Non-commutative differential equations.

Localization, independence and unicity.

V.C. Bùi, Matthieu Deneufchâtel, G.H.E. Duchamp,
Nihar Gargava, Hoang Ngoc Minh, Q.H. Ngô.

Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
N. Behr, K. A. Penson, C. Tollu.

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Plan

Foreword: Goal of this talk

In this talk, I will show tools and sketch proofs about Noncommutative Differential (Evolution) Equations.

The main item of data (not to say the only one) is that of Noncommutative Formal Power Series with variable coefficients which allows to explore in a compact and effective (in the sense of computability, hence needing fraction fields or a localization process) way the Magnus group of proper exponentials and the Hausdorff group of Lie exponentials (i.e. group-like series for the dual of the shuffle product).

Parts of this work are connected with Dyson series and take place within the project: *Evolution Equations in Combinatorics and Physics*. Today we will focus on mathematical motivations and properties (namely localization).

This talk also prepares data structures used in forthcoming works.

Review of the facts

- $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$
- when one multiplies two of these, one gets

$$\zeta(s_1)\zeta(s_2) = \sum_{n_1,n_2\geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}} = \zeta(s_1,s_2) + \zeta(s_1+s_2) + \zeta(s_2,s_1)$$

- leading to the following definition of MultiZeta Values (MZV)

$$\zeta(s_1,\ldots,s_k) := \sum_{n_1>\ldots>n_k\geq 1} \frac{1}{n_1^{s_1}\ldots n_k^{s_k}} \quad ; \quad s_1 > 1 \tag{1}$$

- On the other hand, one has the classical polylogarithms defined, for $k \geq 1, |z| < 1$, by

$$Li_k(z) := \sum_{n\geq 1} \frac{z^n}{n^k}$$

Overview of the facts/2

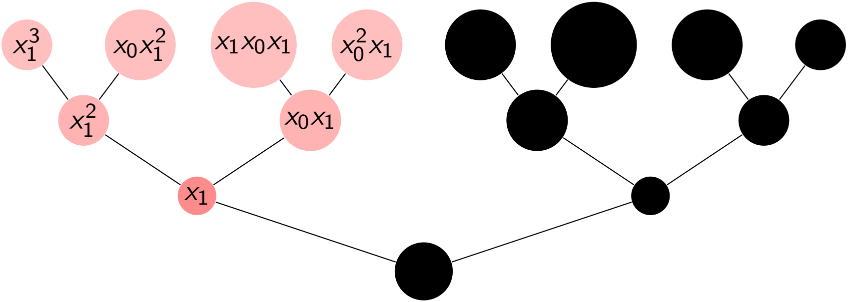
- The analogue of the classical polylogarithms for MZV reads

$$Li_{y_{s_1}\ldots y_{s_k}}(z) := \sum_{n_1>\ldots>n_k\geq 1} \frac{z^{n_1}}{n_1^{s_1}\ldots n_k^{s_k}}.$$

- They satisfy the recursion (ladder stepdown)

$$\begin{aligned} z\frac{d}{dz}Li_{y_{s_1}\ldots y_{s_k}} &= Li_{y_{s_1-1}\ldots y_{s_k}} \text{ if } s_1 > 1 \\ (1-z)\frac{d}{dz}Li_{1.y_{s_2}\ldots y_{s_k}} &= Li_{y_{s_2}\ldots y_{s_k}} \text{ if } k > 1 \\ Li_1(z) &= \log(\frac{1}{1-z}) \end{aligned} \quad (2)$$

- For the next step, we code the moves $z\frac{d}{dz}$ (resp. $(1-z)\frac{d}{dz}$) - or more precisely their sections $\int_{*_0}^{z} \frac{f(s)}{s} ds$ (resp. $\int_{*_1}^{z} \frac{f(s)}{1-s} ds$) - with x_0 (resp. x_1).



Some coefficients with $X = \{x_0, x_1\}$; $u_0(z) = \frac{1}{z}$; $u_1(z) = \frac{1}{1-z}$, $*_0 = 0$

$$\langle S|x_1^n\rangle = \frac{(-log(1-z))^n}{n!} \quad ; \quad \langle S|x_0x_1\rangle = \underbrace{\mathrm{Li}_2(z)}_{cl.not.} = \mathrm{Li}_{x_0x_1}(z) = \sum_{n\geq 1}\frac{z^n}{n^2}$$

$$\langle S|x_0^2x_1\rangle = \underbrace{\mathrm{Li}_3(z)}_{cl.not.} = \mathrm{Li}_{x_0^2x_1}(z) = \sum_{n\geq 1}\frac{z^n}{n^3} \quad ; \quad \langle S|x_1x_0x_1\rangle = \mathrm{Li}_{x_1x_0x_1}(z) = \mathrm{Li}_{[1,2]}(z) = \sum_{n_1>n_2\geq 1}\frac{z^{n_1}}{n_1n_2^2}$$

$$\langle S|x_0x_1^2\rangle = \mathrm{Li}_{x_0x_1^2}(z) = \mathrm{Li}_{[2,1]}(z) = \sum_{n_1>n_2\geq 1}\frac{z^{n_1}}{n_1^2n_2}$$

; above "cl. not." stands for "classical notation"

- Calling S the prospective completed generating series

$$\sum_{w\in X^*} \langle S|w\rangle(z)w \; ; \; X=\{x_0,x_1\} \tag{3}$$

Drinfel'd [1] indirectly proposed a way to complete the tree: from the general theory, the following system

$$\left\{\begin{array}{l} \mathbf{d}(S) = (\frac{x_0}{z}+\frac{x_1}{1-z}).S \\ \lim_{\substack{z\to 0\\ z\in\Omega}} S(z)e^{-x_0 log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \end{array}\right. \tag{4}$$

has a unique solution which is precisely Li (precise meaning of $\mathbf{d}(?)$, the term by term derivation, is to be given below).

