

The generating series $S = \sum_{w \in X^*} Li(w)$ satisfies (and is unique to do so)

$$\begin{cases} \mathbf{d}(S) = (\frac{x_0}{z} + \frac{x_1}{1-z}).S \\ \lim_{\substack{z \to 0 \\ z \in \Omega}} S(z) e^{-x_0 log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle X \rangle\rangle} \end{cases} \quad (2)$$

with $X = \{x_0, x_1\}$. This is, up to the sign of x_1, the solution G_0 of Drinfel'd [1] for KZ3. We define this unique solution as Li. All Li_w are $\mathbb{C}$- and even $\mathbb{C}(z)$-linearly independant (see CAP 17 *Linear independance without monodromy*).

1. V. Drinfel'd, *On quasitriangular quasi-hopf algebra and a group closely connected with Gal($\bar{\mathbb{Q}}/\mathbb{Q}$)*, Leningrad Math. J., 4, 829-860, 1991.

Linear independence without monodromy: theorem

Theorem

Let $S \in \mathcal{H}(\Omega)\langle\langle X \rangle\rangle$ be a solution of the (LM) equation

$$\mathbf{d}(S) = MS \ ; \ \langle S \mid 1_{X^*} \rangle = 1.$$

The following are equivalent :

1. *the family $(\langle S \mid w \rangle)_{w \in X^*}$ of coefficients is independant (linearly) over $\mathcal{C}$.*
2. *the family of coefficients $(\langle S \mid x \rangle)_{x \in X \cup \{1_{X^*}\}}$ is independant (linearly) over $\mathcal{C}$.*
3. *the family $(u_x)_{x \in X}$ is such that, for $f \in \mathcal{C}$ et $\alpha_x \in \mathbb{C}$*

$$\mathrm{d}(f) = \sum_{x \in X} \alpha_x u_x \Longrightarrow (\forall x \in X)(\alpha_x = 0).$$

M. Deneufchâtel, GHED, Hoang Ngoc Minh, A. I. Solomon, *Independence of hyperlogarithms over function fields via algebraic combinatorics*, Lecture Notes in Computer Science (2011), Volume 6742 (2011), 127-139.

Explicit construction of Li

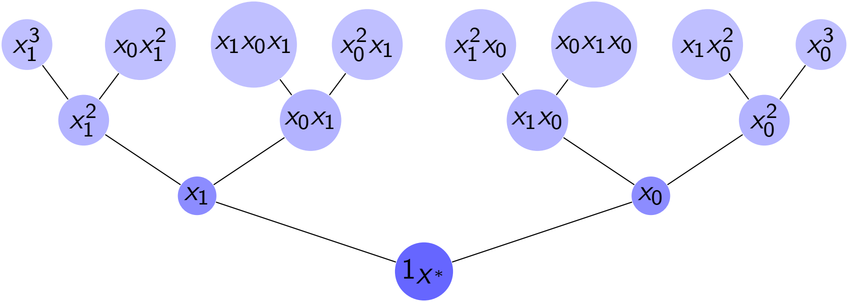
Given a word w, we note $|w|_{x_1}$ the number of occurrences of x_1 within w

$$\alpha_0^z(w) = \begin{cases} 1_\Omega & \text{if} \quad w = 1_{X^*} \\ \int_0^z \alpha_0^s(u)\frac{ds}{1-s} & \text{if} \quad w = x_1 u \\ \int_1^z \alpha_0^s(u)\frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} = 0 \\ \int_0^z \alpha_0^s(u)\frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} > 0 \ . \end{cases} \quad (3)$$

Of course, the third line of this recursion implies

$$\alpha_0^z(x_0^n) = \frac{log(z)^n}{n!}$$

one can check that (a) all the integrals (although improper for the fourth line) are well defined (b) the series $S = \sum_{w \in X^*} \alpha_0^z(w)\, w$ satisfies (2). We then have $\alpha_0^z = \mathrm{Li}$.



As an example, we compute some coefficients

$$\langle \mathrm{Li} \mid x_0^n \rangle = \frac{log(z)^n}{n!} \quad ; \quad \langle \mathrm{Li} \mid x_1^n \rangle = \frac{(-log(1-z))^n}{n!}$$

$$\langle \mathrm{Li} \mid x_0x_1 \rangle = \mathrm{Li}_2(z) = \sum_{n\geq 1} \frac{z^n}{n^2} \quad ; \quad \langle \mathrm{Li} \mid x_1x_0 \rangle = \langle \mathrm{Li} \mid x_1 \shuffle x_0 - x_0x_1 \rangle(z)$$

$$\langle \mathrm{Li} \mid x_0^2x_1 \rangle = \mathrm{Li}_3(z) = \sum_{n\geq 1} \frac{z^n}{n^3} \quad ; \quad \langle \mathrm{Li} \mid x_1x_0 \rangle = (-log(1-z))log(z) - \mathrm{Li}_2(z)$$

$$\langle \mathrm{Li} \mid x_0^{r-1}x_1 \rangle = \mathrm{Li}_r(z) = \sum_{n\geq 1} \frac{z^n}{n^r} \quad ; \quad \langle \mathrm{Li} \mid x_1^2x_0 \rangle = \langle \mathrm{Li} \mid \frac{1}{2}(x_1 \shuffle x_1 \shuffle x_0) - (x_1 \shuffle x_0x_1) + x_0x_1^2 \rangle$$

A simple transition system: weighted graphs

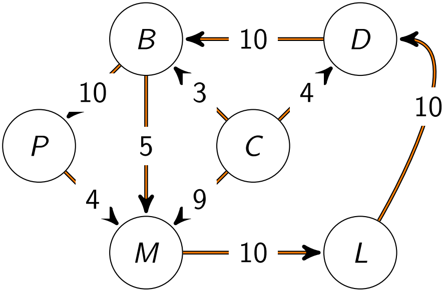


Figure: Directed graph weighted by numbers which can be lengths, time (durations), costs, fuel consumption, probabilities. This graph is equivalent to a square matrix. Coefficients are taken in different semirings (i.e. rings without the "minus" operation, as tropical or [max,+]) according to the type of computations to be done. Tropical mathematics were so called by MPS school because they were founded by the Hungarian-born Brazilian mathematician and computer scientist Imre Simon.

