On universal differential equations

V. Hoang Ngoc Minh
Université Lille, 1 Place Déliot, 59024 Lille, France.

Séminaire Combinatoire, Informatique et Physique
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Outline

INTRODUCTION

Picard-Vessiot theory of ordinary differential equation

$(\mathbf{k}, \partial)$ a commutative differential ring without zero divisors.
$\mathrm{Const}(\mathbf{k}) = \{c \in \mathbf{k} | \partial c = 0\}$ is supposed to be a field.
$(ODE) \quad (a_n\partial^n + a_{n-1}\partial^{n-1} + \ldots + a_0)y = 0, \quad a_0, \ldots, a_{n-1}, a_n \in \mathbf{k}.$
a_n^{-1} is supposed to exist.

Definition 1

1. Let $y_1, \ldots, y_n$ be $\mathrm{Const}(\mathbf{k})$-linearly independent solutions of (ODE). Then $\{y_1, \ldots, y_n\}$ is called a fundamental set of solutions of (ODE) and it generates a $\mathrm{Const}(\mathbf{k})$-vector subspace of dimension at most n.
2. If[1] $M = \mathbf{k}\{y_1, \ldots, y_n\}$ and $\mathrm{Const}(M) = \mathrm{Const}(\mathbf{k})$ then M is called a Picard-Vessiot extension related to (ODE)
3. Let $\mathbf{k} \subset \mathbb{K}_1$ and $\mathbf{k} \subset \mathbb{K}_2$ be differential rings. An isomorphism of rings $\sigma : \mathbb{K}_1 \to \mathbb{K}_2$ is a differential $\mathbf{k}$-isomorphism if
$$\forall a \in \mathbb{K}_1, \quad \partial(\sigma(a)) = \sigma(\partial a) \text{ and, if } a \in \mathbf{k},\ \sigma(a) = a.$$
If $\mathbb{K}_1 = \mathbb{K}_2 = \mathbb{K}$, the differential galois group of $\mathbb{K}$ over $\mathbf{k}$ is by
$$\mathrm{Gal}_{\mathbf{k}}(\mathbb{K}) = \{\sigma | \sigma \text{ is a differential } \mathbf{k}\text{-automorphism of } \mathbb{K}\}.$$

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1. Let R_1, R_2 be differential rings s.t. $R_1 \subset R_2$. Let S be a subset of R_2.
$R_1\{S\}$ denotes the smallest differential subring of R_2 containing R_1.
$R_1\{S\}$ is the ring (over R_1) generated by S and their derivatives of all orders.

Linear differential equations and Dyson series

Let $a_0, \ldots, a_n \in \mathbb{C}(z)$, $\quad (a_n(z)\partial^n + \ldots + a_1(z)\partial + a_0(z))y(z) = 0.$

$$(ED) \quad \left\{ \begin{array}{rcll} \partial q(z) & = & A(z)q(z), & A(z) \in \mathcal{M}_{n,n}(\mathbb{C}(z)), \\ q(z_0) & = & \eta, & \lambda \in \mathcal{M}_{1,n}(\mathbb{C}), \\ y(z) & = & \lambda q(z), & \eta \in \mathcal{M}_{n,1}(\mathbb{C}). \end{array} \right.$$

By successive Picard iterations, with the initial point $q(z_0) = \eta$, we get [2] $y(z) = \lambda U(z_0; z)\eta$, where $U(z_0; z)$ is the following functional expansion

$$U(z_0; z) = \sum_{k \geq 0} \int_{z_0}^{z} A(z_1)dz_1 \int_{z_0}^{z_1} A(z_2)dz_2 \ldots \int_{z_0}^{z_{k-1}} A(z_k)dz_k, \text{(Dyson series)}$$

and $(z_0, z_1 \ldots, z_k, z)$ is a subdivision of the path of integration $z_0 \rightsquigarrow z$.
In order to find the matrix $\Omega(z_0; z)$ s.t.

$$U(z_0; z) = \exp[\Omega(z_0; z)] = \top \exp \int_{z_0}^{z} A(s)ds, \qquad \text{(Feynman's notation)}$$

Magnus computed $\Omega(z_0; z)$ as limit of the following Lie-integral-functionals

$$\begin{array}{rcl} \Omega_1(z_0; z) & = & \displaystyle\int_{z_0}^{z} A(z)ds, \\ \Omega_k(z_0; z) & = & \displaystyle\int_{z_0}^{z} [A(z) + [A(z), \Omega_{k-1}(z_0; s)]/2 \\ & & +[[A(z), \Omega_{k-1}(z_0; s)], \Omega_{k-1}(z_0; s)]/12 + \ldots)ds. \end{array}$$

2. Subject to convergence.

Fuchsian linear differential equations

Let us consider, here, $\sigma = \{s_i\}_{i=0,..,m}$ as set of simple poles of (ED).

$$A(z) = \sum_{i=0}^{m} M_i u_i(z), \quad \text{where} \quad \begin{cases} M_i & \in \mathcal{M}_{n,n}(\mathbb{C}), \\ u_i(z) = (z - s_i)^{-1} & \in \mathbb{C}(z). \end{cases}$$

$$(ED) \quad \begin{cases} \partial q(z) &= \left(\sum_{i=0}^{m} M_i u_i(z)\right) q(z), \\ q(z_0) &= \eta, \\ y(z) &= \lambda q(z). \end{cases}$$

Let $\mathcal{H}(\Omega)$ be the ring of holomorphic functions (1_Ω : neutral element) over the multi-cleft complex plane Ω (from s_i's to infinities without crossing).
Let X^* be the set of words over $X = \{x_0, \ldots, x_m\}$ and

$$\alpha_{z_0}^{z} \otimes \mathcal{M} : \mathbb{C}\langle X \rangle \otimes \mathbb{C}\langle X \rangle \to \mathcal{M}_{n,n}(\mathcal{H}(\Omega))$$

($z_0 \rightsquigarrow z$ is the path of integration previously introduced) s.t.

$$\mathcal{M}(1_{X^*}) = \mathtt{Id}_n \quad \text{and} \quad \mathcal{M}(x_{i_1} \cdots x_{i_k}) = M_{i_1} \ldots M_{i_k},$$

$$\alpha_{z_0}^{z}(1_{X^*}) = 1_{\mathcal{H}(\Omega)} \quad \text{and} \quad \alpha_{z_0}^{z}(x_{i_1} \cdots x_{i_k}) = \int_{z_0}^{z} \frac{dz_1}{z_1 - s_{i_1}} \ldots \int_{z_0}^{z_{k-1}} \frac{dz_k}{z_k - s_{i_k}}.$$

Then [3] $y(z) = \lambda U(z_0; z)\eta$ with

$$U(z_0; z) = \sum_{w \in X^*} \mathcal{M}(w) \alpha_{z_0}^{z}(w) = (\mathcal{M} \otimes \alpha_{z_0}) \sum_{w \in X^*} w \otimes w.$$

3. Subject to convergence.

Examples of linear dynamical systems

Example 2 (Hypergeometric equation)

Let t_0, t_1, t_2 be parameters and

$$z(1-z)\ddot{y}(z) + [t_2 - (t_0 + t_1 + 1)z]\dot{y}(z) - t_0 t_1 y(z) = 0.$$

Let $q_1(z) = -y(z)$ and $q_2(z) = (1-z)\dot{y}(z)$. Hence, one has

$$y(z) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix}$$

and

$$\begin{aligned} \begin{pmatrix} \dot{q}_1(z) \\ \dot{q}_2(z) \end{pmatrix} &= \left(\frac{M_0}{z} + \frac{M_1}{1-z} \right) \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix} \\ &= (u_0(z) M_0 + u_1(z) M_1) \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix}, \end{aligned}$$

where $u_0(z) = z^{-1}, u_1(z) = (1-z)^{-1}$ and

$$M_0 = -\begin{pmatrix} 0 & 0 \\ t_0 t_1 & t_2 \end{pmatrix} \quad \text{and} \quad M_1 = -\begin{pmatrix} 0 & 1 \\ 0 & t_2 - t_0 - t_1 \end{pmatrix}.$$

Nonlinear differential equations

$$(NED)\quad \begin{cases} \partial q(z) &= \left(\sum_{i=0}^{m} T_i(q)u_i(z)\right)(q),\\ q(z_0) &= q_0,\\ y(z) &= f(q(z)), \end{cases}$$

where

- $u_i \in (\mathbf{k}, \partial)$,
- the state $q = (q_1, \ldots, q_n)$ belongs the complex analytic manifold Q of dimension n and q_0 is the initial state,
- the observation $f \in \mathcal{O}$, with $\mathcal{O}$ the ring of analytic functions over Q,
- for $i = 0..1$, $T_i = (T_i^1(q)\partial/\partial q_1 + \cdots + T_i^m(q)\partial/\partial q_m)$ is an analytic vector field over Q, with $T_i^j(q) \in \mathcal{O}$, for $j = 1, \ldots, n$.

With X and $\alpha_{z_0}^z$ given as previously, let the morphism τ be defined by $\tau(1_{X^*}) = \mathtt{Id}$ and $\tau(x_{i_1} \cdots x_{i_k}) = T_{i_1} \ldots T_{i_k}$. Then [4] $y(z) = \mathcal{T} \circ f_{|_{q_0}}$ with

$$\mathcal{T} = \sum_{w \in X^*} \tau(w)\alpha_{z_0}^z(w) = (\tau \otimes \alpha_{z_0}^z) \sum_{w \in X^*} w \otimes w.$$

4. Subject to convergence.

Examples of nonlinear dynamical systems (1/2)

Example 3 (Harmonic oscillator)

Let k_1, k_2 be parameters and $\partial^2 y(z) + k_1 y(z) + k_2 y^2(z) = u_1(z)$ which can be represented by the following state equations (with $n = 1$)

$$\begin{array}{rcl} y(z) &=& q(z), \\ \partial q(z) &=& A_0(q)u_0(z) + A_1(q)u_1(z), \\ \text{where} \quad A_0 &=& -(k_1 q + k_2 q^2)\dfrac{\partial}{\partial q} \quad \text{and} \quad A_1 \;=\; \dfrac{\partial}{\partial q}. \end{array}$$

Example 4 (Duffing equation)

Let a, b, c be parameters and $\partial^2 y(z) + a\partial y(z) + by(z) + cy^3(z) = u_1(z)$ which can be represented by the following state equations (with $n = 2$)

$$\begin{array}{rcl} y(z) &=& q_1(z), \\ \begin{pmatrix} \partial q_1(z) \\ \partial q_2(z) \end{pmatrix} &=& \begin{pmatrix} q_2 \\ -(aq_2 + b^2 q_1 + cq_1^3) \end{pmatrix} u_0(z) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_1(z) \\ &=& A_0(q)u_0(z) + A_1(q)u_1(z), \\ \text{where} \quad A_0 &=& -(aq_2 + b^2 q_1 + cq_1^3)\dfrac{\partial}{\partial q_2} + q_2 \dfrac{\partial}{\partial q_1} \quad \text{and} \quad A_1 \;=\; \dfrac{\partial}{\partial q_2}. \end{array}$$

Examples of nonlinear dynamical systems (2/2)

Example 5 (Van der Pol oscillator)

Let γ, g be parameters and

$$\partial^2 x(z) - \gamma[1 + x(z)^2]\partial x(z) + x(z) = g\cos(\omega z)$$

which can be tranformed into (with C is some constant of integration)

$$\partial x(z) = \gamma[1 + x(z)^2/3]x(z) - \int_{z_0}^{z} x(s)ds + \frac{g}{\omega}\sin(\omega z) + C.$$

Supposing $x = \partial y$ and $u_1(z) = g\sin(\omega z)/\omega + C$, it leads then to

$$\partial^2 y(z) = \gamma[\partial y(z) + (\partial y(z))^3/3] + y(z) + u_1(z)$$

which can be represented by the following state equations (with $n = 2$)

$$\begin{aligned} y(z) &= q_1(z), \\ \begin{pmatrix} \partial q_1(z) \\ \partial q_2(z) \end{pmatrix} &= \begin{pmatrix} q_2 \\ \gamma(q_2 + q_2^3/3) + q_1 \end{pmatrix} u_0(z) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_1(z) \\ &= A_0(q)u_0(z) + A_1(q)u_1(z), \end{aligned}$$

$$\text{where} \quad A_0 = [\gamma(q_2 + q_2^3/3) + q_1]\frac{\partial}{\partial q_2} + q_2\frac{\partial}{\partial q_1} \quad \text{and} \quad A_1 = \frac{\partial}{\partial q_2}.$$

DUAL LAWS AND REPRESENTATIVE SERIES

Dual laws in bialgebras

Startting with a $\mathbf{k} - \mathbf{AAU}$ ($\mathbf{k}$ is a ring) $\mathcal{A}$. Dualizing $\mu : \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \to \mathcal{A}$, we get the transpose ${}^t\mu : \mathcal{A}^\vee \to (\mathcal{A} \otimes_{\mathbf{k}} \mathcal{A})^\vee$ so that we do not get a co-multiplication in general.