- Minh [2] indicated a way to effectively compute this solution through (improper) iterated integrals.

1. V. Drinfel'd, *On quasitriangular quasi-hopf algebra and a group closely connected with Gal($\bar{\mathbb{Q}}/\mathbb{Q}$)*, Leningrad Math. J., 4, 829-860, 1991.
2. H. N. Minh, *Summations of polylogarithms via evaluation transform*, Mathematics and Computers in Simulation, Vol. 42, 4-6, Nov. 1996, pp. 707-728

Explicit construction of Drinfeld's G_0

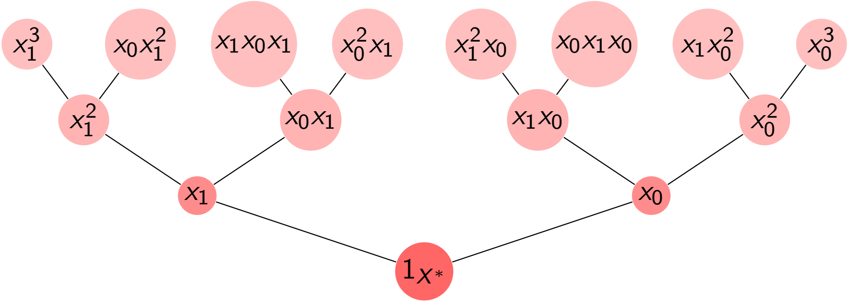
Given a word w, we note $|w|_{x_1}$ the number of occurrences of x_1 within w

$$\alpha_0^z(w) = \begin{cases} 1_\Omega & \text{if} \quad w = 1_{X^*} \\ \int_0^z \alpha_0^s(u) \frac{ds}{1-s} & \text{if} \quad w = x_1 u \\ \int_1^z \alpha_0^s(u) \frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} = 0 \\ \int_0^z \alpha_0^s(u) \frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} > 0 \; . \end{cases} \quad (5)$$

The third line of this recursion implies

$$\alpha_0^z(x_0^n) = \frac{log(z)^n}{n!}$$

one can check that (a) all the integrals (although improper for the fourth line) are well defined (b) the series $S = \sum_{w \in X^*} \alpha_0^z(w)\, w$



Some coefficients with $X = \{x_0, x_1\}$; $u_0(z) = \frac{1}{z}$; $u_1(z) = \frac{1}{1-z}$, $t_0 = 0$

$$\langle S|x_1^n\rangle = \frac{(-log(1-z))^n}{n!} \quad ; \quad \langle S|x_0x_1\rangle = \underbrace{\mathrm{Li}_2(z)}_{cl.not.} = \mathrm{Li}_{x_0x_1}(z) = \sum_{n\geq 1}\frac{z^n}{n^2}$$

$$\langle S|x_0^2x_1\rangle = \underbrace{\mathrm{Li}_3(z)}_{cl.not.} = \mathrm{Li}_{x_0^2x_1}(z) = \sum_{n\geq 1}\frac{z^n}{n^3} \quad ; \quad \langle S|x_1x_0x_1\rangle = \mathrm{Li}_{x_1x_0x_1}(z) = \mathrm{Li}_{[1,2]}(z) = \sum_{n_1>n_2\geq 1}\frac{z^{n_1}}{n_1n_2^2}$$

$$\langle S|x_0x_1^2\rangle = \mathrm{Li}_{x_0x_1^2}(z) = \mathrm{Li}_{[2,1]}(z) = \sum_{n_1>n_2\geq 1}\frac{z^{n_1}}{n_1^2n_2} \quad ; \quad \langle S|x_0^n\rangle = \frac{\log^n(z)}{n!}$$

We need now to do an excursion through the world of iterated integrals and hyperlogarithms but, first, let us recall some definitions of differential algebra.

Some definitions and exemples of differential algebra

Definition [Differential spaces]

A differential ring is a pair $(\mathcal{A}, d)$ with $\mathcal{A}$ a ring (with unit) and $d \in \mathfrak{Der}(\mathcal{A})$ a distinguished derivation i.e. $d \in \mathrm{End}_{\mathbb{Z}}(\mathcal{A})$ s.t.

$$d(f_1 f_2) = d(f_1)f_2 + f_1 d(f_2) \text{ holds identically} .$$

It is an easy exercise to show that $\ker(d)$ is a subring of $\mathcal{A}$.

A differential k-algebra is a differential ring where $\ker(d) = k$ is a field.