A small tribute to MPS or Marco as we used to call him

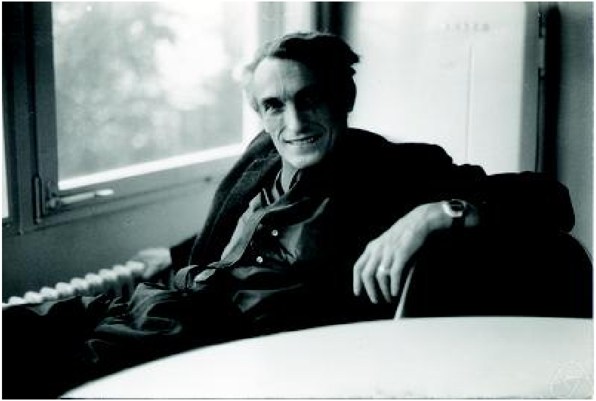
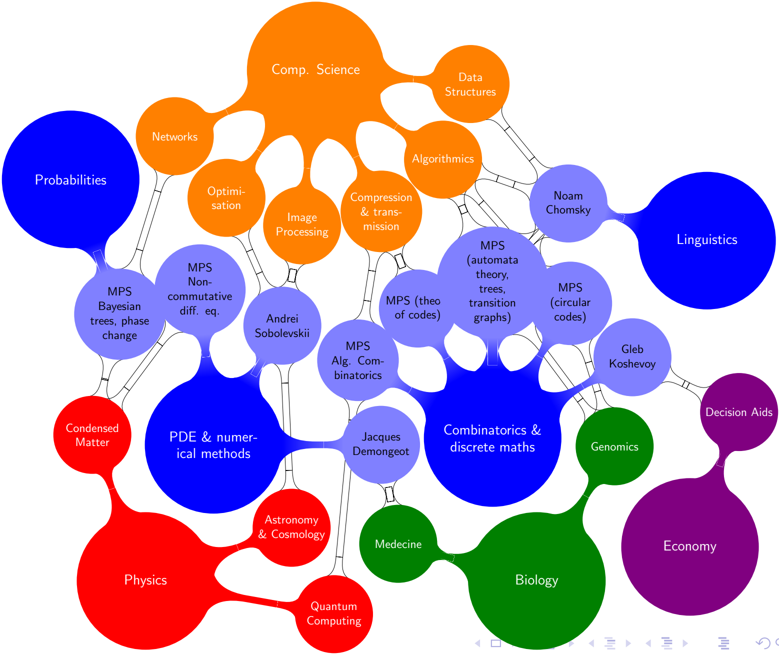
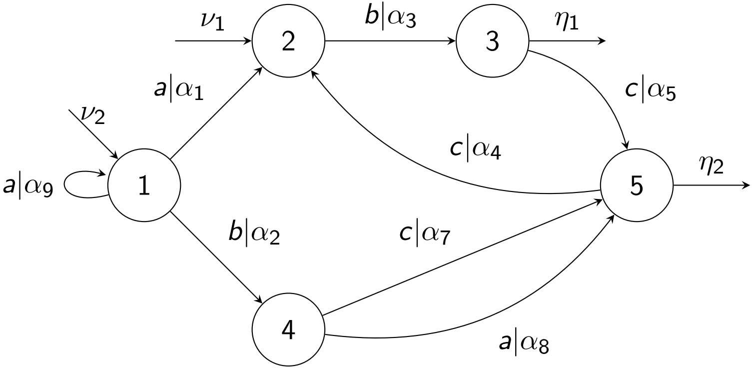


Figure: Marcel-Paul Schützenberger at Oberwolfach (1973)[1]

[1]Contrary to 1972 (Wikipedia)


Comp. Science
Data Structures
Networks
Algorithmics
Optimi-sation
Image Processing
Compression & trans-mission
Probabilities
Noam Chomsky
Linguistics
MPS (automata theory, trees, transition graphs)
MPS Bayesian trees, phase change
MPS Non-commutative diff. eq.
Andrei Sobolevskii
MPS (theo of codes)
MPS (circular codes)
MPS Alg. Com-binatorics
Gleb Koshevoy
Decision Aids
Condensed Matter
PDE & numer-ical methods
Jacques Demongeot
Combinatorics & discrete maths
Genomics
Astronomy & Cosmology
Medecine
Economy
Physics
Biology
Quantum Computing

Multiplicity Automaton (Eilenberg, Schützenberger)



1 S. Eilenberg, *Automata, Languages, and Machines (Vol. A)* Acad. Press, New York, 1974

2 M.P. Schützenberger, *On the definition of a family of automata, Inf. and Contr., 4 (1961)*, 245-270.

Multiplicity automaton (linear representation) & behaviour

Linear representation

$$\nu = \begin{pmatrix} \nu_2 & \nu_1 & 0 & 0 & 0 \end{pmatrix}, \qquad \eta = \begin{pmatrix} 0 & 0 & \eta_1 & 0 & \eta_2 \end{pmatrix}^T$$

$$\mu(a) = \begin{pmatrix} \alpha_9 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 0 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mu(c) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_5 \\ 0 & 0 & 0 & 0 & \alpha_7 \\ 0 & \alpha_4 & 0 & 0 & 0 \end{pmatrix}$$