- Remark that when $\mathbf{k}$ is a field, the following arrow is into (due to the fact that $\mathcal{A}^\vee \otimes_{\mathbf{k}} \mathcal{A}^\vee$ is torsionfree)
$$\Phi : \mathcal{A}^\vee \otimes_{\mathbf{k}} \mathcal{A}^\vee \to (\mathcal{A} \otimes_{\mathbf{k}} \mathcal{A})^\vee.$$
- One restricts the codomain of ${}^t\mu$ to $\mathcal{A}^\vee \otimes_{\mathbf{k}} \mathcal{A}^\vee$ and then the domain to $({}^t\mu)^{-1}\Phi(\mathcal{A}^\vee \otimes_{\mathbf{k}} \mathcal{A}^\vee) =: \mathcal{A}^\circ$.

$$\begin{array}{ccc}
\mathcal{A}^\vee & \xrightarrow{\quad {}^t\mu \quad} & (\mathcal{A} \otimes_{\mathbf{k}} \mathcal{A})^\vee \\
{\scriptstyle can}\uparrow & & \uparrow{\scriptstyle \Phi} \\
\mathcal{A}^\circ & \xrightarrow{\quad \Delta_\mu \quad} & \mathcal{A}^\vee \otimes_{\mathbf{k}} \mathcal{A}^\vee \\
{\scriptstyle can}\uparrow & & \uparrow{\scriptstyle j\otimes j} \\
\mathcal{A}^{\circ\circ} & \xrightarrow{\quad \Delta_\mu \quad} & \mathcal{A}^\circ \otimes_{\mathbf{k}} \mathcal{A}^\circ
\end{array}$$

The descent stops at first step for a field $\mathbf{k}$ and then $\mathcal{A}^{\circ\circ} = \mathcal{A}^\circ$.
The coalgebra $(\mathcal{A}^\circ, \Delta_\mu)$ is called the Sweedler's dual of $(\mathcal{A}, \mu)$.

Case of algebras noncommutative series

- $\mathcal{X}$ denotes the ordered alphabets $Y := \{y_k\}_{k\geq 1}$ or $X := \{x_0, x_1\}$.
  On the free monoid $(\mathcal{X}^*, \mathtt{conc}, 1_{\mathcal{X}^*})$, we use the correspondences
  $$x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \in X^*x_1 \underset{\pi_X}{\overset{\pi_Y}{\rightleftharpoons}} y_{s_1} \ldots y_{s_r} \in Y^* \leftrightarrow (s_1, \ldots, s_r) \in \mathbb{N}_+^r.$$
  Let $\mathcal{L}yn\mathcal{X}$ denote the set of Lyndon words generated by $\mathcal{X}$.
- Let $(\mathcal{L}ie_A\langle\langle\mathcal{X}\rangle\rangle, [.])$ and $(A\langle\langle\mathcal{X}\rangle\rangle, \mathtt{conc})$ (resp. $(\mathcal{L}ie_A\langle\mathcal{X}\rangle, [.])$ and $(A\langle\mathcal{X}\rangle, \mathtt{conc})$) are the algebras of (Lie) series (resp. polynomials).
  $\{P_l\}_{l\in\mathcal{L}yn\mathcal{X}}$ (resp. $\{\Pi_l\}_{l\in\mathcal{L}ynY}$) is a basis of Lie algebra of primitive elements and $\{S_l\}_{l\in\mathcal{L}yn\mathcal{X}}$ (resp. $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$) is a transcendence basis of $(A\langle\mathcal{X}\rangle, ⧢, 1_{\mathcal{X}^*})$ (resp. $(A\langle Y\rangle, \uplus, 1_{Y^*})$).
- $\mathcal{H}_{⧢}(\mathcal{X}) := (A\langle\mathcal{X}\rangle, \mathtt{conc}, 1_{\mathcal{X}^*}, \Delta_{⧢}, \mathrm{e})$ and
  $\mathcal{H}_{\uplus}(Y) := (A\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\uplus}, \mathrm{e})$ with[5] (for $x \in \mathcal{X}, y_i \in Y$)
  $$\begin{aligned} \Delta_{⧢}\, x &= x \otimes 1_{\mathcal{X}^*} + 1_{\mathcal{X}^*} \otimes x, \\ \Delta_{\uplus}\, y_i &= y_i \otimes 1_{Y^*} + 1_{Y^*} \otimes y_i + \textstyle\sum_{k+l=i} y_k \otimes y_l. \end{aligned}$$
- The dual law associated to conc is defined, for $w \in \mathcal{X}^*$, by
  $$\Delta_{\mathtt{conc}}(w) = \sum_{u,v\in\mathcal{X}^*, uv=w} u \otimes v.$$

5. Or equivalently, for $x, y \in \mathcal{X}, y_i, y_j \in Y$ and $u, v \in \mathcal{X}^*$ (resp. Y^*),
$u ⧢ 1_{\mathcal{X}^*} = 1_{\mathcal{X}^*} ⧢ u = u$ and $xu ⧢ yv = x(u ⧢ yv) + y(xu ⧢ v)$,
$u \uplus 1_{Y^*} = 1_{Y^*} \uplus u = u$ and $x_iu \uplus y_jv = y_i(u \uplus y_jv) + y_j(y_iu \uplus v) + y_{i+j}(u \uplus v)$.

Dualizable laws in conc-shuffle bialgebras (1/2)

We can exploit the basis of words as follows

1. Any bilinear law (shuffle, stuffle or any) $\mu : A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle \to A\langle\mathcal{X}\rangle$ can be decribed through its structure constants wrt to the basis of words, *i.e.* for $u, v, w \in \mathcal{X}^*$, $\Gamma_{u,v}^{w} := \langle \mu(u \otimes v) | w \rangle$ so that
$$\mu(u \otimes v) = \sum_{w \in \mathcal{X}^*} \Gamma_{u,v}^{w} w.$$
2. In the case when $\Gamma_{u,v}^{w}$ is locally finite in w, we say that the given law is dualizable, the arrow ${}^t\mu$ restricts nicely to $A\langle\mathcal{X}\rangle \hookrightarrow A\langle\langle\mathcal{X}\rangle\rangle$ and one can define on the polynomials a comultiplication by
$$\Delta_\mu(w) := \sum_{u,v \in \mathcal{X}^*} \Gamma_{u,v}^{w} u \otimes v.$$
3. When the law μ is dualizable, the following arrow Δ_μ is unique to be able to close the rectangle and $\Delta_\mu(P)$ is defined as above,

$$\begin{array}{ccc}
A\langle\langle\mathcal{X}\rangle\rangle & \xrightarrow{\quad {}^t\mu \quad} & A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle \\
{\scriptstyle can}\uparrow & & \uparrow{\scriptstyle \Phi|_{A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle}} \\
A\langle\mathcal{X}\rangle & \xrightarrow{\quad \Delta_\mu \quad} & A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle
\end{array}$$

Dualizable laws in conc-shuffle bialgebras (2/2)

4. Proof that the arrow $A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle \longrightarrow A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle$ is into :
   Let $T = \sum_{i=1}^n P_i \otimes_A Q_i$ such that $\Phi(T) = 0$. Rewriting T as a finitely supported sum $T = \sum_{u,v\in\mathcal{X}^*} c_{u,v} u \otimes v$ (this is indeed the iso between $A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle$ and $A[\mathcal{X}^* \times \mathcal{X}^*]$), $\Phi(T)$ is by definition of Φ the double series (here a polynomial) s.t. $\langle\Phi(T)|u \otimes v\rangle = c_{u,v}$. If $\Phi(T) = 0$, then for all $(u,v) \in \mathcal{X}^* \times \mathcal{X}^*$, $c_{u,v} = 0$ entailing $T = 0$.

We extend by linearity and infinite sums, for $S \in A\langle\langle Y\rangle\rangle$ (resp. $A\langle\langle\mathcal{X}\rangle\rangle$), by

$$\Delta_{\uplus} S = \sum_{w\in Y^*} \langle S|w\rangle \Delta_{\uplus} w \quad \in A\langle\langle Y^* \otimes Y^*\rangle\rangle,$$

$$\Delta_{\mathtt{conc}} S = \sum_{w\in\mathcal{X}^*} \langle S|w\rangle \Delta_{\mathtt{conc}} w \quad \in A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle,$$

$$\Delta_{\sqcup\!\sqcup} S = \sum_{w\in\mathcal{X}^*} \langle S|w\rangle \Delta_{\sqcup\!\sqcup} w \quad \in A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle.$$

<u>$A\langle\langle\mathcal{X}\rangle\rangle \otimes A\langle\langle\mathcal{X}\rangle\rangle$ does not embed injectively in</u>[6] $A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle \cong [A\langle\langle\mathcal{X}\rangle\rangle]\langle\langle\mathcal{X}\rangle\rangle$.

6. $A\langle\langle\mathcal{X}\rangle\rangle \otimes A\langle\langle\mathcal{X}\rangle\rangle$ contains the elements of the form $\sum_{i\in I \text{ finite}} G_i \otimes D_i$ (with $(G_i, D_i) \in A\langle\langle\mathcal{X}\rangle\rangle \times A\langle\langle\mathcal{X}\rangle\rangle$) which can be interpreted as double series. But, a priori, the images of different dual laws cannot be, in general reduced to such sums. Furthermore, the arrow `tensor products of series`$\rightarrow$`double series` may not be into, when A is only a ring.

Extended Ree's theorem

Let $S \in A\langle\langle Y \rangle\rangle$ (resp. $A\langle\langle \mathcal{X} \rangle\rangle$), A is a commutative ring containing $\mathbb{Q}$. The series S is said to be

1. a $\uplus$ (resp. $\mathtt{conc}$, $\shuffle$)-character iff, for any $w, v \in Y^*$ (resp. $\mathcal{X}^*$), $\langle S|w\rangle\langle S|v\rangle = \langle S|w \uplus v\rangle$ (resp. $\langle S|wv\rangle, \langle S|w \shuffle v\rangle$) and $\langle S|1\rangle = 1$.
2. an infinitesimal $\uplus$ (resp. $\mathtt{conc}$, $\shuffle$)-character iff, for any $w, v \in Y^*$ (resp. $\mathcal{X}^*$), $\langle S|w \uplus v\rangle = \langle S|w\rangle\langle v|1_{Y^*}\rangle + \langle w|1_{Y^*}\rangle\langle S|v\rangle$ (resp. $\langle S|wv\rangle = \langle S|w\rangle\langle v|1_{\mathcal{X}^*}\rangle + \langle w|1_{\mathcal{X}^*}\rangle\langle S|v\rangle$, $\langle S|w \shuffle v\rangle = \langle S|w\rangle\langle v|1_{\mathcal{X}^*}\rangle + \langle w|1_{\mathcal{X}^*}\rangle\langle S|v\rangle$).
3. a group-like series iff $\langle S|1_{\mathcal{X}^*}\rangle = 1$ and $\Delta_{\uplus} S = \Phi(S \otimes S)$ (resp. $\Delta_{\mathtt{conc}} S = \Phi(S \otimes S), \Delta_{\uplus} S = \Phi(S \otimes S)$).
4. a primitive series iff $\Delta_{\uplus} S = 1_{Y^*} \otimes S + S \otimes 1_{Y^*}$ (resp. $\Delta_{\mathtt{conc}} S = 1_{\mathcal{X}^*} \otimes S + S \otimes 1_{\mathcal{X}^*}, \Delta_{\shuffle} S = 1_{\mathcal{X}^*} \otimes S + S \otimes 1_{\mathcal{X}^*}$).

Then the following assertions are equivalent

1. S is a $\uplus$ (resp. $\mathtt{conc}$ and $\shuffle$)-character.
2. $\log S$ an infinitesimal $\uplus$ (resp. $\mathtt{conc}$ and $\shuffle$)-character.
3. S is group-like, for $\Delta_{\uplus}$ (resp. $\Delta_{\mathtt{conc}}$ and $\Delta_{\shuffle}$).
4. $\log S$ is primitive, for $\Delta_{\uplus}$ (resp. $\Delta_{\mathtt{conc}}$ and $\Delta_{\shuffle}$).