Examples

Ex1) $(C^\infty(I, \mathbb{R}), \frac{d}{dx})$ ($I \subset \mathbb{R}$, an open interval) is a differential $\mathbb{R}$-algebra.
Ex2) $(C^\infty(I_1, I_2, \mathbb{R}), \frac{d}{dx})$ (I_i, as above, but $I_1 \cap I_2 = \emptyset$) is a differential ring.
Ex3) $(\mathcal{H}(\Omega), \frac{d}{dz})$ (Ω is a connected open subset of $\mathbb{C}$) is a differential $\mathbb{C}$-algebra.

Some definitions/2 and a construction.

A construction

Let $(\mathcal{A}, d)$ be a commutative differential ring, X an alphabet and $\mathcal{A}\langle\langle X\rangle\rangle$ be the corresponding algebra of (noncommutative, if $|X| \geq 2$) series. We extend d to series by

$$\mathbf{d}(S) := \sum_{w \in X^*} d(\langle S|w\rangle)w \tag{6}$$

It is an easy exercise to show that $\mathbf{d}$ is a derivation of $\mathcal{A}\langle\langle X\rangle\rangle$ and that $\ker(\mathbf{d}) = k\langle\langle X\rangle\rangle$ (here $k = \ker(d)$).

Recall that an operator $s \in \mathrm{End}_{\mathbb{Z}}(\mathcal{A})$ is a section of d iff $d \circ s = Id_{\mathcal{A}}$.
For example, in $(\mathcal{H}(\Omega), \frac{d}{dz})$ has a section iff Ω is simply connected. In this case, all the sections read

$$f \mapsto \lambda + \int_{z_0}^{z} f(s)ds \; ; \; \lambda \in \mathbb{C}$$

Lappo-Danilevskij setting

J. A. Lappo-Danilevskij (J. A. Lappo-Danilevsky), Mémoires sur la théorie des systémes des équations différentielles linéaires. Vol. I, *Travaux Inst. Physico-Math. Stekloff*, 1934, Volume 6, 1–256

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§ 2. Hyperlogarithmes. En abordant la résolution algorithmique du problème de Poincaré, nous introduisons le système des fonctions

$$L_b(a_{j_1}, a_{j_2}, \ldots, a_{j_\nu}|x), \qquad (j_1, j_2, \ldots, j_\nu = 1, 2, \ldots, m; \nu = 1, 2, 3 \ldots)$$

définies par les relations de récurrence:

$$L_b(a_{j_1}|x) = \int_b^x \frac{dx}{x - a_{j_1}} = \log \frac{x - a_{j_1}}{b - a_{j_1}};$$

(10)

$$L_b(a_{j_1} a_{j_2} \ldots a_{j_\nu}|x) = \int_b^x \frac{L_b(a_{j_1} \ldots a_{j_{\nu-1}}|x)}{x - a_{j_\nu}}\, dx,$$

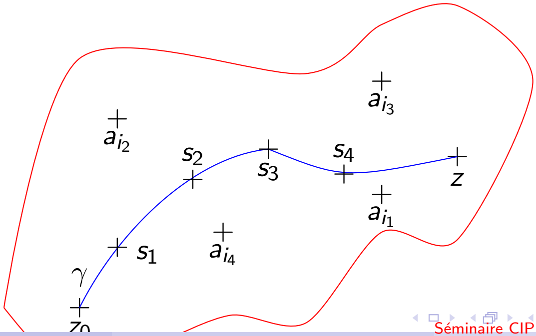
où b est un point fixe à distance finie, distinct des points $a_1, a_2, \ldots, a_m$. Ces fonctions seront nommées *hyperlogarithmes de la première espèce de*

Lappo-Danilevskij setting/2

Let $(a_i)_{1\le i\le n}$ be a family of complex numbers (all different) and $z_0 \notin \{a_i\}_{1\le i\le n}$, then

Definition [Lappo-Danilevskij, 1928]

$$L(a_{i_1},\ldots,a_{i_n}|z_0 \overset{\gamma}{\rightsquigarrow} z) = \int_{z_0}^{z}\int_{z_0}^{s_n}\ldots\Big[\int_{z_0}^{s_1}\frac{ds}{s-a_{i_1}}\Big]\ldots\frac{ds_n}{s_n-a_{i_n}}.$$



Remarks

1. The result depends only on the homotopy class of the path and then the result is a holomorphic function on $\widetilde{B}$ $(B = \mathbb{C} \setminus \{a_1, \cdots, a_n\})$
2. We can also study them in an open (simply connected) subset like the cleft plane

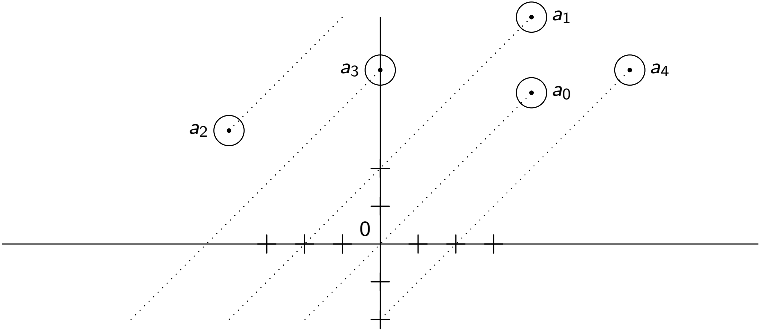


Figure: The cleft plane.