Behaviour

$$\mathcal{A}(w) = \nu\,\mu(w)\,\eta = \sum_{\substack{i,j \\ \text{states}}} \nu(i) \underbrace{\left(\sum weight(p)\right)}_{\text{weight of all paths } (i) \to (j) \text{ with label } w} \eta(j)$$

Operations and definitions on series

Addition, Scaling: as for functions because $R\langle\langle X\rangle\rangle = R^{X^*}$
Concatenation: $f.g(w) = \sum_{w=uv} f(u)g(v)$
Polynomials: Series s.t. $supp(f) = \{w\}_{f(w)\neq 0}$ is finite.
The set of polynomials will be denoted $R\langle X\rangle$.
Pairing: $\langle S \mid P\rangle = \sum_{w\in X^*} S(w)P(w)$ (S series, P polynomial)
Summation: A family $\sum_{i\in I} S_i$ is said summable iff f or all $w \in X^*$, $i \mapsto \langle S_i \mid w\rangle$ is finitely supported. This corresponds to the product topology (with R discrete). In particular, we have

$$\sum_{i\in I} S_i := \sum_{w\in X^*} \Big(\sum_{i\in I}\langle S_i \mid w\rangle\Big)\, w$$

Star: For all series S s.t. $\langle S \mid 1_{X^*}\rangle = 0$, the family $(S^n)_{n\geq 0}$ is summable and we set $S^* := \sum_{n\geq 0} S^n = 1 + S + S^2 + \cdots \ (= (1-S)^{-1})$.
Shifts: $\langle u^{-1}S \mid w\rangle = \langle S \mid uw\rangle$, $\langle Su^{-1} \mid w\rangle = \langle S \mid wu\rangle$

Rational series (Sweedler & Schützenberger)

Theorem [A]

Let $\mathbf{k}$ be a field, X a finite set and $S \in k\langle\langle X \rangle\rangle$ TFAE

i) The family $(Su^{-1})_{u \in X^*}$ is of finite rank.

ii) The family $(u^{-1}S)_{u \in X^*}$ is of finite rank.

iii) The family $(u^{-1}Sv^{-1})_{u,v \in X^*}$ is of finite rank.

iv) It exists $n \in \mathbb{N}$, $\lambda \in \mathbf{k}^{1\times n}$, $\mu : X^* \to \mathbf{k}^{n\times n}$ (a multiplicative morphism) and $\tau \in \mathbf{k}^{n\times 1}$ such that, for all $w \in X^*$

$$(S, w) = \lambda\mu(w)\tau \tag{4}$$

v) The series S is in the closure of $\mathbf{k}\langle X \rangle$ for $(+, conc, ^*)$ within $\mathbf{k}\langle\langle X \rangle\rangle$.

Definition

A series which fulfill one of the conditions of Theorem [A] will be called *rational*. The set of these series will be denoted by $k^{rat}\langle\langle X \rangle\rangle$.

Sweedler's duals

Remarks

1. (i $\leftrightarrow$ iii) needs k to be a field (see below for an extension).
2. (iv) needs X to be finite, (iv $\leftrightarrow$ v) is known as the theorem of Kleene-Schützenberger (M.P. Schützenberger, *On the definition of a family of automata, Inf. and Contr., 4 (1961)*, 245-270.)
3. For the sake of Combinatorial Physics (where the alphabets can be infinite), **(iv)** has been extended to infinite alphabets and replaced by **iv')** The series S is in the rational closure of k^X (linear series) within $k\langle\langle X\rangle\rangle$.
4. This theorem is linked to the following: Representative functions on X^* (see Eichii Abe, Chari & Pressley), Sweedler's duals &c.
5. In the vein of (v) expressions like ab^* or identities like $(ab^*)^*a^* = (a+b)^*$ (Lazard's elimination) will be called rational.

Sweedler's duals/2

Extension of the transpose of a law

We start with a $\mathbf{k}-\mathbf{AAU}$ ($\mathbf{k}$ a field) $\mathcal{A}$, dualizing $\mu:\ \mathcal{A}\otimes_{\mathbf{k}}\mathcal{A}\to\mathcal{A}$, we have

$$\begin{array}{ccc}
\mathcal{A}^{\vee} & \xrightarrow{\ {}^{t}\mu\ } & (\mathcal{A}\otimes_{\mathbf{k}}\mathcal{A})^{\vee} \\
{\scriptstyle can}\uparrow & & \uparrow{\scriptstyle \Phi} \\
\mathcal{A}^{\circ} & \xrightarrow{\ \Delta_{\mu}\ } & \mathcal{A}^{\vee}\otimes_{\mathbf{k}}\mathcal{A}^{\vee} \\
{\scriptstyle can}\uparrow & & \uparrow{\scriptstyle j\otimes j} \\
\mathcal{A}^{\circ\circ} & \xrightarrow{\ \Delta_{\mu}\ } & \mathcal{A}^{\circ}\otimes_{\mathbf{k}}\mathcal{A}^{\circ}
\end{array}$$

In fact, one can prove that the “descent” stops at first step and then $\mathcal{A}^{\circ\circ}=\mathcal{A}^{\circ}$.

Sweedler's duals/3

Proof

1. Due to the fact that **k** is a field, the arrow

$$\mathcal{A}^\vee \otimes_{\mathbf{k}} \mathcal{A}^\vee \xrightarrow{\Phi} (\mathcal{A} \otimes_{\mathbf{k}} \mathcal{A})^\vee$$

is into, we will return to this crucial point in the case when **k** is a ring. When one bluntly writes $\Delta_{conc}(R) = \sum_{i=1}^{n} S_i \otimes T_i$, there is already an indentification. If we set, as usual, $\Delta_\mu =^t \mu$, in perfect rigor one should write

$$\Delta_{conc}(R) = \Phi(\sum_{i=1}^{n} S_i \otimes T_i)$$

But, in the case when Φ is into (**k** field or noetherian ring), one can define $(\mathcal{A}^\circ, \Delta_\mu)$ as the only arrow (with dom) that closes the upper square.