Extension by continuity (infinite sums)

Now, suppose that the ring A (containing $\mathbb{Q}$) is a field $\mathbf{k}$. Then

$$\Delta_{\sqcup\!\sqcup} : \mathbf{k}\langle\mathcal{X}\rangle \to \mathbf{k}\langle\mathcal{X}\rangle \otimes \mathbf{k}\langle\mathcal{X}\rangle \text{ and } \Delta_{\uplus\!\!\sqcup} : \mathbf{k}\langle Y\rangle \to \mathbf{k}\langle Y\rangle \otimes \mathbf{k}\langle Y\rangle$$

are graded for the multidegree. Then $\Delta_{\uplus\!\!\sqcup}$ is graded for the length. Their extension to the completions (*i.e.* $\mathbf{k}\langle\!\langle\mathcal{X}\rangle\!\rangle$ and $\mathbf{k}\langle\!\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\!\rangle$) are continuous and then, when exist, commute with infinite sums. Hence[7, 8],

$$\forall c \in \mathbf{k}, \quad \Delta_{\sqcup\!\sqcup}(cx)^* = \sum_{n\geq 0} c^n \Delta_{\sqcup\!\sqcup} x^n = \sum_{n\geq 0} c^n \sum_{j=0}^{n} \binom{n}{j} x^j \otimes x^{n-j}.$$

For $c \in \mathbb{N}_{\geq 2}$ which is neither a field nor a ring (containing $\mathbb{Q}$), we also get

$$(cx)^* = (c-1)^{-1} \sum_{a,b\in\mathbb{N}_{\geq 1}, a+b=c} (ax)^* \sqcup\!\sqcup (bx)^* \quad \in \mathbb{N}_{\geq 2}\langle\!\langle\mathcal{X}\rangle\!\rangle,$$

$$\Delta_{\sqcup\!\sqcup}(cx)^* \neq (c-1)^{-1} \sum_{a,b\in\mathbb{N}_{\geq 1}, a+b=c} (ax)^* \otimes (bx)^* \quad \in \mathbb{Q}\langle\!\langle\mathcal{X}\rangle\!\rangle \otimes \mathbb{Q}\langle\!\langle\mathcal{X}\rangle\!\rangle,$$

because

$$\langle \mathrm{LHS} | x \otimes 1_{\mathcal{X}^*}\rangle = c \quad \text{and} \quad \langle \mathrm{RHS} | x \otimes 1_{\mathcal{X}^*}\rangle = (c-1)^{-1} \sum_{a=1}^{c-1} a = \frac{c}{2}.$$

For $c \in \mathbb{Z}$ (or even $\mathbb{Q}, \mathbb{R}, \mathbb{C}$), the such decomposition is not finite.

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7. For $S \in A\langle\!\langle\mathcal{X}\rangle\!\rangle$ s.t. $\langle S|1_{\mathcal{X}^*}\rangle = 0$, $S^* = \sum_{n\geq 0} S^n$ is called Kleene star of S.
8. $\Delta_{\sqcup\!\sqcup} x^n = (\Delta_{\sqcup\!\sqcup} x)^n = (1_{\mathcal{X}^*} \otimes x + x \otimes 1_{\mathcal{X}^*})^n = \sum_{j=0}^{n} \binom{n}{j} x^j \otimes x^{n-j}$.

Case of rational series and of $\Delta_{\texttt{conc}}$

$A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ denotes the algebraic closure by[9] $\{\texttt{conc}, +, *\}$ of $\widehat{A.\mathcal{X}}$ in $A\langle\langle\mathcal{X}\rangle\rangle$.

$$\begin{array}{ccc}
A\langle\langle\mathcal{X}\rangle\rangle & \xrightarrow{\ {}^t\texttt{conc}\ } & A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle \\
{\scriptstyle can}\uparrow & & \uparrow{\scriptstyle \Phi|_{A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \otimes_A A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle}} \\
A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle & \dashrightarrow & A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \otimes_A A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle
\end{array}$$

The dashed arrow may not exist in general, but for any $R \in A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ admitting (λ, μ, η) as linear representation of dimension n, we can get

$$ {}^t\texttt{conc}(R) = \Phi(\textstyle\sum_{i=1}^n G_i \otimes D_i).$$

Indeed, since $\langle R|xy\rangle = \lambda\mu(xy)\eta = \lambda\mu(x)\mu(y)\eta$ $(x, y \in \mathcal{X})$ then, letting e_i is the vector such that ${}^t e_i = \begin{pmatrix}0 & \ldots & 0 & 1 & 0 & \ldots & 0\end{pmatrix}$, one has

$$\langle R|xy\rangle = \sum_{i=1}^n \lambda\mu(x)e_i\,{}^t e_i\mu(y)\eta = \sum_{i=1}^n \langle G_i|x\rangle\langle D_i|y\rangle = \sum_{i=1}^n \langle G_i \otimes D_i|x \otimes y\rangle.$$

G_i (resp. D_i) admits then (λ, μ, e_i) (resp. $({}^t e_i, \mu, \eta)$) as linear representation. If $A = \mathbf{k}$ being a field then, due to the injectivity of Φ, all expressions of the type $\sum_{i=1}^n G_i \otimes D_i$, of course, coincide. Hence, the dashed arrow (a restriction of $\Delta_{\texttt{conc}}$) in the above diagram is well-defined.

9. $A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ is closed under $⧢$. $A^{\mathrm{rat}}\langle\langle Y\rangle\rangle$ is also closed under $\stackrel{-}{⊎}$.

Representative series and Sweedler's dual

Theorem 6 (representative series)

Let $S \in A\langle\langle\mathcal{X}\rangle\rangle$. The following assertions are equivalent

1. *The series S belongs to $A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$.*
2. *There exists a linear representation (ν, μ, η), of rank n, for S with $\nu \in M_{1,n}(A), \eta \in M_{n,1}(A)$ and a morphism of monoids $\mu : \mathcal{X}^* \to M_{n,n}(A)$ s.t., for any $w \in \mathcal{X}^*$, $\langle S|w\rangle = \nu\mu(w)\eta$.*
3. *The shifts[10] $\{S \triangleleft w\}_{w\in\mathcal{X}^*}$ (resp. $\{w \triangleright S\}_{w\in\mathcal{X}^*}$) lie within a finitely generated shift-invariant A-module.*

Moreover, if A is a field $\mathbf{k}$, the previous assertions are equivalent to

4. *There exist $(G_i, D_i)_{i\in F\,finite}$ s.t. $\Delta_{\mathtt{conc}}(S) = \sum_{i\in F\,finite} G_i \otimes D_i$.*

Hence, $\mathcal{H}^{\circ}_{\sqcup\!\sqcup}(\mathcal{X}) = (\mathbf{k}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*}, \Delta_{\mathtt{conc}}, \mathrm{e})$ *and* $\mathcal{H}^{\circ}_{\uplus}(Y) = (\mathbf{k}^{\mathrm{rat}}\langle\langle Y\rangle\rangle, \uplus, 1_{\mathcal{X}^*}, \Delta_{\mathtt{conc}}, \mathrm{e})$.

Now, let $A_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ (resp. $A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$) be the set of exchangeable[11] series (resp. series admitting a linear representation with commuting matrices).

10. The *left* (resp. *right*) shift of S by P is $P \triangleright S$ (resp. $S \triangleleft P$) defined by, for $w \in \mathcal{X}^*$, $\langle P \triangleright S|w\rangle = \langle S|wP\rangle$ (resp. $\langle S \triangleleft P|w\rangle = \langle S|Pw\rangle$).
11. *i.e.* if $S \in A_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ then $(\forall u, v \in \mathcal{X}^*)((\forall x \in \mathcal{X})(|u|_x = |v|_x) \Rightarrow \langle S|u\rangle = \langle S|v\rangle)$.

Kleene stars of the plane and conc-characters

For any $S \in A\langle\langle\mathcal{X}\rangle\rangle$, let ∇S denotes $S - 1_{\mathcal{X}^*}$.

Theorem 7 (rational exchangeable series)

1. $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \subset A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \cap A_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$. *If A is a field then the equality holds and* $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle X\rangle\rangle = A^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup A^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$ *and, for the algebra of series over subalphabets* $A_{\mathrm{fin}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle := \cup_{F\subset_{finite} Y} A^{\mathrm{rat}}\langle\langle F\rangle\rangle$, *we get*[12] $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle \cap A_{\mathrm{fin}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle = \cup_{k\ge 0} A^{\mathrm{rat}}\langle\langle y_1\rangle\rangle \sqcup\!\sqcup \ldots \sqcup\!\sqcup A^{\mathrm{rat}}\langle\langle y_k\rangle\rangle \subsetneq A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle$.
2. $\forall x \in \mathcal{X}, A^{\mathrm{rat}}\langle\langle x\rangle\rangle = \{P(1-xQ)^{-1}\}_{P,Q\in A[x]}$. *If $\mathbf{k}$ is an algebraically closed field then* $\mathbf{k}^{\mathrm{rat}}\langle\langle x\rangle\rangle = \mathrm{span}_{\mathbf{k}}\{(ax)^* \sqcup\!\sqcup \mathbf{k}\langle x\rangle | a \in K\}$.
3. *If A is a $\mathbb{Q}$-algebra,* $\{x^*\}_{x\in\mathcal{X}}$ *(resp.* $\{y^*\}_{y\in Y}$*) are* conc-character *and alg. free over* $(A\langle\mathcal{X}\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*})$ *(resp.* $(A\langle Y\rangle, \uplus, 1_{Y^*})$*) within* $(A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*})$ *(resp.* $(A^{\mathrm{rat}}\langle\langle Y\rangle\rangle, \sqcup\!\sqcup, 1_{Y^*})$*).*
4. *Let* $S \in A\langle\langle\mathcal{X}\rangle\rangle$. *If* $A = \mathbf{k}$, *a field, then t.f.a.e.*
   a) S *is groupe-like, for* $\Delta_{\mathtt{conc}}$.
   b) *There exists* $M := \sum_{x\in\mathcal{X}} c_x x \in \widehat{\mathbf{k}.\mathcal{X}}$ *s.t.* $S = M^*$.
   c) *There exists* $M := \sum_{x\in\mathcal{X}} c_x x \in \widehat{\mathbf{k}.\mathcal{X}}$ *s.t.* $\nabla S = MS = SM$.

12. The following identity lives in $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle$ but not in $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle \cap A_{\mathrm{fin}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle$,
$(y_1 + \ldots)^* = \lim_{k\to+\infty}(y_1 + \ldots + y_k)^* = \lim_{k\to+\infty} y_1^* \sqcup\!\sqcup \ldots \sqcup\!\sqcup y_k^* = \sqcup\!\sqcup_{k\ge1} y_k$.

Linear representations and automata

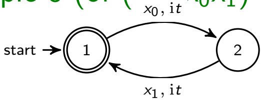
For $i = 1, 2$, let $R_i \in \mathbb{C}^{\mathrm{rat}}\langle\langle \mathcal{X} \rangle\rangle$ and (ν_i, μ_i, η_i) be, respectively, representations of dimension n_i. Then the linear representation of

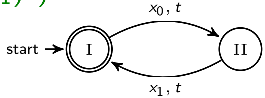
$$R_1 + R_2 \quad \text{is} \quad \left(\begin{pmatrix} \nu_1 & \nu_2 \end{pmatrix}, \left\{ \begin{pmatrix} \mu_1(x) & \mathbf{0} \\ \mathbf{0} & \mu_2(x) \end{pmatrix} \right\}_{x \in \mathcal{X}}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right),$$

$$R_1 \shuffle R_2 \quad \text{is} \quad (\nu_1 \otimes \nu_2, \{\mu_1(x) \otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1} \otimes \mu_2(x)\}_{x \in \mathcal{X}}, \eta_1 \otimes \eta_2),$$

$$R_1 \stuffle R_2 \quad \text{is} \quad (\nu_1 \otimes \nu_2, \{\mu_1(y_k) \otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1} \otimes \mu_2(y_k) + \textstyle\sum_{i+j=k} \mu_1(y_i) \otimes \mu_2(y_j)\}_{k \geq 1}, \eta_1 \otimes \eta_2).$$

Example 8 (of $(-t^2 x_0 x_1)^*$ and $(t^2 x_0 x_1)^*$)





$(-t^2 x_0 x_1)^*$ $\qquad\qquad$ $(t^2 x_0 x_1)^*$

$$\nu_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \eta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mu_1(x_0) = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \quad \mu_1(x_1) = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix},$$

$$\nu_2 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mu_2(x_0) = \begin{pmatrix} 0 & \mathrm{i}t \\ 0 & 0 \end{pmatrix}, \quad \mu_2(x_1) = \begin{pmatrix} 0 & 0 \\ \mathrm{i}t & 0 \end{pmatrix}$$

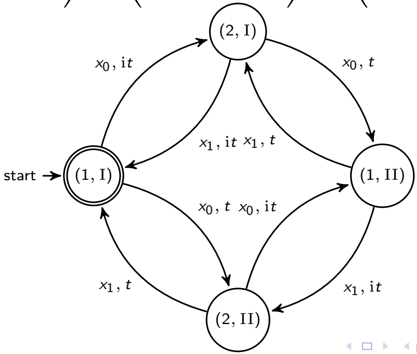
$$(\nu, \{\mu(x_0), \mu(x_1)\}, \eta) = (\nu_1 \otimes \nu_2, \{\mu_1(x_0) \otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1} \otimes \mu_2(x_0), \mu_1(x_1) \otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1} \otimes \mu_2(x_1), \eta_1 \otimes \eta_2).$$