Remarks/2

3. The set of functions $\alpha_{z_0}^{z}(\lambda) = L(a_{i_1}, \ldots, a_{i_n} | z_0 \overset{\gamma}{\rightsquigarrow} z)$ (or 1_Ω if the list is void) has a lot of nice combinatorial properties
   - Noncommutative ED with left multiplier
   - Linear independence
   - Shuffle property
   - Factorisation
   - Possibility of left or right multiplicative renormalization at a neighbourhood of the singularities
   - Extension to rational functions

Now, in order to use the rich allowance of notations invented by algebraists, computer scientists, combinatorialists and physicists about NonCommutative Formal Power Series (NCFPS[1]), we code the lists by words which will permit to do linear algebra and topology on the indexing.

[1] This was the initial intent of the series of conferences FPSAC.

Coding by words

Consider again the mapping

$$\alpha_{z_0}^{z}(\lambda) = L(a_{i_1}, \ldots, a_{i_n} | z_0 \overset{\gamma}{\rightsquigarrow} z) =: \alpha_{z_0}^{z}(x_{i_1} \ldots x_{i_n})$$

Lappo-Danilevskij recursion is from left to right, we will use here right to left indexing to match with[2]

1. P. Cartier, *Jacobiennes généralisées, monodromie unipotente et intégrales itérées*, Séminaire Bourbaki, Volume 30 (1987-1988) , Talk no. 687 , p. 31-52

2. V. Drinfel'd, *On quasitriangular quasi-hopf algebra and a group closely connected with Gal($\bar{\mathbb{Q}}/\mathbb{Q}$)*, Leningrad Math. J., 4, 829-860, 1991.

3. H.J. Susmann, *A product expansion for Chen Series*, in Theory and Applications of Nonlinear Control Systems, C.I. Byrns and Lindquist (eds). 323-335, 1986

4. P. Deligne, *Equations Différentielles à Points Singuliers Réguliers*, Lecture Notes in Math, 163, Springer-Verlag (1970).

[2] Data structures are Letters (1,2), Vector fields (3) and Matrices (4).

Words

We recall basic definitions and properties of the free monoid [1]:

- An alphabet is a set X (of variables, indeterminates etc.
- Words of length n (set X^n) are mappings $w : [1 \cdots n] \to X$. The letter at place j is $w[j]$, the empty word 1_{X^*} is the sole mapping $\emptyset \to X$ (i.e. of length 0). As such, we get, by composition
  - of $\mathfrak{S}_n$ on the right (noted $w.\sigma$) and
  - of the transformation monoid X^X on the left
- Words concatenate by shifting and union of domains, this law is noted *conc*
- $(X^*, conc, 1_{X^*})$ is the free monoid of base X.
- Given a total order on X, (X^*) is totally ordered by the graded lexicographic ordering $\prec_{glex}$ (length first and then lexicographic from left to right). This ordering is compatible with the monoid structure.

[1] M. Lothaire, *Combinatorics on Words*, 2nd Edition, Cambridge Mathematical Library (1997)

Lyndon words and factorizations

- Let $c = [2 \cdots n, 1]$ be the large cycle
- a Lyndon word is a word which is strict minimum of its conjugacy class (as a family) i.e. $(\forall 1 \leq k < n)(l \prec_{lex} l.\sigma^k)$
- Each word w factorizes uniquely as $w = l_1^{\alpha_1} \cdots l_n^{\alpha_n}$ with $l_i \in \mathcal{L}yn(X)$ and $l_1 \prec \cdots \prec l_n$ (strict). We write

$$X^* = \prod_{l \in \mathcal{L}yn(X)}^{\searrow} l^* \tag{7}$$

- If $(P_l)_{l \in \mathcal{L}yn(X)}$ is any multihomogeneous basis of $Lie_R\langle X\rangle$ (R a $\mathbb{Q}$-algebra) then

$$\sum_{w \in X^*} w \otimes w = \prod_{l \in \mathcal{L}yn(X)}^{\searrow} e^{S_l \otimes P_l}$$

where P_w is computed after eq. 7 and S_w is such that $\langle S_u | P_v\rangle = \delta_{u,v}$.

Noncommutative generating series

Let X^* be the set of words constructed on the alphabet X. We now have a function $w \mapsto \alpha_{z_0}^{z}(w)$ which maps words to holomorphic functions on Ω. This is a noncommutative series of variables in X and coefficients in $\mathcal{H}(\Omega)$. It is convenient here to use the "sum notation".

$$S = \sum_{w \in X^*} \alpha_{z_0}^{z}(w)\, w$$

and it is not difficult to see that S is the unique solution of

$$\begin{cases} \mathbf{d}(S) & = & M.S \text{ with } M = \sum_{i=1}^{n} \frac{x_i}{z-a_i} \\ S(z_0) & = & 1_{\mathcal{H}(\Omega)\langle\langle X \rangle\rangle} \end{cases} \tag{8}$$