2. Let us define left-right actions $\triangleright(?)\triangleleft$ of the **k**-algebra $\mathcal{A}$ on $\mathcal{A}^\vee$ by setting

$$\langle u \triangleright (f) \triangleleft w \mid v\rangle = \langle f \mid vuw\rangle \qquad \text{for all } f \in \mathcal{A}^\vee \text{ and } u, v, w \in \mathcal{A}.$$

Sweedler's duals/4

Proof/2

3. These two actions commute, thus, $\mathcal{A}^\vee$ is a $\mathcal{A}-\mathcal{A}$ bimodule.
4. For a given $u\in\mathcal{A}$, we shall refer to the operator $\mathcal{A}^\vee\to\mathcal{A}^\vee,\ f\mapsto u\triangleright f$ as shifting by u or the u-left shift operator; it generalizes Schützenberger's right u^{-1} in automata theory [BeRe88, Schütz61].
5. We first prove (exercise) that TFAE
   1. $f\in\mathcal{A}^\circ$
   2. $(u\triangleright f)_{u\in\mathcal{A}}$ is of finite rank
   3. There exists a linear representation (i.e. $\lambda\in\mathbf{k}^{1\times n},\ \tau\in\mathbf{k}^{n\times 1}$ and $\mu:\ \mathcal{A}\to\mathbf{k}^{n\times n}$ a morphism of $\mathbf{k}-\mathbf{AAU}$) of dimension n (λ,μ,τ) such that for all $v\in\mathcal{A}$, $\langle f\mid v\rangle=\lambda\mu(v)\tau$.
6. Let us $e_i^*:=(\cdots\underbrace{,1,}_{\text{place } i}\cdots)\in\mathbf{k}^{1\times n}$ and $e_i={}^t(e_i^*)$. One remarks that

$$I_{n\times n}=\sum_{i=1}^n e_ie_i^*.$$

Sweedler's duals/5

Proof/3

7. (Minh's trick) Now, for $u, v \in \mathcal{A}$

$$\langle f \mid uv\rangle = \lambda\mu(uv)\tau = \lambda\mu(u)\mu(v)\tau = \lambda\mu(u)I_{n\times n}\mu(v)\tau =$$
$$\lambda\mu(u)(\sum_{i=1}^{n} e_i e_i^*)\mu(v)\tau = \sum_{i=1}^{n}(\lambda\mu(u)e_i)(e_i^*\mu(v)\tau) =$$
$$\sum_{i=1}^{n}\langle g_i \mid u\rangle\langle h_i \mid v\rangle = \sum_{i=1}^{n}\langle g_i \otimes h_i \mid u \otimes v\rangle^{\otimes 2} \tag{5}$$

in other words $\Delta_\mu(f) = \sum_{i=1}^{n} g_i \otimes h_i$ where

$$g_i : \ (\lambda, \mu, e_i) \text{ and } h_i : \ (e_i^*, \mu, \tau)$$

this proves that, for all $f \in \mathcal{A}^\circ$, in fact $\Delta_\mu(f) \in \mathcal{A}^\circ \otimes_{\mathbf{k}} \mathcal{A}^\circ$ □

Sweedler's duals/6

After proof

8. $(\mathcal{A}^{\circ}, \Delta_{\mu})$ is called the Sweedler's dual of $(\mathcal{A}, \mu)$ (it is a coalgebra)
9. Please check throughly the Threefold Conditions above 5 .
10. **Remark**. — In the construction, the arrow Φ plays a crucial rôle, but it may not be into. There is a old lemma by Gérard Jacob which can salvage partially the situation in the case of a general semiring and of $\mathcal{A} = \mathbf{k}\langle X\rangle$. In this case, condition 5.2 is replaced by "$(u \triangleright f)_{u\in\mathcal{A}}$ lies within a finite-type left $\mathbf{k}\langle X\rangle$-module" (similar condition can be stated with a finite-type right module).

From theory to practice: Schützenberger's calculus

From series to automata

Starting from a series S, one has a way to construct an automaton (finite-stated iff the series is rational) providing that we know how to compute on shifts and one-letter-shifts will be sufficient due to the formula $u^{-1}v^{-1}S = (vu)^{-1}S$. In automata theory, we note $u^{-1}S$ for $S \triangleleft u$

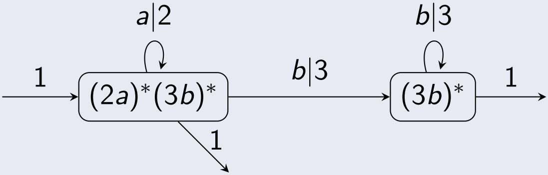
Calculus on rational expressions

In the following, x is a letter, E, F are rational expressions (i.e. expressions built from letters by scalings, concatenations and stars)

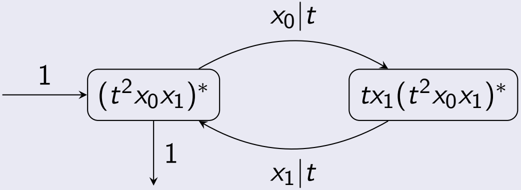
1. x^{-1} is (left and right) linear
2. $x^{-1}(E.F) = x^{-1}(E).F + \langle E \mid 1_{X^*}\rangle x^{-1}(F)$
3. $x^{-1}(E^*) = x^{-1}(E).E^*$

Examples

With $(2a)^*(3b)^*$; $X = \{a, b\}$



With $(t^2x_0x_1)^*$; $X = \{x_0, x_1\}$



From theory to practice: construction starting from S.