Example of $(-t^2x_0x_1)^* \sqcup\!\sqcup (t^2x_0x_1)^* = (-4t^4x_0^2x_1^2)^*$

$$\nu = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}, \quad \eta = {}^T\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix},$$

$$\mu(x_0) = \begin{pmatrix}0 & 0 & t & 0\\ 0 & 0 & 0 & t\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & \mathrm{i}t & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \mathrm{i}t\\ 0 & 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & \mathrm{i}t & t & 0\\ 0 & 0 & 0 & t\\ 0 & 0 & 0 & \mathrm{i}t\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$\mu(x_1) = \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ t & 0 & 0 & 0\\ 0 & t & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0 & 0\\ \mathrm{i}t & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \mathrm{i}t & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 0\\ \mathrm{i}t & 0 & 0 & 0\\ t & 0 & 0 & 0\\ 0 & t & \mathrm{i}t & 0\end{pmatrix}.$$



Triangular sub bialgebras of $(A^{\mathrm{rat}}\langle\langle X\rangle\rangle, ш, 1_{X^*}, \Delta_{\texttt{conc}}, \mathrm{e})$

Let (ν,μ,η) be a linear representation of $R \in A^{\mathrm{rat}}\langle\langle X\rangle\rangle$ and $\mathcal{L}$ be the Lie algebra generated by $\{\mu(x)\}_{x\in X}$.
Let $M(x) := \mu(x)x$, for $x \in X$. Then $R = \nu M(X^*)\eta$. If $\{\mu(x)\}_{x\in X}$ are triangular then let $D(X)$ (resp. $N(X)$) be the diagonal (resp. nilpotent) letter matrix s.t. $M(X) = D(X) + N(X)$ then
$M(X^*) = ((D(X^*)T(X))^*D(X^*))$. Moreover, if $X = \{x_0, x_1\}$ then
$M(X^*) = (M(x_1^*)M(x_0))^*M(x_1^*) = (M(x_0^*)M(x_1))^*M(x_0^*)$.

If A is an algabraically closed field, the modules generated by the following families are closed by $\texttt{conc}$, ш and coproducts :

$$\begin{array}{llll}(F_0) & E_1x_1\ldots E_jx_1E_{j+1}, & \text{where} & E_k \in A^{\mathrm{rat}}\langle\langle x_0\rangle\rangle,\\ (F_1) & E_1x_0\ldots E_jx_0E_{j+1}, & \text{where} & E_k \in A^{\mathrm{rat}}\langle\langle x_1\rangle\rangle,\\ (F_2) & E_1x_{i_1}\ldots E_jx_{i_j}E_{j+1}, & \text{where} & E_k \in A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle, x_{i_k} \in X.\end{array}$$

It follows then that

1. R is a linear combination of expressions in the form (F_0) (resp. (F_1)) iff $M(x_1^*)M(x_0)$ (resp. $M(x_0^*)M(x_1)$) is nilpotent,
2. R is a linear combination of expressions in the form (F_2) iff $\mathcal{L}$ is solvable. Thus, if $R \in A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle$ ш $A\langle X\rangle$ then $\mathcal{L}$ is nilpotent.

CONTINUITY OVER CHEN SERIES

Iterated integrals over $\omega_i(z) = u_{x_i}(z)dz$ and along $z_0 \rightsquigarrow z$

Now, let Ω be a simply connected domain admitting 1_Ω as neutral element. Let $\mathcal{A} := (\mathcal{H}(\Omega), \partial)$ and let $\mathcal{C}_0$ be a differential subring of $\mathcal{A}$ ($\partial\mathcal{C}_0 \subset \mathcal{C}_0$) which is an integral domain containing $\mathbb{C}$.

$\mathbb{C}\{\{(g_i)_{i\in I}\}\}$ denotes the differential subalgebra of $\mathcal{A}$ generated by $(g_i)_{i\in I}$, *i.e.* the $\mathbb{C}$-algebra generated by g_i's and their derivatives

$\{u_x\}_{x\in\mathcal{X}}$: elements[13] in $\mathcal{C}_0 \cap \mathcal{A}^{-1}$, correspondent to $\{\theta_x\}_{x\in\mathcal{X}}$ ($\theta_x = u_x^{-1}\partial$).

The iterated integral[14] associated to $x_{i_1} \ldots x_{i_k} \in \mathcal{X}^*$, over the differential forms $\omega_i(z) = u_{x_i}(z)dz, i \geq 1$, and along a path $z_0 \rightsquigarrow z$ on Ω, is defined by

$$\begin{aligned}
\alpha_{z_0}^z(1_{\mathcal{X}^*}) &= 1_\Omega, \\
\alpha_{z_0}^z(x_{i_1} \ldots x_{i_k}) &= \int_{z_0}^z \omega_{i_1}(z_1) \ldots \int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k). \\
\partial\alpha_{z_0}^z(x_{i_1} \ldots x_{i_k}) &= u_{x_{i_1}}(z) \int_{z_0}^z \omega_{i_2}(z_2) \ldots \int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k).
\end{aligned}$$

$$\begin{aligned}
\mathrm{span}_{\mathbb{C}}\{\partial^l\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*, l\geq 0} &\subset \mathrm{span}_{\mathbb{C}\{\{(u_x)_{x\in\mathcal{X}}\}\}}\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*} \\
&\subset \mathrm{span}_{\mathbb{C}\{\{(u_x^{\pm 1})_{x\in\mathcal{X}}\}\}}\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*} \\
&\cong \mathbb{C}\{\{(u_x^{\pm 1})_{x\in\mathcal{X}}\}\} \otimes_{\mathbb{C}} \mathrm{span}_{\mathbb{C}}\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*}\ ?
\end{aligned}$$

13. In control theory, these are called "inputs" and they may vary (see bellow).
14. The value of $\alpha_{z_0}^z(x_{i_1} \ldots x_{i_k})$ depends on $\{\omega_i\}_{i\geq 1}$, or equivalently on $\{u_x\}_{x\in\mathcal{X}}$.

Iterated integrals and integro differential operators

Let $\mathcal{C} = \mathbb{C}\{\{(u_x^{\pm 1})_{x\in\mathcal{X}}\}\}$. One has $\theta_x \in \mathcal{C}\langle\partial\rangle$, for $x \in \mathcal{X}$, and
$$\forall x, y \in \mathcal{X}, \quad \forall w \in \mathcal{X}^*, \quad \theta_x \alpha_{z_0}^z(yw) = u_x^{-1}(z)u_y(z)\alpha_{z_0}^z(w).$$
Now, let Θ be the morphism $\mathbb{C}\langle\mathcal{X}\rangle \longrightarrow \mathcal{C}\langle\partial\rangle$ defined as follows
$$\Theta(w) = \begin{cases} \mathrm{Id} & \text{if} \quad w = 1_{\mathcal{X}^*}, \\ \Theta(u)\theta_x & \text{if} \quad w = ux \in \mathcal{X}^*\mathcal{X}. \end{cases}$$
One has, for any $w \in \mathcal{X}^*$,

1. $\Theta(\tilde{w})\alpha_{z_0}^z(w) = 1_\Omega$, and then $\partial(\Theta(\tilde{w})\alpha_{z_0}^z(w)) = 0$.
2. $L_w \alpha_{z_0}^z(\tilde{w}) = 0$, where $L_w := \partial\Theta(w) \in \mathcal{C}\langle\partial\rangle$.

For any $x_i \in \mathcal{X}$, let us consider a section of θ_{x_i} : $\theta_{x_i}\iota_{x_i}^{z_0} = \mathrm{Id}$, *i.e.*
$$\forall f \in \mathcal{H}(\Omega), \quad \iota_{x_i}^{z_0} f(z) = \int_{z_0}^{z} \omega_i(s) f(s).$$
The operator $\theta_y \iota_x^{z_0}$, for $x \neq y$, admits $u_y u_x^{-1}$ as eigenvalue, *i.e.*
$$\forall f \in \mathcal{H}(\Omega), \quad (\theta_y \iota_x^{z_0})f = u_y u_x^{-1} f, \quad \text{in particular,} \quad (\theta_y \iota_x^{z_0})1_\Omega = u_y u_x^{-1}.$$
Now, let $\Im^{z_0}$ be the morphism defined as follows
$$\Im^{z_0}(w) = \begin{cases} \mathrm{Id} & \text{if} \quad w = 1_{\mathcal{X}^*}, \\ \Im^{z_0}(u)\iota_x^{z_0} & \text{if} \quad w = ux \in \mathcal{X}^*\mathcal{X}. \end{cases}$$
Hence, for any $w \in X^*, \Im^{z_0}(w)1_\Omega = \alpha_{z_0}^z(w)$.

Practical example (polylogarithms)

For $X = \{x_0, x_1\}$ and $\Omega = \mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$, let us consider

$$u_{x_0}(z) = z^{-1} \quad \text{and} \quad u_{x_1}(z) = (1-z)^{-1}.$$

Then, on the other hand,

$$\omega_0(z) = u_{x_0}(z)dz = z^{-1}dz \quad \text{and} \quad \omega_1(z) = u_{x_1}(z)dz = (1-z)^{-1}dz,$$

$$\theta_{x_0} = u_{x_0}^{-1}(z)\partial = z\partial \qquad \text{and} \quad \theta_{x_1} = u_{x_1}^{-1}(z)\partial = (1-z)\partial.$$

On the other hand [15], $\mathcal{C} = \mathbb{C}\{\{(u_x^{\pm 1})_{x \in X}\}\} = \mathbb{C}[z, z^{-1}, (1-z)^{-1}]$ being closed by $\theta_{x_0}, \theta_{x_1}$ and then by $\partial = \theta_{x_0} + \theta_{x_1} = \Theta(x_0 + x_1)$. One also has

1. $\Theta([x_1, x_0]) = [\theta_{x_1}, \theta_{x_0}] = \partial$.
2. $\forall w \in X^* x_1, \Im^0(w)1_\Omega = \alpha_0^z(w) = \mathrm{Li}_w(z)$.
3. $(\theta_{x_0}\iota_{x_1}^{z_0})1_\Omega = z(1-z)^{-1}$ and $(\theta_{x_1}\iota_{x_0}^{z_0})1_\Omega = z^{-1} - 1$.
4. $[\theta_{x_0}\iota_{x_1}^{z_0}, \theta_{x_1}\iota_{x_0}^{z_0}] = 0$.
5. $(\theta_{x_0}\iota_{x_1}^{z_0})(\theta_{x_1}\iota_{x_0}^{z_0}) = (\theta_{x_1}\iota_{x_0}^{z_0})(\theta_{x_0}\iota_{x_1}^{z_0}) = \mathrm{Id}$.

For any $L \in \mathcal{C}\langle\partial\rangle$, there is $P \in \mathcal{C}\langle X\rangle$ s.t $L = \Theta(P)$, meaning that Θ is surjective and non injective. Moreover, $\ker\Theta$ is the left principal ideal generated by $[x_1, x_0] - x_0 - x_1$.

15. Any $p \in \mathcal{C}$ is polynomial on z, z^{-1} and $(1-z)^{-1}$ and admits 0 and 1 as poles.

Structure of iterated integrals

Proposition 1

The following assertions are equivalent

1. *The morphism* $(\mathcal{C}_0\langle\mathcal{X}\rangle, ⧢, 1_{\mathcal{X}^*}) \to (\mathrm{span}_{\mathcal{C}_0}\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*}, \times, 1_\Omega)$ *is injective.*
2. $\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*}$ *is* $\mathcal{C}_0$*-linearly independent.*
3. $\{\alpha_{z_0}^z(l)\}_{l\in\mathcal{L}yn\mathcal{X}}$ *is* $\mathcal{C}_0$*-algebraically independent.*
4. $\{\alpha_{z_0}^z(x)\}_{x\in\mathcal{X}}$ *is* $\mathcal{C}_0$*-algebraically independent.*
5. $\{\alpha_{z_0}^z(x)\}_{x\in\mathcal{X}\cup\{1_{\mathcal{X}^*}\}}$ *is* $\mathcal{C}_0$*-linearly independent.*

If one of the above assertions holds then

1. $\mathcal{C}_0[\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*}]$ forms the universal $\mathcal{C}_0$-module of solutions of all differential equations $Ly = 0$,
2. $\mathcal{C}_0\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*}$ forms the universal Picard-Vessiot extension related to all differential equations $Ly = 0$,

where[16] L's are linear differential operators belonging to $\mathcal{C}_0\langle\partial\rangle$.