The series $S_{Pic}^{z_0}$

The series S can be computed by Picard's process

$$S_0 = 1_{X^*} \ ; \ S_{n+1} = 1_{X^*} + \int_{z_0}^{z} M.S_n$$

and its limit is $S_{Pic}^{z_0} := \lim_{n\to\infty} S_n = \sum_{w\in X^*} \alpha_{z_0}^{z}(w)\, w$. One has,

Proposition

i) Series $S_{Pic}^{z_0}$ is the unique solution of

$$\left\{ \begin{array}{rcl} \mathbf{d}(S) & = & M.S \textit{ with } M = \sum_{i=1}^{n} \frac{x_i}{z-a_i} \\ S(z_0) & = & 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \end{array} \right. \quad (9)$$

ii) The complete set of solutions of $\mathbf{d}(S) = M.S$ is $S_{Pic}^{z_0}.\mathbb{C}\langle\langle X\rangle\rangle$.

These (Noncommutative) Differential Equations with Multipliers (as eq. 9) admit a powerful calculus and set of properties .

Main facts about Non Comm. Diff. Eq.

Theorem

Let us coonsider the following two-sided multiplier equation (TSME)

$$(TSME) \qquad \mathbf{d}S = M_1 S + S M_2 \ . \tag{10}$$

with $S \in \mathcal{H}(\Omega)\langle\langle X\rangle\rangle,\ M_i \in \mathcal{H}(\Omega)_+\langle\langle X\rangle\rangle$

(i) *Solutions of* $(TSME)$ *form a* $\mathbb{C}$*-vector space.*

(ii) *Solutions of* $(TSME)$ *have their constant term (as coefficient of* 1_{X^*}*) which are constant functions (on* Ω*); there exists solutions with constant coefficient* 1_Ω *(hence invertible).*

(iii) *If two solutions coincide at one point* $z_0 \in \Omega$ *(or asymptotically), they coincide everywhere.*

Theorem (cont'd)

(iv) *Let be the following one-sided equations*

$$(LM_1) \quad \mathbf{d}S = M_1 S \qquad (RM_2) \quad \mathbf{d}S = SM_2. \tag{11}$$

and let S_1 (resp. S_2) be a solution of (LM_1) (resp. (LM_2)), then $S_1 S_2$ is a solution of $(TSME)$. Conversely, every solution of $(TSME)$ can be constructed so.

(v) *Let $S^{z_0}_{Pic,LM_1}$ (resp. $S^{z_0}_{Pic,RM_2}$) the unique solution of (LM_1) (resp. (RM_2)) s.t. $S(z_0) = 1_{\mathcal{H}(\Omega)_+\langle\langle X \rangle\rangle}$ then, the space of all solutions of $(TSME)$ is*

$$S^{z_0}_{Pic,LM_1}.\mathbb{C}\langle\langle X \rangle\rangle.S^{z_0}_{Pic,RM_2}$$

(vi) *If M_i, $i = 1, 2$ are primitive for $\Delta_{Ш}$ and if S, a solution of $(TSME)$, is group-like at one point (or asymptotically), it is group-like everywhere (over Ω).*

Linear independence without monodromy: Abstract theorem

Theorem (DDMS, 2011)

Let $(\mathcal{A}, d)$ be a k-commutative associative differential algebra with unit ($\ker(d) = k$ is a field) and $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e. $d(\mathcal{C}) \subset \mathcal{C}$). We suppose that $S \in \mathcal{A}\langle\langle X \rangle\rangle$ is a solution of the differential equation

$$\mathbf{d}(S) = MS \ ; \ \langle S | 1_{X^*} \rangle = 1_{\mathcal{A}} \tag{12}$$

where the multiplier M is a homogeneous series (a polynomial in the case of finite X) of degree 1, i.e.

$$M = \sum_{x \in X} u_x x \in \mathcal{C}\langle\langle X \rangle\rangle \ . \tag{13}$$

The following conditions are equivalent :

Abstract theorem/2

Theorem (cont'd)

i) *The family* $(\langle S|w\rangle)_{w\in X^*}$ *of coefficients of* S *is free over* $\mathcal{C}$.

ii) *The family of coefficients* $(\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}$ *is free over* $\mathcal{C}$.

iii) *The family* $(u_x)_{x\in X}$ *is such that, for* $f\in\mathcal{C}$ *and* $\alpha_x\in k$

$$d(f)=\sum_{x\in X}\alpha_x u_x \Longrightarrow (\forall x\in X)(\alpha_x=0)\ . \tag{14}$$

iv) *The family* $(u_x)_{x\in X}$ *is free over* k *and*

$$d(\mathcal{C})\cap span_k\Big((u_x)_{x\in X}\Big)=\{0\}\ . \tag{15}$$

Independence of Hyperlogarithms over Function Fields via Algebraic Combinatorics, **M. Deneufchâtel, GHED, V. Hoang Ngoc Minh and A. I. Solomon**, 4th International Conference on Algebraic Informatics, Linz (2011). Proceedings, Lecture Notes in Computer Science, 6742, Springer.