- **States** $\boxed{u^{-1}S}$ (constructed step by step)
- **Edges** We shift every state by letters (length) level by level (knowing that $x^{-1}(u^{-1}S) = (ux)^{-1}S$). Two cases:
  Returning state: The state is a linear combination of the already created ones i.e. $x^{-1}(u^{-1}S) = \sum_{v \in F} \alpha(ux, v) v^{-1}S$ (with F finite), then we set the edges

$$\boxed{u^{-1}S} \xrightarrow{x|\alpha_v} \boxed{v^{-1}S}$$

  The created state is new: Then

$$\boxed{u^{-1}S} \xrightarrow{x|1} \boxed{x^{-1}(u^{-1}S)}$$

- **Input** $\boxed{S}$ with the weight 1
- **Outputs** All states $\boxed{T}$ with weight $\langle T \mid 1_{X^*} \rangle$

Link with conc-bialgebras (CAP 17)

We call here conc-bialgebras, structures such that $\mathcal{B} = (k\langle X\rangle, conc, 1_{X^*}, \Delta, \epsilon)$ is a bialgebra and $\Delta(X) \subset (k.X \oplus k.1_{X^*})^{\otimes 2}$. For this, as $k\langle X\rangle$ is a free algebra, it suffices to define Δ and check the axioms on letters. Below, some examples

Shuffle: X is arbitrary $\Delta(x) = x \otimes 1 + 1 \otimes x$ and

$$\Delta(w) = \sum_{I+J=[1\cdots|w|]} w[I] \otimes w[J]$$

Stuffle: $Y = \{y_i\}_{i\geq 1}$, $\Delta(y_k) = y_k \otimes 1 + 1 \otimes y_k + \sum_{i+j=k} y_i \otimes y_j$
q-**infiltration**: X is arbitrary, $\Delta(x) = x \otimes 1 + 1 \otimes x + q\, x \otimes x$ and

$$\Delta(w) = \sum_{I\cup J=[1\cdots|w|]} q^{|I\cap J|} w[I] \otimes w[J]$$

Link with conc-bialgebras/2

In case $\epsilon(P) = \langle P \mid 1_{X^*} \rangle$[a], the restricted (graded) dual is $\mathcal{B}^\vee = (k\langle X \rangle, *, 1_{X^*}, \Delta_{conc}, \epsilon)$ and we can write, for $x \in X$

$$\Delta(x) = x \otimes 1_{X^*} + 1_{X^*} \otimes x + \Delta_+(x) \tag{6}$$

then, the dual law $*$ $(=^t \Delta)$ can be defined by recursion

$$\begin{aligned} w * 1_{X^*} &= 1_{X^*} * w = w \\ au * bv &= a(u * bv) + b(au * v) + \varphi(a, b)(u * v) \end{aligned} \tag{7}$$

where $\varphi =^t \Delta_+ : k.X \otimes k.X \to k.X$ is an associative law.

[a] which covers all usual combinatorial cases, save Hadamard

Some dual laws

Name	Formula (recursion)	φ	Type
Shuffle [21]	$au \shuffle bv = a(u \shuffle bv) + b(au \shuffle v)$	$\varphi \equiv 0$	I
Stuffle [19]	$x_i u \stuffle x_j v = x_i(u \stuffle x_j v) + x_j(x_i u \stuffle v) + x_{i+j}(u \stuffle v)$	$\varphi(x_i, x_j) = x_{i+j}$	I
Min-stuffle [7]	$x_i u \minstuffle x_j v = x_i(u \minstuffle x_j v) + x_j(x_i u \minstuffle v) - x_{i+j}(u \minstuffle v)$	$\varphi(x_i, x_j) = -x_{i+j}$	III
Muffle [14]	$x_i u \muffle x_j v = x_i(u \muffle x_j v) + x_j(x_i u \muffle v) + x_{i\times j}(u \muffle v)$	$\varphi(x_i, x_j) = x_{i\times j}$	I
q-shuffle [3]	$x_i u \stuffle_q x_j v = x_i(u \stuffle_q x_j v) + x_j(x_i u \stuffle_q v) + q x_{i+j}(u \stuffle_q v)$	$\varphi(x_i, x_j) = q x_{i+j}$	III
q-shuffle$_2$	$x_i u \stuffle_q x_j v = x_i(u \stuffle_q x_j v) + x_j(x_i u \stuffle_q v) + q^{i.j} x_{i+j}(u \stuffle_q v)$	$\varphi(x_i, x_j) = q^{i.j} x_{i+j}$	II
$\mathtt{LDIAG}(1, q_s)$ [10] (non-crossed, non-shifted)	$au \shuffle bv = a(u \shuffle bv) + b(au \shuffle v) + q_s^{\lvert a\rvert \lvert b\rvert} a.b(u \shuffle v)$	$\varphi(a,b) = q_s^{\lvert a\rvert \lvert b\rvert}(a.b)$	II
q-Infiltration [12]	$au \uparrow bv = a(u \uparrow bv) + b(au \uparrow v) + q\delta_{a,b} a(u \uparrow v)$	$\varphi(a,b) = q\delta_{a,b} a$	III
AC-stuffle	$au \shuffle_\varphi bv = a(u \shuffle_\varphi bv) + b(au \shuffle_\varphi v) + \varphi(a,b)(u \shuffle_\varphi v)$	$\varphi(a,b) = \varphi(b,a)$ $\varphi(\varphi(a,b),c) = \varphi(a,\varphi(b,c))$	IV
Semigroup-stuffle	$x_t u \shuffle_\perp x_s v = x_t(u \shuffle_\perp x_s v) + x_s(x_t u \shuffle_\perp v) + x_{t\perp s}(u \shuffle_\perp v)$	$\varphi(x_t, x_s) = x_{t\perp s}$	I
φ-shuffle	$au \shuffle_\varphi bv = a(u \shuffle_\varphi bv) + b(au \shuffle_\varphi v) + \varphi(a,b)(u \shuffle_\varphi v)$	$\varphi(a,b)$ law of AAU	V