16. Let $\mathcal{I}_w := \{L \in \mathcal{C}_0\langle\partial\rangle \text{ s.t. } L\alpha_{z_0}^z(w) = 0\}$, for $w \in X^*$. Then $\mathcal{I}_w$ is a left ideal.

Examples of linear differential equation

Example 9 (with $\mathcal{C} = \mathbb{C}(z)$)

$$(\partial - z)y = 0. \qquad (1)$$

1. $e^{z^2/2}$ is solution of (1).
2. $ce^{z^2/2} = e^{z^2/2}e^{\log c}$ is an other solution ($c \in \mathbb{R} \setminus \{0\}$).
3. $\{e^{z^2/2}\}$ is a fundamental set of solutions of (1).
4. $\mathcal{C}\{e^{z^2/2}\}$ is a Picard-Vessiot extension related to (1).

For $\theta_{x_0} = z\partial$ and $\theta_{x_1} = (1-z)\partial$, since $L_{x_1x_0} = \partial\theta_{x_1}\theta_{x_0} \in \mathcal{C}\langle\partial\rangle$ then let

$$L_{x_1x_0}y = (z(1-z)\partial^3 + (2-3z)\partial^2 - \partial)y = 0. \qquad (2)$$

1. $L_{x_1x_0}\,\mathrm{Li}_2 = 0$ meaning that Li_2 is solution of (2).
2. $c\,\mathrm{Li}_2 = \mathrm{Li}_2\,e^{\log c}$ is an other solution ($c \in \mathbb{R} \setminus \{0\}$) but it is not independent to Li_2.
3. $\{\mathrm{Li}_2, \log, 1_\Omega\}$ is a fundamental set of solutions of (2).
4. $\mathcal{C}\{\mathrm{Li}_2, \log, 1_\Omega\}$ is a Picard-Vessiot extension [17] related to (2).

17. $\mathcal{C}\{\mathrm{Li}_2(z)\} = \mathcal{C} \otimes \mathbb{C}[\mathrm{Li}_2(z), \log(1-z), \log(z)]$.

Chen series of $\{\omega_i\}_{i\geq 1}$ and along $z_0 \rightsquigarrow z$

We get on the bialgebras $\mathcal{H}_{ш}(\mathcal{X})$ and $\mathcal{H}_{ш\!\!\!+}(Y)$ (over a commutative ring A containing $\mathbb{Q}$)

$$\mathcal{D}_{\mathcal{X}} := \sum_{w\in\mathcal{X}^*} w\otimes w = \prod_{l\in\mathcal{L}yn\mathcal{X}}^{\searrow} e^{S_l\otimes P_l} \text{ and } \mathcal{D}_Y := \sum_{w\in Y^*} w\otimes w = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\Sigma_l\otimes\Pi_l}.$$

Hence, since $\alpha_{z_0}^z(u ш v) = \alpha_{z_0}^z(u)\alpha_{z_0}^z(v)$, for $u, v \in \mathcal{X}^*$, then the Chen series, $C_{z_0\rightsquigarrow z} \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle$, is given by

$$C_{z_0\rightsquigarrow z} := \sum_{w\in\mathcal{X}^*} \alpha_{z_0}^z(w)w = (\alpha_{z_0}^z \otimes \mathtt{Id})\mathcal{D}_{\mathcal{X}} = \prod_{l\in\mathcal{L}yn\mathcal{X}}^{\searrow} e^{\alpha_{z_0}^z(S_l)P_l}$$

and then [18] $\Delta_{ш} C_{z_0\rightsquigarrow z} = C_{z_0\rightsquigarrow z} \otimes C_{z_0\rightsquigarrow z}$ and $\langle C_{z_0\rightsquigarrow z}|1_{\mathcal{X}^*}\rangle = 1$.

Note that $C_{z_0\rightsquigarrow z}$ only depends on the homotopy class of $z_0 \rightsquigarrow z$ and the endpoints z_0, z. One has $C_{z_0\rightsquigarrow z}C_{z_1\rightsquigarrow z_0} = C_{z_1\rightsquigarrow z}$. Or equivalently,

$$\forall w \in \mathcal{X}^*, \quad \langle C_{z_1\rightsquigarrow z}|w\rangle = \sum_{u,v\in\mathcal{X}^*, uv=w} \langle C_{z_0\rightsquigarrow z}|u\rangle\langle C_{z_1\rightsquigarrow z_0}|v\rangle.$$

Although $\Delta_{\mathrm{conc}} w = \sum_{u,v\in\mathcal{X}^*, uv=w} u\otimes v$ but $\Delta_{\mathrm{conc}} C_{z_1\rightsquigarrow z} \neq C_{z_0\rightsquigarrow z} \otimes C_{z_1\rightsquigarrow z_0}$.

18. $\langle C_{z_0\rightsquigarrow z}|u ш v\rangle = \langle C_{z_0\rightsquigarrow z}|u\rangle\langle C_{z_0\rightsquigarrow z}|v\rangle$ and on the other hand, $\langle C_{z_0\rightsquigarrow z}|u ш v\rangle = \langle \Delta_{ш} C_{z_0\rightsquigarrow z}|u\otimes v\rangle, \langle C_{z_0\rightsquigarrow z}|u\rangle\langle C_{z_0\rightsquigarrow z}|v\rangle = \langle C_{z_0\rightsquigarrow z} \otimes C_{z_0\rightsquigarrow z}|u\otimes v\rangle$.

More about Chen series

Note also that, for $g \in \mathcal{H}(\Omega)$, one has $C_{g(z_0)\rightsquigarrow g(z)} = g_* C_{z_0 \rightsquigarrow z}$, *i.e.* the Chen series of $\{g^*\omega_i\}_{i\geq 1}$ along the path $g^*(z_0 \rightsquigarrow z)$.

Example 10 (with $\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1-z)^{-1}dz$)

$g(z)$	z	z^{-1}	$(z-1)z^{-1}$	$z(z-1)^{-1}$	$(1-z)^{-1}$	$1-z$
$g^*\omega_0$	ω_0	$-\omega_0$	$-\omega_1-\omega_0$	$\omega_1+\omega_0$	ω_1	$-\omega_1$
$g^*\omega_1$	ω_1	$\omega_1+\omega_0$	$-\omega_0$	$-\omega_1$	$-\omega_1-\omega_0$	$-\omega_0$

For any $n \geq 0$, one has

$$\mathbf{d}^n C_{z_0\rightsquigarrow z} = p_n C_{z_0\rightsquigarrow z},$$

where, for any $S \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle$, $\mathbf{d}S \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle$ is defined as follows

$$\mathbf{d}S = \sum_{w\in\mathcal{X}^*} (\partial\langle S|w\rangle)w,$$

$p_n \in \mathcal{C}\langle\mathcal{X}\rangle$ is defined as follows

$$p_n = \sum_{\mathrm{wgt}\,\mathbf{r}=n} \sum_{w\in\mathcal{X}^n} \prod_{i=1}^{\deg \mathbf{r}} \binom{\sum_{j=1}^{i} r_j + j - 1}{r_i} \tau_{\mathbf{r}}(w)$$

and, for $w = x_{i_1}\ldots x_{i_k} \in \mathcal{X}^*$ associated to the derivation multiindex $\mathbf{r} = (r_1,\ldots,r_k) \in \mathbb{N}^k$ of weight $\mathrm{wgt}\,\mathbf{r} = |w| + \sum_{i=1}^k r_i$ and of degree $\deg \mathbf{r} = |w|$, $\tau_{\mathbf{r}}(w) := \tau_{r_1}(x_{i_1})\ldots\tau_{r_k}(x_{i_k}) = (\partial^{r_1}u_{x_{i_1}})x_{i_1}\ldots(\partial^{r_k}u_{x_{i_k}})x_{i_k}$.

Continuity, indiscernability and growth condition

For $i = 0, 2$, let $(\mathbf{k}_i, \|.\|_i)$ be a semi-normed space and $g_i \in \mathbb{Z}$.

Definition 11

1. Let $\mathcal{C}l$ be a class of $\mathbf{k}_1\langle\langle\mathcal{X}\rangle\rangle$. Let $S \in \mathbf{k}_2\langle\langle\mathcal{X}\rangle\rangle$ and it is said to be

   a) *continuous* over $\mathcal{C}l$ if, for $\Phi \in \mathcal{C}l$, the following sum is convergent
   $$\sum_{w\in\mathcal{X}^*} \|\langle S|w\rangle\|_2 \|\langle\Phi|w\rangle\|_1.$$
   We will denote $\langle S\|\Phi\rangle$ the sum $\sum_{w\in\mathcal{X}^*}\langle S|w\rangle\langle\Phi|w\rangle$ and $\mathbf{k}_2\langle\langle\mathcal{X}\rangle\rangle^{\mathrm{cont}}$ the set of continuous power series over $\mathcal{C}l$.

   b) *indiscernable* over $\mathcal{C}l$ iff, for any $\Phi \in \mathcal{C}l$, $\langle S\|\Phi\rangle = 0$.

2. Let χ_1 and χ_2 be real positive functions over $\mathcal{X}^*$. Let $S \in \mathbf{k}_1\langle\langle\mathcal{X}\rangle\rangle$.

   a) S satisfies the χ_1-*growth condition* of order g_1 if it satisfies
   $$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in \mathcal{X}^{\geq n}, \quad \|\langle S|w\rangle\|_1 \leq K\chi_1(w)\ |w|!^{g_1}.$$
   We denote by $\mathbf{k}_1^{(\chi_1,g_1)}\langle\langle\mathcal{X}\rangle\rangle$ the set of formal power series in $\mathbf{k}_1\langle\langle\mathcal{X}\rangle\rangle$ satisfying the χ_1-growth condition of order g_1.

   b) If S is continuous over $\mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle$ then it will be said to be (χ_2, g_2)-*continuous*. The set of formal power series which are (χ_2, g_2)-continuous is denoted by $\mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle^{\mathrm{cont}}$.

Convergence condition

Proposition 2

Let χ_1 and χ_2 be real positive functions over $\mathcal{X}^*$.
Let g_1 and $g_2 \in \mathbb{Z}$ such that $g_1 + g_2 \leq 0$.

1. *Let $\mathbf{k}_1^{(\chi_1,g_1)}\langle\langle\mathcal{X}\rangle\rangle, g_1 \geq 0$, and let $P \in \mathbf{k}_1\langle\mathcal{X}\rangle$.*
   The right residual of S by P belongs to $\mathbf{k}_1^{(\chi_1,g_1)}\langle\langle\mathcal{X}\rangle\rangle$.
2. *Let $R \in \mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle, g_2 < 0$, and let $Q \in \mathbf{k}_2\langle\mathcal{X}\rangle$.*
   The concatenation QR belongs to $\mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle$.
3. *χ_1, χ_2 are morphisms over $\mathcal{X}^*$ satisfying $\sum_{x\in\mathcal{X}} \chi_1(x)\chi_2(x) < 1$.*
   If $F_1 \in \mathbf{k}_1^{(\chi_1,g_1)}\langle\langle\mathcal{X}\rangle\rangle$ (resp. $F_2 \in \mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle$) then F_1 (resp. F_2) is continuous over $\mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle$ (resp. $\mathbf{k}_1^{(\chi_1,g_1)}\langle\langle\mathcal{X}\rangle\rangle$).

Proposition 3

Let $\mathcal{C}l \subset \mathbf{k}_1\langle\langle\mathcal{X}\rangle\rangle$ be a monoid containing $\{e^{tx}\}_{x\in\mathcal{X}}^{t\in\mathbf{k}_1}$. Let $S \in \mathbf{k}_2\langle\langle\mathcal{X}\rangle\rangle^{cont}$.