Need for localization

In practical cases, we have a differential subalgebra of $\mathcal{C}_0 \subset \mathcal{H}(\Omega)$

- $\mathbb{C}[z]$
- $\mathbb{C}[z, z^{-1}, (1-z)^{-1}]$
- $\mathbb{C}[z^\alpha (1-z)^{-\beta}]_{\alpha,\beta\in\mathbb{C}}$

Realizing the fraction field $Fr(\mathcal{C}_0)$ as (differential) field of germs makes the computation difficult to handle. It is easier to check the freeness of the " basic triangle" directly with the algebra. For instance, for the polylogarithms, we just have to show that, given 3 polynomials,

$$P_1(z) + P_2(z)\log(z) + P_3(z)(\log(\frac{1}{1-z})) = 0_\Omega \Longrightarrow P_i \equiv 0 \tag{16}$$

which is straightforward (with the freeness of the comparison scale $x \mapsto x^u \log(x)^v$ in $]0,1[$).

Localization

Theorem (Thm1 in "Linz", Localized form)

Let $(\mathcal{A}, d)$ be a commutative associative differential ring ($\ker(d) = k$ being a field) and $\mathcal{C}$ be a differential subring (i.e. $d(\mathcal{C}) \subset \mathcal{C}$) of $\mathcal{A}$ which is an integral domain containing the field of constants.
We suppose that, for all $x \in X$, $u_x \in \mathcal{C}$ and that $S \in \mathcal{A}\langle\langle X \rangle\rangle$ is a solution of the differential equation (12) and that $(u_x)_{x \in X} \in \mathcal{C}^X$.
The following conditions are equivalent :

i) *The family $(\langle S|w\rangle)_{w \in X^*}$ of coefficients of S is free over $\mathcal{C}$.*

ii) *The family of coefficients $(\langle S|y\rangle)_{y \in X \cup \{1_{X^*}\}}$ is free over $\mathcal{C}$.*

iii') *For all $f_1, f_2 \in \mathcal{C}$, $f_2 \neq 0$ and $\alpha \in k^{(X)}$, we have the property*

$$W(f_1, f_2) = f_2^2 \Big(\sum_{x \in X} \alpha_x u_x\Big) \Longrightarrow (\forall x \in X)(\alpha_x = 0)\ . \qquad (17)$$

where $W(f_1, f_2)$, the wronskian, stands for $d(f_1)f_2 - f_1 d(f_2)$.

In fact, in the localized form and with $\mathcal{C}$ not a differential field, (iii) is strictly weaker than (iii'), as shows the following family of counterexamples

1. $\Omega = \mathbb{C} \setminus (]-\infty, 0])$
2. $X = \{x_0\}$, $u_0 = z^\beta$, $\beta \notin \mathbb{Q}$
3. $\mathcal{C}_0 = \mathbb{C}\{\{z^\beta\}\} = \mathbb{C}.1_\Omega \oplus span_{\mathbb{C}}\{z^{(k+1)\beta - l}\}_{k,l\geq 0}$
4. $S = 1_\Omega + (\sum_{n\geq 1} \frac{z^{n(\beta+1)}}{(\beta+1)^n n!})$

Let us show that, for these data (iii) holds but not (i).
Firstly, we show that $\mathcal{C}_0 = \mathbb{C}\{\{z^\beta\}\}$ corresponds to the given direct sum.
We remark that the family $(z^\alpha)_{\alpha\in\mathbb{C}}$ is $\mathbb{C}$-linearly free (within $\mathcal{H}(\Omega)$), which is a consequence of the fact that they are eigenfunctions, for different eigenvalues, of the Euler operator $z\frac{d}{dz}$.

Then

$$\mathbb{C}\{\{z^\beta\}\} = \mathbb{C}1_\Omega \oplus span_{\mathbb{C}}\{z^{(k+1)\beta-l}\}_{k,l\geq 0} = span_{\mathbb{C}}\{z^{(k')\beta-l}\}_{k',l\geq 0}$$

comes from the fact that the RHS is a subset of the LHS as, for all, $k,l\geq 0$, $z^{(k+1)\beta-l}\in\mathbb{C}\{\{z^\beta\}\}$. Finally $1_\Omega\in\mathbb{C}\{\{z^\beta\}\}$ by definition ($\mathbb{C}\{\{X\}\}$ is a $\mathbb{C}$-AAU).

(iii) is fulfilled. Here

$u_0(z)=z^\beta$ is such that, for any $f\in\mathcal{C}_0$ and c_0 in $\mathbb{C}$, we have

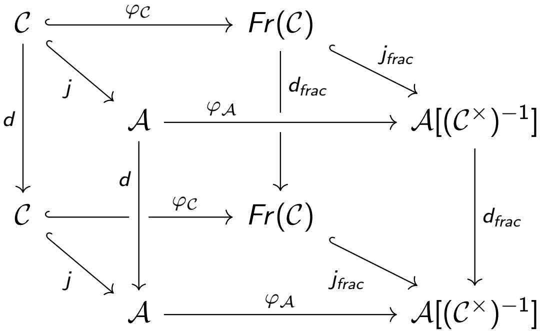
$$c_0u_0=\partial_z(f)\Longrightarrow(c_0=0) \tag{18}$$

But (i) is not Because we have the following relation

$$(\beta+1)z^{\beta-1}\langle S|x_0\rangle - z^{2\beta}.1_\Omega = 0$$

Sketch of the proof

After some technicalities, we show that (12) can be transported in $\mathcal{A}[(\mathcal{C}^\times)^{-1}$ by means of the following commutative diagram and back.