Of course, the q-shuffle is equal to the (classical) shuffle when $q = 0$. As for the q-infiltration, when $q = 1$, one recovers the infiltration product defined in [6]

A useful property

Proposition B

Let $\mathcal{B} = (k\langle X\rangle, conc, 1_{X^*}, \Delta, \epsilon)$ be a conc-bialgebra, then

1. The space $k^{rat}\langle X\rangle$ is closed by the convolution product $\diamond$ (here ${}^t\Delta$) given by
$$\langle S \diamond T \mid w\rangle = \langle S \otimes T \mid \Delta(w)\rangle \tag{8}$$
2. If k is a $\mathbb{Q}$-algebra and $\Delta_+ \ : \ k.X \to k.X \otimes k.X$ cocommutative, $\mathcal{B}$ is an enveloping algebra iff Δ_+ is moderate[a].
3. If, moreover k is without zero divisors, the characters $(x^*)_{x\in X}$ are algebraically independant over $(k\langle X\rangle, \diamond, 1_{X^*})$ within $(k\langle\langle X\rangle\rangle, \diamond, 1_{X^*})$.

[a] See CAP 2017

A useful property/2

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Independence of characters with respect to polynomials

5

1

I came across the following property :

Let $\mathfrak{g}$ be a Lie algebra over a ring k without zero divisors, $\mathcal{U}=\mathcal{U}(\mathfrak{g})$ be its enveloping algebra. As such, $\mathcal{U}$ is a Hopf algebra and ϵ, its counit, is the only character of $\mathcal{U}\to k$ which vanishes on $\mathfrak{g}$.

Set $\mathcal{U}_+=ker(\epsilon)$. We build the following filtrations $(N\geq 1)$

$$\mathcal{U}_N=\mathcal{U}_+^N=\underbrace{\mathcal{U}_+.\ldots.\mathcal{U}_+}_{N\text{ times}} \qquad (1)$$

and

$$\mathcal{U}_N^*=\mathcal{U}_{N+1}^{\perp}=\{f\in\mathcal{U}^*|(\forall u\in\mathcal{U}_{N+1})(f(u)=0)\} \qquad (2)$$

the first one is decreasing and the second one increasing. One shows easily that (with $\diamond$ as the convolution product)

$$\mathcal{U}_p^*\diamond\mathcal{U}_q^*\subset\mathcal{U}_{p+q}^*$$

so that $\mathcal{U}_\infty^*=\cup_{n\geq 1}\mathcal{U}_n^*$ is a convolution subalgebra of $\mathcal{U}^*$.

Now, we can state the

Theorem : The set of characters of $(\mathcal{U},.,1_\mathcal{U})$ is linearly free w.r.t. $\mathcal{U}_\infty^*$.

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A useful property/3

Remark

Property (3) is no longer true if Δ is not moderate. For example with the Hadamard coproduct and $x \neq y$, one has $y \odot (x)^* = 0$.

Examples

Shuffle: $(\alpha x)^* ⧢ (\beta y)^* = (\alpha x + \beta y)^*$

Stuffle: $(\alpha y_i)^* \uplus (\beta y_j)^* = (\alpha y_i + \beta y_j + \alpha\beta y_{i+j})^*$

***q*-infiltration**: $(\alpha x)^* \uparrow_q (\beta y)^* = (\alpha x + \beta y + \alpha\beta\delta_{x,y} x)^*$

Hadamard: $(\alpha a)^* \odot (\beta b)^* = 1_{X^*}$ if $a \neq b$ and $(\alpha a)^* \odot (\beta a)^* = (\alpha\beta a)^*$

Starting the ladder

$$\begin{array}{ccc} (\mathbb{C}\langle X\rangle, ⧢, 1_{X^*}) & \xrightarrow{\mathrm{Li}_\bullet} & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\ \downarrow & & \downarrow \\ (\mathbb{C}\langle X\rangle, ⧢, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] & \xrightarrow{\mathrm{Li}_\bullet^{(1)}} & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*} \end{array}$$

Domain of Li (definition)

In order to extend Li to series, we define $Dom(Li;\Omega)$ (or $Dom(Li)$) if the context is clear) as the set of series $S=\sum_{n\geq 0} S_n$ (decomposition by homogeneous components) such that $\sum_{n\geq 0} Li_{S_n}(z)$ converges for the compact convergence in Ω. One sets

$$Li_S(z) := \sum_{n\geq 0} Li_{S_n}(z) \tag{9}$$

Examples

$$Li_{x_0^*}(z)=z,\ \ Li_{x_1^*}(z)=(1-z)^{-1}\ ;\ Li_{(\alpha x_0+\beta x_1)^*}(z)=z^{\alpha}(1-z)^{-\beta}$$