1. *If S is indiscernable over $\mathcal{C}l$ then for any $x \in \mathcal{X}$, $x \triangleleft S$ and $S \triangleright x$ belong to $\mathbf{k}_2\langle\langle\mathcal{X}\rangle\rangle^{cont}$ and they are indiscernable over $\mathcal{C}l$.*
2. *S is indiscernable over $\mathcal{C}l$ iff $S = 0$.*

Chen series and differential equations

Let K be a compact on Ω. There is $c_K \in \mathbb{R}_{\geq 0}$ and a morphism M_K s.t.

$$\forall w \in \mathcal{X}^*, \quad \|\langle C_{z_0 \rightsquigarrow z} | w \rangle\|_K \leq c_K M_K(w) \, |w|!^{-1}.$$

Let $R \in \mathbb{C}^{\mathrm{rat}}\langle\langle X \rangle\rangle$ of minimal representation (λ, μ, η) of dimension n. Then

$$\forall w \in \mathcal{X}^*, \quad |\langle R | w \rangle| \leq \|\lambda\|_\infty^{1,n} \|\mu(w)\|_\infty^{n,n} \|\eta\|_\infty^{n,1}.$$

With these data, we have

Theorem 12

If $c_K \|\lambda\|_\infty^{1,n} \|\eta\|_\infty^{n,1} \sum_{x \in \mathcal{X}} M_K(x) \|\mu(x)\|_\infty^{n,n} < 1$ then $\alpha_{z_0}^z(R) = \langle R \| C_{z_0 \rightsquigarrow z} \rangle$ and

$$\forall x \in \mathcal{X}, \quad \theta_x \alpha_{z_0}^z(R) = \sum_{x' \in \mathcal{X}} u_x^{-1}(z) u_{x'}(z) \alpha_{z_0}^z(R \triangleleft x').$$

Letting $y(z_0, z) := \langle R \| C_{z_0 \rightsquigarrow z} \rangle$, the following assertions are equivalent :

1. *There is $p \in \mathcal{C}_0\langle \mathcal{X} \rangle$ s.t. $\langle R \| p C_{z_0 \rightsquigarrow z} \rangle = \langle R \triangleleft p \| C_{z_0 \rightsquigarrow z} \rangle = 0$.*
2. *There is $l = 0, .., n-1$ s.t. $\{\partial^k y\}_{0 \leq k \leq l}$ is $\mathcal{C}_0$-linearly independent and $a_l, \ldots, a_1, a_0 \in \mathcal{C}_0$ s.t. $(a_l \partial^l + \ldots + a_1 \partial + a_0) y = 0$.*

Proposition 4

Let $G \in \mathbb{C}\langle\langle X \rangle\rangle$ and $H \in \mathbb{C}_{\mathrm{exc}}\langle\langle X \rangle\rangle$ s.t. $\alpha_{z_0}^z(G) = \langle G \| C_{z_0 \rightsquigarrow z} \rangle$ and $h(\alpha_{z_0}^z(x_0), \alpha_{z_0}^z(x_1)) := \alpha_{z_0}^z(H) = \langle H \| C_{z_0 \rightsquigarrow z} \rangle$ exist ($X = \{x_0, x_1\}$). Then

$$\alpha_{z_0}^z(HG) = \langle G | 1_{X^*} \rangle \alpha_{z_0}^z(H) + \int_{z_0}^z h(\alpha_s^z(x_0), \alpha_s^z(x_1)) d\alpha_{z_0}^s(G).$$

Practical examples (eulerian functions)

For any $z \in \Omega = \mathbb{C}, |z| < 1$, in all the sequel, let us consider

$$\ell_1(z) := \gamma z - \sum_{k\geq 2} \zeta(k) \frac{(-z)^k}{k} \quad \text{and } \forall r \geq 2, \quad \ell_r(z) := - \sum_{k\geq 1} \zeta(kr) \frac{(-z^r)^k}{k}.$$

Recall that $y^n = y^{\sqcup\!\sqcup\, n}/n!$, for $y \in \mathcal{X}^*, n \in \mathbb{N}$ and $t \in \mathbb{C}, |t| < 1$. Then

$$\alpha_{z_0}^z(y^n) = \frac{[\alpha_{z_0}^z(y)]^n}{n!} \quad \text{and} \quad \alpha_{z_0}^z((ty)^*) = e^{t\alpha_{z_0}^z(y)}.$$

Example 13 (extension of eulerian functions)

For any $z \in \Omega = \mathbb{C}, |z| < 1$ and $k \geq 1$, one has

u_{y_k}	$\alpha_0^z(y_k)$	$\alpha_0^z(y_k^*)$
1_Ω	z	e^z
$(1-z)^{-1}$	$-\log(1-z)$	$(1-z)^{-1}$
$\partial\ell_k$	$\ell_k(z)$	$e^{\ell_k(z)} =: \Gamma_{y_k}^{-1}(1+z)$
$e^{\ell_k}\partial\ell_k$	$e^{\ell_k(z)} =: \Gamma_{y_k}^{-1}(1+z)$	$e^{e^{\ell_k(z)}-1}$

The function ℓ_1 is already considered by Legendre for studying the eulerian Gamma function, Γ, noted here by Γ_{y_1} (Legendre cited Euler). What are $\{\alpha_0^z(w)\}_{w\in Y^*Y}$? Similarly, in the case of $\{\alpha_0^z(w)\}_{w\in(Y\cup\{y_0\})^*}$ and with the new input $u_{y_0}(z) = z^{-1}dz$?

First properties of extended eulerian functions

Let G_r (resp. $\mathcal{G}_r$) denote the set (resp. group) of solutions, $\{\xi_0, \ldots, \xi_{r-1}\}$, of $z^r = (-1)^{r-1}$ (resp. $z^r = 1$), for $r \geq 1$. If r is odd, it is a group as $G_r = \mathcal{G}_r$ otherwise it is an orbit as $G_r = \xi\mathcal{G}_r$, where ξ is any solution of $\xi^r = -1$ (or equivalently, $\xi \in \mathcal{G}_{2r}$ and $\xi \notin \mathcal{G}_r$).

Proposition 5 (Weierstrass factorization)

1. *For $r \geq 1, \chi \in \mathcal{G}_r$ and $z \in \mathbb{C}, |z| < 1$, the functions ℓ_r and e^{ℓ_r} have the symmetry, $\ell_r(z) = \ell_r(\chi z)$ and $e^{\ell_r(z)} = e^{\ell_r(\chi z)}$. In particular, for r even, as $-1 \in \mathcal{G}_r$, these functions are even.*
2. *For $|z| < 1$, we have*
$$\ell_r(z) = \sum_{\chi \in G_r} \log \frac{1}{\Gamma(1+\chi z)} \text{ and } e^{\ell_r(z)} = \prod_{\chi \in G_r} e^{\gamma \chi z} \prod_{n \geq 1} (1 + \frac{\chi z}{n}) e^{-\frac{\chi z}{n}}.$$
3. *For any odd $r \geq 2$, $\Gamma_{y_r}^{-1}(1+z) = e^{\ell_r(z)} = \Gamma^{-1}(1+z) \prod_{\chi \in G_r \setminus \{1\}} e^{\ell_1(\chi z)}$.*
4. *In general, for any odd or even $r \geq 2$,*
$$e^{\ell_r(z)} = \prod_{\chi \in G_r} e^{\ell_1(\chi z)} = \prod_{n \geq 1} (1 + \frac{z^r}{n^r}).$$

Other practical examples (1/2)

Example 14 ($\omega_1(z) = (1-z)^{-1}dz$ and $\omega_0(z) = z^{-1}dz$)

1. For any $a, z \in \mathbb{C}$ s.t. $|a| < 1, |z| < 1$, one has
$$\begin{aligned}\mathrm{Li}_{(ax_0)^*x_1}(z) &= \alpha_0^z((ax_0)^*x_1)\\ &= \int_0^z e^{a\log(\frac{z}{s})}\omega_1(s) = z^a\int_0^z \sum_{n\geq 0} s^{n-a}ds = \sum_{n\geq 1}\frac{z^n}{n-a}.\end{aligned}$$

2. For any $n \in \mathbb{N}$ and $a, b \in \mathbb{C}$ s.t. $|a| < 1, |b| < 1$, one has
$$\mathrm{Li}_{x_0^n}(z) = \alpha_1^z(x_0^n) = \log^n(z)/n!, \quad \mathrm{Li}_{x_1^n}(z) = \alpha_0^z(x_1^n) = \log^n((1-z)^{-1})/n!,$$
$$\mathrm{Li}_{(ax_0)^*}(z) = \alpha_1^z((ax_0)^*) = z^a, \quad \mathrm{Li}_{(bx_1)^*}(z) = \alpha_0^z((bx_1)^*) = (1-z)^{-b}.$$
Let $\mathcal{C} = \mathbb{C}[z^a, (1-z)^b]_{a,b\in\mathbb{C}}$ and $S \in \mathbb{C}_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle X\rangle\rangle \sqcup\!\sqcup\, \mathbb{C}\langle X\rangle$ (resp. $\mathbb{C}_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle X\rangle\rangle = \mathbb{C}_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup\, \mathbb{C}_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$), we get
$$\mathrm{Li}_S(z) \in \mathcal{C}[\{\mathrm{Li}_l\}_{l\in\mathcal{L}ynX}] \text{ (resp. } \mathcal{C}[\log(z), \log(1-z)]).$$

3. For any $z, a, b \in \mathbb{C}$ s.t. $|z| < 1$ and $\Re a > 0, \Re b > 0$, we get the partial Beta function and the eulerian Beta function, $\mathrm{B}(a,b) = \mathrm{B}(1;a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, as follows [19]
$$\mathrm{B}(z;a,b) := \int_0^z dt\, t^{a-1}(1-t)^{b-1} = \left\{\begin{array}{c}\mathrm{Li}_{x_0[(ax_0)^* \sqcup\!\sqcup\, ((1-b)x_1)^*]}(z)\\ \mathrm{Li}_{x_1[((a-1)x_0)^* \sqcup\!\sqcup\, (-bx_1)^*]}(z)\end{array}\right\}.$$

19. $x_0[(ax_0)^* \sqcup\!\sqcup\, ((1-b)x_1)^*$ and $x_1[((a-1)x_0)^* \sqcup\!\sqcup\, (-bx_1)^*]$ are of the form (F_2). What is $\alpha_0^z(S)$, for S of the form (F_2) ?

Other practical examples (2/2)

Example 15 (Polylogarithms indexed by non positive integers)

Now, let us use the noncommutative multivariate exponential transforms, *i.e.*, for any rational exchangeable series, we get the following transform

$$\sum_{i_0,i_1\geq 0} s_{i_0,i_1} x_0^{i_0} ⧢ x_1^{i_1} \longmapsto \sum_{i_0,i_1\geq 0} \frac{s_{i_0,i_1}}{i_0!\,i_1!} \log^{i_0}(z)\log^{i_1}((1-z)^{-1}).$$

In particular, for any $n \in \mathbb{N}$, we have $x_0^n \mapsto \log^n(z)/n!$ and $x_1^n \mapsto \log^n((1-z)^{-1})/n!$. Then $(tx_0)^* \mapsto z^t$ and $(tx_1)^* \mapsto (1-z)^{-t}$.
We then obtain the following polylogarithms indexed by rational series

$$\mathrm{Li}_{x_0^*}(z) = z, \quad \mathrm{Li}_{x_1^*}(z) = (1-z)^{-1}, \quad \mathrm{Li}_{(ax_0+bx_1)^*}(z) = z^a(1-z)^{-b}$$

Thus, for any $(s_1,\ldots,s_r) \in \mathbb{N}_+^r$, there exists an unique series $R_{y_{s_1}\ldots y_{s_r}}$ belonging to $(\mathbb{Z}[x_1^*], ⧢, 1_{X^*})$ s.t. $\mathrm{Li}_{-s_1,\ldots,-s_r} = \mathrm{Li}_{R_{y_{s_1}\ldots y_{s_r}}}$. More precisely,

$$R_{y_{s_1}\ldots y_{s_r}} = \sum_{k_1=0}^{s_1} \ldots \sum_{k_r=0}^{\substack{(s_1+\ldots+s_r)-\\(k_1+\ldots+k_{r-1})}} \binom{s_1}{k_1} \ldots \binom{\sum_{i=1}^{r} s_i - \sum_{i=1}^{r-1} k_i}{k_r} \rho_{k_1} ⧢ \ldots ⧢ \rho_{k_r},$$

where, for any $i = 1,\ldots,r$, if $k_i = 0$ then $\rho_{k_i} = x_1^* - 1_{X^*}$ else

$$\rho_{k_i} = x_1^* ⧢ \sum_{j=1}^{k_i} S_2(k_i,j)j!(x_1^* - 1_{X^*})^{⧢ j}$$

the $S_2(k_i,j)$ being the Stirling numbers of second kind.

NONCOMMUTATIVE PV THEORY AND INDEPENDENCE VIA WORDS

First step of noncommutative PV theory

The Chen series $C_{z_0 \rightsquigarrow z}$ of $\{\omega_k\}_{k\geq 1}$ and along the path $z_0 \rightsquigarrow z$ over Ω satisfies the following differential equation

$$(NCDE) \quad \mathbf{d}S = MS, \quad \text{with} \quad M = \sum_{x\in\mathcal{X}} u_x x \quad \text{and} \quad u_x \in \mathcal{C}_0 \cap \mathcal{A}^{-1}.$$

$$\Delta_{\sqcup\!\sqcup} M = \sum_{x\in\mathcal{X}} u_x(1_{\mathcal{X}^*} \otimes x + x \otimes 1_{\mathcal{X}^*}) = 1_{\mathcal{X}^*} \otimes M + M \otimes 1_{\mathcal{X}^*}.$$

The space of solutions of $(NCDE)$ is a right free $\mathbb{C}\langle\langle X\rangle\rangle$-module of rank 1. By a theorem of Ree, $C_{z_0 \rightsquigarrow z}$ is a $\sqcup\!\sqcup$ $-$group-like solution [20] of $(NCDE)$. Moreover, if G, H are $\sqcup\!\sqcup$ $-$group-like solutions there is a constant Lie series C s.t. $G = He^C$ (and conversely). From this, it follows that

- the Hausdorff group $\{e^C\}_{C\in\mathcal{L}ie_{\mathbb{C}}\langle\langle\mathcal{X}\rangle\rangle}$, group of characters of $\mathcal{H}_{\sqcup\!\sqcup}(\mathcal{X})$, plays the role of the differential Galois group of $(NCDE)+$ $\sqcup\!\sqcup$ $-$group-like.