(19)

Concrete theorem

Theorem (DDMS, 2011)

Let $S \in \mathcal{H}(\Omega)\langle\langle X \rangle\rangle$ be a solution of the (Left Multiplier) equation

$$\mathbf{d}(S) = MS\ ;\ \langle S | 1_{X^*} \rangle = 1_{\Omega}.$$

The following are equivalent :

i) *the family $(\langle S|w\rangle)_{w \in X^*}$ of coefficients is independant (linearly) over $\mathcal{C}$.*

ii) *the family of coefficients $(\langle S|x\rangle)_{x \in X \cup \{1_{X^*}\}}$ is independant (linearly) over $\mathcal{C}$.*

iii) *the family $(u_x)_{x \in X}$ is such that, for $f \in \mathcal{C}$ et $\alpha_x \in \mathbb{C}$*

$$\mathrm{d}(f) = \sum_{x \in X} \alpha_x u_x \Longrightarrow (\forall x \in X)(\alpha_x = 0).$$

Solutions as ш-characters with values in $\mathcal{H}(\Omega)$

We have seen that solutions of systems like that of hyperlogarithms (8) possess the shuffle property i.e. defining the shuffle product by the recursion

$$\begin{aligned} u \,ш\, 1_{Y^*} &= 1_{Y^*} \,ш\, u = u \text{ and} \\ au \,ш\, bv &= a(u \,ш\, bv) + b(au \,ш\, v) \end{aligned}$$

one has

$$\langle S_{Pic}^{z_0} | u \,ш\, v\rangle = \langle S_{Pic}^{z_0} | u\rangle \langle S_{Pic}^{z_0} | v\rangle ;\ \langle S | 1_{X^*}\rangle = 1_{\Omega} \tag{20}$$

(product in $\mathcal{H}(\omega)$).

Now it is not difficult to check that the characters of type (20) form a group (these are characters on a Hopf algebra, see below). I would be interesting to have at our disposal a system of local coordinates in order to perform estimates in neighbourhood of the singularities.

Applying MRS to a shuffle character

Now, remarking that this factorization lives within the subalgebra

$$Iso(X) = \{T \in R\langle\langle X^* \otimes X^*\rangle\rangle | (u \otimes v \in supp(T) \Longrightarrow |u| = |v|)\}$$

if Z is any shuffle character, one has

$$Z = (Z \otimes Id)(\sum_{w \in X^*} w \otimes w) = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\langle Z|S_l\rangle P_l}$$

We would like to get such a factorisation at our disposal for other types of (deformed) shuffle products, this will be done in the second part of the talk. Let us first, with this factorization (MRS) at hand, construct explicitely Drinfeld's solution G_0.

Extensions of MRS to other shuffles

Name	Formula (recursion)	φ	Type
Shuffle [21]	$au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v)$	$\varphi \equiv 0$	I
Stuffle [19]	$x_iu \uplus x_jv = x_i(u \uplus x_jv) + x_j(x_iu \uplus v) + x_{i+j}(u \uplus v)$	$\varphi(x_i, x_j) = x_{i+j}$	I
Min-stuffle [7]	$x_iu \sqcup\!\sqcup_- x_jv = x_i(u \sqcup\!\sqcup_- x_jv) + x_j(x_iu \sqcup\!\sqcup_- v) - x_{i+j}(u \sqcup\!\sqcup_- v)$	$\varphi(x_i, x_j) = -x_{i+j}$	III
Muffle [14]	$x_iu \sqcup\!\sqcup_\bullet x_jv = x_i(u \sqcup\!\sqcup_\bullet x_jv) + x_j(x_iu \sqcup\!\sqcup_\bullet v) + x_{i\times j}(u \sqcup\!\sqcup_\bullet v)$	$\varphi(x_i, x_j) = x_{i\times j}$	I
q-shuffle [3]	$x_iu \uplus_q x_jv = x_i(u \uplus_q x_jv) + x_j(x_iu \uplus_q v) + qx_{i+j}(u \uplus_q v)$	$\varphi(x_i, x_j) = qx_{i+j}$	III
q-shuffle$_2$	$x_iu \uplus_q x_jv = x_i(u \uplus_q x_jv) + x_j(x_iu \uplus_q v) + q^{i.j}x_{i+j}(u \uplus_q v)$	$\varphi(x_i, x_j) = q^{i.j}x_{i+j}$	II
$\mathtt{LDIAG}(1, q_s)$ [10] (non-crossed, non-shifted)	$au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v) + q_s^{\lvert a\rvert\lvert b\rvert}a.b(u \sqcup\!\sqcup v)$	$\varphi(a, b) = q_s^{\lvert a\rvert\lvert b\rvert}(a.b)$	II
q-Infiltration [12]	$au \uparrow bv = a(u \uparrow bv) + b(au \uparrow v) + q\delta_{a,b}a(u \uparrow v)$	$\varphi(a, b) = q\delta_{a,b}a$	III
AC-stuffle	$au \sqcup\!\sqcup_\varphi bv = a(u \sqcup\!\sqcup_\varphi bv) + b(au \sqcup\!\sqcup_\varphi v) + \varphi(a, b)(u \sqcup\!\sqcup_\varphi v)$	$\varphi(a, b) = \varphi(b, a)$; $\varphi(\varphi(a, b), c) = \varphi(a, \varphi(b, c))$	IV
Semigroup-stuffle	$x_tu \sqcup\!\sqcup_\perp x_sv = x_t(u \sqcup\!\sqcup_\perp x_sv) + x_s(x_tu \sqcup\!\sqcup_\perp v) + x_{t\perp s}(u \sqcup\!\sqcup_\perp v)$	$\varphi(x_t, x_s) = x_{t\perp s}$	I
φ-shuffle	$au \sqcup\!\sqcup_\varphi bv = a(u \sqcup\!\sqcup_\varphi bv) + b(au \sqcup\!\sqcup_\varphi v) + \varphi(a, b)(u \sqcup\!\sqcup_\varphi v)$	$\varphi(a, b)$ law of AAU	V