Properties of the extended Li

Proposition

With this definition, we have

1. $Dom(Li)$ is a shuffle subalgebra of $\mathbb{C}\langle\langle X\rangle\rangle$ and then so is $Dom^{rat}(Li) := Dom(Li) \cap \mathbb{C}^{rat}\langle\langle X\rangle\rangle$
2. For $S, T \in Dom(Li)$, we have

$$\mathrm{Li}_{S \sqcup\!\sqcup T} = \mathrm{Li}_S \,.\, \mathrm{Li}_T$$

Examples and counterexamples

For $|t| < 1$, one has $(tx_0)^* x_1 \in Dom(Li, D)$ (D is the open unit slit disc), whereas $x_0^* x_1 \notin Dom(Li, D)$.
Indeed, we have to examine the convergence of $\sum_{n\geq 0} \mathrm{Li}_{x_0^n x_1}(z)$, but, for $z \in]0, 1[$, one has $0 < z < \mathrm{Li}_{x_0^n x_1}(z) \in \mathbb{R}$ and therefore, for these values $\sum_{n\geq 0} \mathrm{Li}_{x_0^n x_1}(z) = +\infty$.
In fact, in this case ($|t| < 1$)

Coefficients in the Ladder

$$\begin{array}{ccc}
(\mathbb{C}\langle X\rangle, ⧢, 1_{X^*}) & \xhookrightarrow{\ \mathrm{Li}_\bullet\ } & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\
\big\downarrow & & \big\downarrow \\
(\mathbb{C}\langle X\rangle, ⧢, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] & \xrightarrow{\ \mathrm{Li}_\bullet^{(1)}\ } & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\big\downarrow & & \big\downarrow \\
\mathbb{C}\langle X\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle & \xrightarrow{\ \mathrm{Li}_\bullet^{(2)}\ } & \mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*}
\end{array}$$

Were, for every additive subgroup $(H,+) \subset (\mathbb{C},+)$, $\mathcal{C}_H$ has been set to the following subring of $\mathbb{C}$

$$\mathcal{C}_H := \mathbb{C}\{z^\alpha(1-z)^{-\beta}\}_{\alpha,\beta\in H} \ . \tag{11}$$

Examples

$$Li_{x_0^*}(z) = z,\ Li_{x_1^*}(z) = (1-z)^{-1}\ ;\ Li_{\alpha x_0^* + \beta x_1^*}(z) = z^\alpha(1-z)^{-\beta}$$

The arrow $\mathrm{Li}_\bullet^{(1)}$

Proposition

i. The family $\{x_0^*, x_1^*\}$ is algebraically independent over $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})$ within $(\mathbb{C}\langle\langle X\rangle\rangle^{\mathrm{rat}}, ш, 1_{X^*})$.

ii. $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, x_1^*, (-x_0)^*]$ is a free module over $\mathbb{C}\langle X\rangle$, the family $\{(x_0^*)^{ш\, k} ш (x_1^*)^{ш\, l}\}_{(k,l)\in\mathbb{Z}\times\mathbb{N}}$ is a $\mathbb{C}\langle X\rangle$-basis of it.

iii. As a consequence, $\{w ш (x_0^*)^{ш\, k} ш (x_1^*)^{ш\, l}\}_{\substack{w\in X^* \\ (k,l)\in\mathbb{Z}\times\mathbb{N}}}$ is a $\mathbb{C}$-basis of it.

iv. $\mathrm{Li}_\bullet^{(1)}$ is the unique morphism from $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*]$ to $\mathcal{H}(\Omega)$ such that

$$x_0^* \to z,\ (-x_0)^* \to z^{-1} \text{ and } x_1^* \to (1-z)^{-1}$$

v. $\mathrm{Im}(\mathrm{Li}_\bullet^{(1)}) = \mathcal{C}_\mathbb{Z}\{\mathrm{Li}_w\}_{w\in X^*}$.

vi. $\ker(\mathrm{Li}_\bullet^{(1)})$ is the (shuffle) ideal generated by $x_0^* ш x_1^* - x_1^* + 1_{X^*}$.

Sketch of the proof for vi.

Let $\mathcal{J}$ be the ideal generated by $x_0^* ⧢ x_1^* - x_1^* + 1_{X^*}$. It is easily checked, from the following formulas[a], for $k \geq 1$,

$$\begin{aligned} w ⧢ x_0^* ⧢ (x_1^*)^{⧢ k} &\equiv w ⧢ (x_1^*)^{⧢ k} - w ⧢ (x_1^*)^{⧢ k-1} \; [\mathcal{J}], \\ w ⧢ (-x_0)^* ⧢ (x_1^*)^{⧢ k} &\equiv w ⧢ (-x_0)^* ⧢ (x_1^*)^{⧢ k-1} + w ⧢ (x_1^*)^{⧢ k} [\mathcal{J}], \end{aligned}$$

that one can rewrite $[\mathrm{mod}\,\mathcal{J}]$ any monomial $w ⧢ (x_0^*)^{⧢ l} ⧢ (x_1^*)^{⧢ k}$ as a linear combination of such monomials with $kl = 0$. Observing that the image, through $\mathrm{Li}_\bullet^{(1)}$, of the following family is free in $\mathcal{H}(\Omega)$

$$\{w ⧢ (x_1^*)^{⧢ l} ⧢ (x_0^*)^{⧢ k}\}_{(w,l,k) \in (X^* \times \mathbb{N} \times \{0\}) \sqcup (X^* \times \{0\} \times \mathbb{Z})} \tag{12}$$

we get the result.

[a] In the Figure below, (w, l, k) codes the element $w ⧢ (x_0^*)^{⧢ l} ⧢ (x_1^*)^{⧢ k}$.

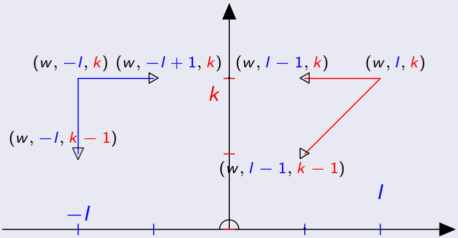


Figure: Rewriting $\mathcal{J}$-**Mod** of $\{w ⧢ (x_0^*)^{⧢ l} ⧢ (x_1^*)^{⧢ k}\}_{k\in\mathbb{N}, l\in\mathbb{Z}, w\in X^*}$.