Which leads us to the following definition

- the PV extension related to $(NCDE)$ is $\widehat{\mathcal{C}_0.\mathcal{X}}\{C_{z_0 \rightsquigarrow z}\}$.

It, of course, is such that $\mathrm{Const}(\mathcal{C}_0\langle\langle\mathcal{X}\rangle\rangle) = \ker \mathbf{d} = \mathbb{C}.1_\Omega\langle\langle\mathcal{X}\rangle\rangle$.

20. It can be obtained as the limit of a convergent Picard iteration, initialized at $\langle C_{z_0 \rightsquigarrow z} | 1_{\mathcal{X}^*}\rangle = 1_{\mathcal{H}(\Omega)}$, for ultrametric distance.

Basic triangular theorem over a differential ring (BTT)

If $S \in \mathcal{A}\langle\langle\mathcal{X}\rangle\rangle$ is a group-like solution of $(NCDE)$, given as follows[21]

$$S = \sum_{w \in \mathcal{X}^*} \langle S|w\rangle w = \sum_{w \in \mathcal{X}^*} \langle S|S_w\rangle P_w = \prod_{l \in \mathcal{L}yn\mathcal{X}}^{\searrow} e^{\langle S|S_l\rangle P_l}$$

then

1. If $H \in \mathcal{A}\langle\langle\mathcal{X}\rangle\rangle$ is another grouplike solution then there exists $C \in \mathcal{L}ie_{\mathcal{A}}\langle\langle\mathcal{X}\rangle\rangle$ such that $S = He^C$ (and conversely).
2. The following assertions are equivalent
   a) $\{\langle S|w\rangle\}_{w \in \mathcal{X}^*}$ is $\mathcal{C}_0$-linearly independent,
   b) $\{\langle S|S_l\rangle\}_{l \in \mathcal{L}yn\mathcal{X}}$ is $\mathcal{C}_0$-algebraically independent,
   c) $\{\langle S|x\rangle\}_{x \in \mathcal{X}}$ is $\mathcal{C}_0$-algebraically independent,
   d) $\{\langle S|x\rangle\}_{x \in \mathcal{X} \cup \{1_{\mathcal{X}^*}\}}$ is $\mathcal{C}_0$-linearly independent,
   e) $\{u_x\}_{x \in \mathcal{X}}$ is such that, for $f \in \mathrm{Frac}(\mathcal{C}_0)$ and $(c_x)_{x \in \mathcal{X}} \in \mathbb{C}^{(\mathcal{X})}$,
   $$\sum_{x \in \mathcal{X}} c_x u_x = \partial f \quad \Longrightarrow \quad (\forall x \in \mathcal{X})(c_x = 0).$$
   f) $(u_x)_{x \in \mathcal{X}}$ is free over $\mathbb{C}$ and $\partial \mathrm{Frac}(\mathcal{C}_0) \cap \mathrm{span}_{\mathbb{C}}\{u_x\}_{x \in \mathcal{X}} = \{0\}$.

21. For instance, $S = C_{z_0 \rightsquigarrow z} = \sum_{w \in \mathcal{X}^*} \alpha_{z_0}^z(w) w$.

Examples of positive cases over $\mathcal{X} = \{x\}, \mathcal{A} = (\mathcal{H}(\Omega), \partial)$

1. $\Omega = \mathbb{C}, u_x(z) = 1_\Omega, \mathcal{C}_0 = \mathbb{C}\{\{u_x^{\pm 1}\}\} = \mathbb{C}.$
   $\alpha_0^z(x^n) = z^n/n!$, for $n \geq 1$. Thus, $\mathbf{d}S = xS$ and

   $$S = \sum_{n\geq 0} \alpha_0^z(x^n)x^n = \sum_{n\geq 0} \frac{z^n}{n!}x^n = e^{zx}.$$

   Moreover, $\alpha_0^z(x) = z$ which is transcendent over $\mathcal{C}_0$
   and the family $\{\alpha_0^z(x^n)\}_{n\geq 0}$ is $\mathcal{C}_0$-free. Let $f \in \mathcal{C}_0$ then $\partial f = 0$. Thus,
   if $\partial f = cu_x$ then $c = 0$.

2. $\Omega = \mathbb{C}\backslash]-\infty, 0], u_x(z) = z^{-1}, \mathcal{C}_0 = \mathbb{C}\{\{z^{\pm 1}\}\} = \mathbb{C}[z^{\pm 1}] \subset \mathbb{C}(z).$
   $\alpha_1^z(x^n) = \log^n(z)/n!$, for $n \geq 1$. Thus $\mathbf{d}S = z^{-1}xS$ and

   $$S = \sum_{n\geq 0} \alpha_1^z(x^n)x^n = \sum_{n\geq 0} \frac{\log^n(z)}{n!}x^n = z^x.$$

   Moreover, $\alpha_1^z(x) = \log(z)$ which is transcendent over $\mathbb{C}(z)$ then
   over $\mathbb{C}[z^{\pm 1}]$. The family the family $\{\alpha_1^z(x^n)\}_{n\geq 0}$ is $\mathbb{C}(z)$-free and
   then $\mathcal{C}_0$-free. Let $f \in \mathcal{C}_0$ then $\partial f \in \mathrm{span}_{\mathbb{C}}\{z^{\pm n}\}_{n\neq 1}$. Thus,
   if $\partial f = cu_x$ then $c = 0$.

Examples of negative cases over $\mathcal{X} = \{x\}, \mathcal{A} = (\mathcal{H}(\Omega), \partial)$

1. $\Omega = \mathbb{C}, u_x(z) = e^z, \mathcal{C}_0 = \mathbb{C}\{\{e^{\pm z}\}\} = \mathbb{C}[e^{\pm z}]$.
   $\alpha_0^z(x^n) = (e^z - 1)^n/n!$, for $n \geq 1$. Thus, $\mathbf{d}S = e^z x S$ and
   $$S = \sum_{n\geq 0} \alpha_0^z(x^n)x^n = \sum_{n\geq 0} \frac{(e^z-1)^n}{n!} x^n = e^{(e^z-1)x}.$$
   Moreover, $\alpha_0^z(x) = e^z - 1$ which is not transcendent over $\mathcal{C}_0$ and and $\{\alpha_0^z(x^n)\}_{n\geq 0}$ is not $\mathcal{C}_0$-free. If $f(z) = ce^z \in \mathcal{C}_0$ $(c \neq 0)$ then $\partial f(z) = ce^z = cu_x(z)$.
2. $\Omega = \mathbb{C}\backslash]-\infty, 0], u_x(z) = z^a (a \notin \mathbb{Q})$, $\mathcal{C}_0 = \mathbb{C}\{\{z, z^{\pm a}\}\} = \mathrm{span}_{\mathbb{C}}\{z^{ka+l}\}_{k,l\in\mathbb{Z}}$.
   $\alpha_0^z(x^n) = (a+1)^{-n} z^{n(a+1)}/n!$, for $n \geq 1$. Thus, $\mathbf{d}S = z^a x S$ and
   $$S = \sum_{n\geq 0} \alpha_0^z(x^n)x^n = \sum_{n\geq 0} \frac{z^{n(a+1)}}{(a+1)^n n!} x^n = e^{(a+1)^{-1} z^{(a+1)} x}.$$
   Moreover, $\alpha_0^z(x) = z^{a+1}/(a+1)$ which is not transcendent over $\mathcal{C}_0$ and $\{\alpha_0^z(x^n)\}_{n\geq 0}$ is not $\mathcal{C}_0$-free. If $f(z) = cz^{a+1}/(a+1) \in \mathcal{C}_0$ $(c \neq 0)$ then $\partial f(z) = cz^a = cu_x(z)$.

Independence over $\mathbb{C}$ of extended eulerian functions

Let $L := \mathrm{span}_{\mathbb{C}}\{\ell_r\}_{r\geq 1}$ and $E := \mathrm{span}_{\mathbb{C}}\{e^{\ell_r}\}_{r\geq 1}$.
Let $\mathbb{C}[L]$ and $\mathbb{C}[E]$ be their respective algebra.

Proposition 6

1. *The families $(\ell_r)_{r\geq 1}$ and $(e^{\ell_r})_{r\geq 1}$ are $\mathbb{C}$-lin. free and free from 1_Ω.*
2. *The families $(\ell_r)_{r\geq 1}$ and $(e^{\ell_r})_{r\geq 1}$ are $\mathbb{C}$-algebraically independent.*
3. *For any $r \geq 1$, one has*
   a) *The functions ℓ_r and e^{ℓ_r} $\mathbb{C}$-algebraically independent.*
   b) *The function ℓ_r is holomorphic on the open unit disc, $D_{<1}$,*
   c) *The function e^{ℓ_r} (resp. $e^{-\ell_r}$) is entire (resp. meromorphic), and admits a countable set of isolated zeroes (resp. poles) on the complex plane which is expressed as $\biguplus_{\chi\in G_r} \chi\mathbb{Z}_{\leq -1}$.*
4. *One has $E \cap L = \{0\}$ and, more generally, $\mathbb{C}[E] \cap \mathbb{C}[L] = \mathbb{C}.1_\Omega$.*

By Theorem 7 and Propositions 1, 6, one deduces then

Corollary 16

The morphism $\alpha_0^z : (\mathbb{C}\langle\langle Y\rangle\rangle, ⧢, 1_{Y^*}) \to (\mathrm{span}_{\mathbb{C}}\{\alpha_0^z(w)\}_{w\in Y^*}, \times, 1_\Omega)$, is injective, using the inputs $\{\partial\ell_r\}_{r\geq 1}$ (resp. $\{e^{\ell_r}\partial\ell_r\}_{r\geq 1}$).

Sketched proof of Proposition 6

1. $(\ell_r)_{r\geq 1}$ is triangular[22]. So is $(e^{\ell_r}-e^{\ell_r(0)})_{r\geq 1}$. Hence, $(\ell_r)_{r\geq 1}$ and $(e^{\ell_r})_{r\geq 1}$ are $\mathbb{C}$-lin. free. Moreover, $(e^{\ell_r})_{r\geq 1}$ is free from 1_Ω.
2. Using Chen series of $\{\omega_r\}_{r\geq 1}$ defined, as in Ex. 13, by $u_{x_r}=e^{\ell_r}\partial\ell_r$ (resp. $u_{x_r}=\partial\ell_r$), via BTT, $\{e^{\ell_r}\}_{r\geq 1}$ (resp. $\{\ell_r\}_{r\geq 1}$) is the $\mathbb{C}$-alg. free.
3. a) Since $\ell_r(0)=0, \partial e^{\ell_r}=e^{\ell_r}\partial\ell_r$ then ℓ_r and e^{ℓ_r} are $\mathbb{C}$-alg. free.
   b) One has $e^{\ell_1(z)}=\Gamma^{-1}(1+z)$ which proves the claim for $r=1$. For $r\geq 2$, note that $1\leq\zeta(r)\leq\zeta(2)$ which implies that the radius of convergence of the exponent is 1 and means that ℓ_r is holomorphic on the open unit disc. This proves the claim.
   c) $e^{\ell_r(z)}=\Gamma_{y_r}^{-1}(1+z)$ (resp. $e^{-\ell_r(z)}=\Gamma_{y_r}(1+z)$) is entire (resp. meromorphic) as finite product of entire (resp. meromorphic) functions and Weierstrass factorization yields zeroes (resp. poles).
4. $\mathbb{C}[L]$ (resp. $\mathbb{C}[E]$) is generated freely by $(\ell_r)_{r\geq 1}$ (resp. $(e^{\ell_r})_{r\geq 1}$) which is holomorphic on $D_{<1}$ (resp. entire) function. Moreover, any $f\in\mathbb{C}[L]$ (resp. $g\in\mathbb{C}[E]$), $\neq 1_\Omega$, is holomorphic (resp. entire). Thus, $f\notin\mathbb{C}[E]$ (resp. $g\notin\mathbb{C}[L]$). It follows then the expected result.