Of course, the q-shuffle is equal to the (classical) shuffle when $q = 0$. As for the q-

About Drinfeld's solutions G_0, G_1

We give below the computational construction of a solution with an asymptotic condition.
In his paper (2. above), V. Drinfel'd states that there is a unique solution (called G_0) of

$$\begin{cases} \mathbf{d}(S) = (\frac{x_0}{z} + \frac{x_1}{1-z}).S \\ \lim_{\substack{z\to 0\\ z\in\Omega}} S(z)e^{-x_0 log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \end{cases}$$

and a unique solution (called G_1) of

$$\begin{cases} \mathbf{d}(S) = (\frac{x_0}{z} + \frac{x_1}{1-z}).S \\ \lim_{\substack{z\to 1\\ z\in\Omega}} e^{x_1 log(1-z)}S(z) = 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \end{cases}$$

Let us give here, as an example, a construction of G_0 (G_1 can be derived or checked by symmetry).

satisfies the one sided evolution equation (LM)

$$\mathbf{d}(S) = (\frac{x_0}{z} + \frac{x_1}{1-z}).S$$

hence $T = (\sum_{w\in X^*} \alpha_0^z(w)\, w)e^{-x_0 log(z)}$ satisfies the two sided evolution equation (TSME)

$$\mathbf{d}(T) = (\frac{x_0}{z} + \frac{x_1}{1-z}).T + T.(-\frac{x_0}{z})$$

Now, using Radford's theorem, one proves that S is group-like, factorizes through (MRS) and that $\lim_{z\to 0} T(z) = 1$.
This asymptotic condition on T implies that $S = G_0$.

Conclusion

- For Series with variable coefficients, we have a theory of Noncommutative Evolution Equation sufficiently powerful to cover iterated integrals and multiplicative renormalisation
- MRS factorisation allows to remove singularities with simple counterterms
- MRS factorisation can be performed in many other cases (like stuffle for harmonic functions)
- Use of combinatorics on words gives a necessary and sufficient condition on the "inputs" to have linear independance of the solutions over higher function fields.
- Picard (Chen) solutions admit enlarged indexing w.r.t. compact convergence on Ω (polylogarithmic case) but Drinfeld's G_0 has a domain which includes only some rational series.
- Localization is possible (under certain conditions).

Thank you for your attention.

Combinatorics and Arithmetic for Physics 2019 (IHES)

(fifth edition) will take place

07-08 November 2019

At IHES, Bures-sur-Yvette (France)

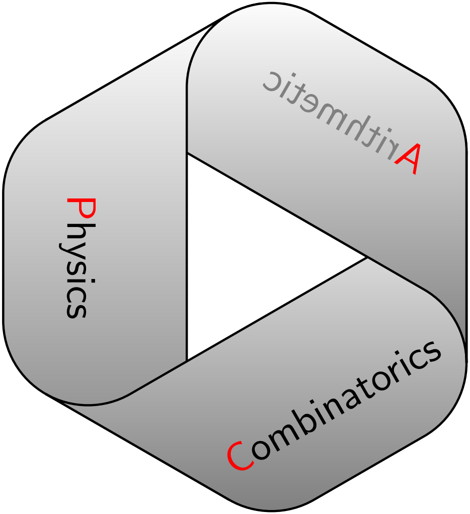
Organisers: Vincel Hoang Ngoc Minh, Maxim Kontsevitch, Gleb Koshevoy, Gérard Duchamp.

Scope: The meeting's focus is on questions of discrete mathematics and number theory with an emphasis on computability. Problems are drawn mainly from theoretical physics (renormalisation, combinatorial physics, geometry) or related to its models.

Computation, based on combinatorial structures (graphs, trees, words, automata, semirings, bases) or classic structures (operators, Hopf algebras, evolution equations, special functions, categories) are good candidates for computer-based implantation and experimentation.

- **web announcements (coming)**
`http://www-lipn.univ-paris13.fr/~duchamp/Conferences/CAP6_2019.html`
- **emails**
`Hoang.Ngocminh@lipn.univ-paris13.fr`
`Gerard.Duchamp@lipn.univ-paris13.fr`
`gheduchamp@gmail.com`

Figure: CAP18 & CAP19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