End of the ladder: pushing coefficients to $\mathcal{C}_{\mathbb{C}}$

$$\begin{array}{ccc}
(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*}) & \xrightarrow{\mathrm{Li}_\bullet} & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] & \xrightarrow{\mathrm{Li}_\bullet^{(1)}} & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
\mathbb{C}\langle X\rangle \sqcup\!\sqcup \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle & \xrightarrow{\mathrm{Li}_\bullet^{(2)}} & \mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*}
\end{array}$$

Exchangeable (rational) series

The power series S belongs to $\mathbb{C}_{\mathrm{exc}}\langle X\rangle$, iff

$$(\forall u, v \in X^*)((\forall x \in X)(|u|_x = |v|_x) \;\Rightarrow\; \langle S|u\rangle = \langle S|v\rangle). \qquad (13)$$

We will note $\mathbb{C}_{exc}^{rat}\langle X\rangle$, the set of exchangeable rational series i.e.

$$\mathbb{C}_{exc}^{rat}\langle X\rangle := \mathbb{C}_{exc}\langle X\rangle \cap \mathbb{C}^{rat}\langle X\rangle \qquad (14)$$

Lemma (D., HNM, Ngô, 2016)

1. $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle := \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle \bigcap \mathbb{C}_{\mathrm{exc}}\langle\langle X\rangle\rangle = \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$.
2. *For any $x \in X$, from a theorem by Kronecker, one has* $\mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle = \mathrm{span}_{\mathbb{C}}\{(ax)^* \sqcup\!\sqcup \mathbb{C}\langle x\rangle | a \in \mathbb{C}\}$ *and*
$$\{(ax)^* \sqcup\!\sqcup x^n\}_{(a,n)\in\mathbb{C}\times\mathbb{N}} \tag{15}$$
is a basis of it. When restricted to $(\mathbb{C}^* \times \mathbb{N}) \cup \{(0,0)\}$ this family spans $\mathbb{C}^{rat}_{const}\langle\langle x\rangle\rangle$ (fractions being constant at infinity)
3. $\mathbb{C}\langle X\rangle \sqcup\!\sqcup \mathbb{C}^{rat}_{exc}\langle\langle X\rangle\rangle \simeq \mathbb{C}\langle X\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}_{const}\langle\langle x_0\rangle\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}_{const}\langle\langle x_1\rangle\rangle$
4. $\mathrm{Im}(\mathrm{Li}^{(2)}_\bullet) = \mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*}$.
5. $\ker(\mathrm{Li}^{(2)}_\bullet)$ *is the (shuffle) ideal generated by* $x_0^* \sqcup\!\sqcup x_1^* - x_1^* + 1_{X^*}$ *(prospective).*

Concluding remarks/1

1. We have coded classical (and extended) polylogarithms with words obtaining a Noncommutative generating series which is a shuffle character
2. This character can be extended by continuity to certain series forming a shuffle subalgebra of Noncommutative formal power series.
3. We have found some remarkable subalgebras of $Dom^{rat}(Li)$, given their bases and described the kernel of the so extended $\mathrm{Li}_\bullet$.

4. Definition of $Dom(Li)$ and $Dom^{rat}(Li)$ have to be refined and their exploration pushed further.
5. Combinatorics of discrete Dyson integrals for various sets of differential forms has to be implemented

Concluding remarks/2

6. Drinfeld-Kohno Lie algebras i.e. algebras presented by

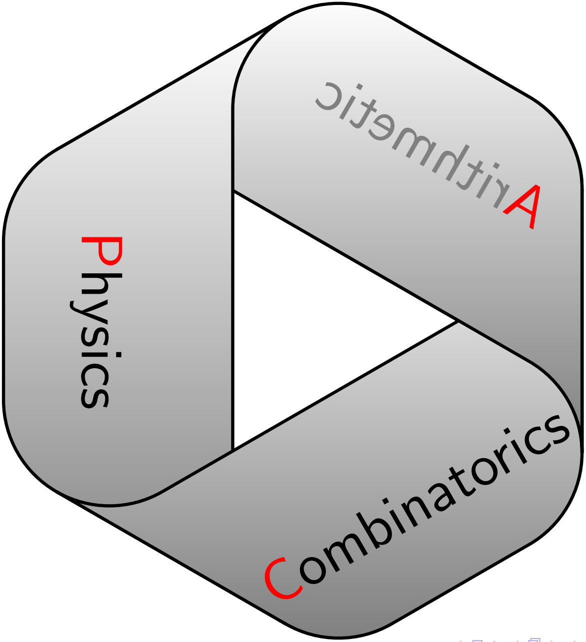
$$DK(A;k) = \langle A \times A \; ; \; \mathbf{R_A} \, \rangle_{k-\text{Lie algebras}} \tag{16}$$

with $\mathbf{R_A}$, the relator

$$\mathbf{R_A} = \begin{cases} (a,a) &= 0 \text{ for } a \in A \\ (a,b) &= (b,a) \text{ for } a,b \in A \\ [(a,c),(a,b)+(b,c)] &= 0 \text{ for } |\{a,b,c\}| = 3, \\ [(a,b),(c,d)] &= 0 \text{ for } |\{a,b,c,d\}| = 4 \end{cases} \tag{17}$$

can be decomposed in several ways as a direct sum of Free Lie algebras giving rise to product of MRS factorisations

$$\chi = \prod_{l \in \mathcal{L}yn(X)}^{\searrow} e^{\chi(S_l)\, P_l} \tag{18}$$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THANK YOU FOR YOUR ATTENTION !

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