22. $(g_i)_{i\geq 1}$ is said to be *triangular* if the valuation of $g_i, \varpi(g_i)$, equals $i\geq 1$. It is easy to check that such a family is $\mathbb{C}$-lin. free and that is also the case of families s.t. $(g_i-g(0))_{i\geq 1}$ is triangular.

Independence of $\{e^{\ell_r}\}_{k\geq 1}$ over differential subalgebra

Let $\mathcal{L} := \mathbb{C}\{\{(\ell_r^{\pm 1})_{r\geq 1}\}\} = \mathbb{C}[\{\ell_r^{\pm 1}, \partial^i \ell_r\}_{r,i\geq 1}]$ and $\mathcal{E} := \mathbb{C}\{\{(e^{\pm\ell_r})_{r\geq 1}\}\}$.
Let $\mathcal{L}^+ := \mathbb{C}[\{\partial^i \ell_r\}_{r,i\geq 1}]$. $\mathrm{Frac}(\mathcal{L}^+)$ is generated then by meromorphic functions. Since, for any $i, l, k \geq 1$, there is $0 \neq q_{i,l,k} \in \mathcal{L}^+$ s.t. $(\partial^i e^{\pm\ell_k})^l = q_{i,l,k} e^{\pm l\ell_k}$ then let

$$\begin{aligned}\mathcal{E}^+ &:= \mathrm{span}_{\mathbb{C}}\{(\partial^{i_1} e^{\pm\ell_{r_1}})^{l_1} \ldots (\partial^{i_k} e^{\pm\ell_{r_k}})^{l_k}\}_{(i_1,l_1,r_1),\ldots,(i_k,l_k,r_k)\in(\mathbb{N}_{\geq 1})^3, k\geq 1}\\ &= \mathrm{span}_{\mathbb{C}}\{q_{i_1,l_1,r_1} \ldots q_{i_k,l_k,r_k} e^{l_1\ell_{r_1}+\ldots+l_k\ell_{r_k}}\}_{(i_1,l_1,r_1),\ldots,(i_k,l_k,r_k)\in\mathbb{N}_{\geq 1}\times\mathbb{Z}_{\neq 0}\times\mathbb{N}_{\geq 1}, k\geq 1}\\ &\subset \mathrm{span}_{\mathcal{L}^+}\{e^{l_1\ell_{r_1}+\ldots+l_k\ell_{r_k}}\}_{(l_1,r_1),\ldots,(l_k,r_k)\in\mathbb{Z}^*\times\mathbb{N}_{\geq 1}, k\geq 1} =: \mathcal{C}.\end{aligned}$$

Note that $\mathcal{E}^+ \cap E = \{0\}$ and $\mathcal{C}$ is a differential subring of $\mathcal{A} = \mathcal{H}(\Omega)$.
Hence, $\mathrm{Frac}(\mathcal{C})$ is a differential subfield of $\mathrm{Frac}(\mathcal{A})$.

Theorem 17

1. *The family $(e^{\ell_r})_{r\geq 1}$ (resp. $(\ell_r)_{r\geq 1}$) is alg. free over $\mathcal{E}^+$ (resp. $\mathcal{L}^+$).*
2. *$\mathbb{C}[E]$ and $\mathbb{C}[L]$ are alg. disjoint, within $\mathcal{A}$.*

By Theorems 7, 17 and Proposition 1, one deduces then

Corollary 18

The morphism $\alpha_0^z : (\mathcal{C}\langle\langle Y\rangle\rangle, ⧢, 1_{Y^*}) \to (\mathrm{span}_{\mathcal{C}}\{\alpha_0^z(w)\}_{w\in Y^*}, \times, 1_\Omega)$, is injective, where $\mathcal{C} = \mathcal{L}^+$ (resp. $\mathcal{E}^+$) using the inputs $\{\partial\ell_r\}_{r\geq 1}$ (resp. $\{e^{\ell_r}\partial\ell_r\}_{r\geq 1}$).

Sketched proof of Theorem 17

1. Using the Chen series of $\{\omega_r\}_{r\geq 1}$ defined by $u_{y_r} = e^{\ell_r}\partial\ell_r$, let $Q \in \mathrm{Frac}(\mathcal{L})$ (resp. $\mathrm{Frac}(\mathcal{C})$) and let $\{c_y\}_{y\in Y} \in \mathbb{C}^{(Y)}$, non simultaneously vanishing, s.t.
$$\partial Q = \sum_{y\in Y} c_y u_y = \sum_{r\geq 1} c_{y_r} e^{\ell_r}\partial\ell_r.$$
If $\partial Q \neq 0$ then, integrating, $Q \in E$ and then
$$E \supset \mathrm{Frac}(\mathcal{L}) \supset \mathcal{L} \supset \mathbb{C}[L] \text{ (resp. } E \supset \mathrm{Frac}(\mathcal{C}) \supset \mathcal{C} \supset \mathcal{E}^+)$$
contradicting with $E \cap \mathbb{C}[L] = \{0\}$ (resp. $E \cap \mathcal{E}^+ = \{0\}$). It remains that $\partial Q = 0$. Since $\{e^{\ell_k}\}_{k\geq 1}$ and then $\{\partial e^{\ell_k}\}_{k\geq 1}$ are $\mathbb{C}$-lin. free then, for any $r \geq 1$, $c_{y_r} = 0$.
By BTT, $\{\alpha_0^z(S_l)\}_{l\in\mathcal{L}ynY}$ and then $\{\alpha_0^z(S_y)\}_{y\in Y}$ are alg. free over $\mathcal{L}$ (resp. $\mathcal{C}$). Thus, $(e^{\ell_k})_{k\geq 1}$ is alg. free over $\mathbb{C}[L]$ (resp. $\mathcal{E}^+$).
Now, suppose there is an alg. relation among $(\ell_k)_{k\geq 1}$ over $\mathcal{L}^+$ in which, by differentiating and substituting $\partial\ell_k$ by $e^{-\ell_k}\partial e^{\ell_k}$, we get an alg. relation among $\{e^{\ell_k}\}_{k\geq 1}$ over $\mathbb{C}[L]$ and $\mathcal{E}^+$ contradicting with previous results. It follows then $(\ell_k)_{k\geq 1}$ is $\mathcal{L}^+$-alg. free.
2. $\{e^{\ell_k}\}_{k\geq 1}$ (resp. $\{\ell_k\}_{k\geq 1}$) is alg. free over $\mathbb{C}[L]$ (resp. $\mathbb{C}[E]$). Thus, $\{e^{\ell_k}, \ell_k\}_{k\geq 1}$ generates freely $\mathbb{C}[E+L]$ and $\mathbb{C}[E] \cap \mathbb{C}[L] = \mathbb{C}.1_\Omega$. Hence, $\mathbb{C}[E]$ and $\mathbb{C}[L]$ are alg. disjoint, within $\mathcal{A}$.

Dom(Li$_\bullet$) AND Dom(H$_\bullet$)

Chen series of $\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1-z)^{-1}dz$

Let $\gamma_0(\varepsilon)$ and $\gamma_1(\varepsilon)$ be the circular paths of radius ε encircling 0 and 1 clockwise, respectively. In particular, letting $\beta = \beta_1 - \beta_0$, one considers

$$\begin{array}{rcccl} \gamma_0(\varepsilon,\beta) & = & \varepsilon e^{\mathrm{i}\beta_0} \rightsquigarrow \varepsilon e^{\mathrm{i}\beta_1} & \subset & \gamma_0(\varepsilon), \\ \gamma_1(\varepsilon,\beta) & = & 1-\varepsilon e^{\mathrm{i}\beta_0} \rightsquigarrow 1-\varepsilon e^{\mathrm{i}\beta_1} & \subset & \gamma_1(\varepsilon). \end{array}$$

On the one hand, one has, for any $i = 0$ or 1 and $w \in X^+$,

$$|\langle C_{\gamma_i(\varepsilon,\beta)}|w\rangle| \leq \varepsilon^{|w|_{x_i}} \beta^{|w|} |w|!^{-1}.$$

It follows then

$$C_{\gamma_i(\varepsilon,\beta)} = e^{\mathrm{i}\beta x_i} + o(\varepsilon) \quad \text{and} \quad C_{\gamma_i(\varepsilon)} = e^{2\mathrm{i}\pi x_i} + o(\varepsilon).$$

Hence[23], for $R \in \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ of minimal representation (λ,μ,η), one has

$$\langle R \| C_{\gamma_i(\varepsilon,\beta)}\rangle = \lambda\Big(\prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\alpha_{\gamma_i(\varepsilon,\beta)}(S_l)\mu(P_l)}\Big)\eta,$$

$$\langle R \| C_{\gamma_i(\varepsilon)}\rangle = \lambda\Big(\prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\alpha_{\gamma_i(\varepsilon)}(S_l)\mu(P_l)}\Big)\eta.$$

23. Recall that the map $\alpha_{z_0}^z : \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle \to \mathcal{H}(\Omega)$ is not injective. For example, $\alpha_{z_0}^z(z_0x_0^* + (1-z_0)(-x_1)^* - 1_{X^*}) = 0$.

Back to polylogrithms : $u_{x_0}(z) = z^{-1}, u_{x_1}(z) = (1-z)^{-1}$

Here, $\mathcal{A} = (\mathcal{H}(\Omega), \partial)$ with $\Omega = \mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$.
Let us consider the character $\mathrm{Li}_\bullet : (\mathbb{C}\langle X \rangle, ⧢, 1_{X^*}) \to (\mathcal{H}(\Omega), \times, 1_\Omega)$
defined by $\mathrm{Li}_{x_0}(z) = \log(z), \mathrm{Li}_{x_1}(z) = -\log(1-z)$ and

$$\forall x_i v \in \mathcal{L}ynX - X, \quad \mathrm{Li}_{x_i v}(z) = \int_0^z \omega_i(s) \, \mathrm{Li}_v(s).$$

Hence, the n.g.s. of $\{\mathrm{Li}_w\}_{w \in X^*}$, L, is group-like, for $\Delta_{⧢}$, and

$$\mathrm{L} := \sum_{w \in X^*} \mathrm{Li}_w \, w = (\mathrm{Li}_\bullet \otimes \mathrm{Id}) \mathcal{D}_X = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\mathrm{Li}_{S_l} \, P_l}.$$

L satisfies the following differential equation

$$(DE) \qquad \mathbf{d}\mathrm{L} = (u_{x_0} x_0 + u_{x_1} x_1) \mathrm{L}$$

and then $\mathrm{L}(z) = C_{z_0 \rightsquigarrow z} \mathrm{L}(z_0)$. It follows the definition of

$$Z_{⧢} := \mathrm{L}_{\mathbf{reg}}(1), \quad \text{where} \quad \mathrm{L}_{\mathbf{reg}} := \prod_{l \in \mathcal{L}ynX - X}^{\searrow} e^{\mathrm{Li}_{S_l} \, P_l}.$$

Theorem 19

$\mathrm{Li}_\bullet$ *is injective. It follows then* $\{\mathrm{Li}_w\}_{w \in X^*}$ *is* $\mathbb{C}$*-lin. free and* $\{\mathrm{Li}_l\}_{l \in \mathcal{L}ynX}$ *(resp.* $\{\mathrm{Li}_{S_l}\}_{l \in \mathcal{L}ynX}$*) is alg. free.*

Back to harmonic sums

Let $\pi_Y : (\mathbb{C}\langle\langle X\rangle\rangle, .) \to (\mathbb{C}\langle\langle Y\rangle\rangle, .)$, maps $x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1$ to $y_{s_1}\ldots y_{s_r}$.

$$\forall w \in X^*x_1, \quad \forall z \in \mathbb{C}, |z| < 1, \quad \frac{\mathrm{Li}_w(z)}{1-z} = \sum_{n\geq 0} \mathrm{H}_{\pi_Y w}(n) z^n.$$

Theorem 20

The morphism of algebras $\mathrm{H}_\bullet : (\mathbb{C}\langle Y\rangle, \uplus, 1_{Y^*}) \to (\mathbb{C}\{\mathrm{H}_w\}_{w\in Y^*}, ., 1)$, *mapping* u *to*[24] H_u, *is injective. Hence,* $\{\mathrm{H}_w\}_{w\in Y^*}$ *is lin. free. It follows then* $\{\mathrm{H}_l\}_{l\in\mathcal{L}ynY}$ *(resp.* $\{\mathrm{H}_{\Sigma_l}\}_{l\in\mathcal{L}ynY}$*) is alg. free.*

Hence, the n.g.s. of $\{\mathrm{H}_w\}_{w\in Y^*}$, H, is group-like, for $\Delta_{\uplus}$, and

$$\mathrm{H} := \sum_{w\in Y^*} \mathrm{H}_w w = (\mathrm{H}_\bullet \otimes \mathrm{Id})\mathcal{D}_Y = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\mathrm{H}_{\Sigma_l}\Pi_l}.$$

It follows then the definition of

$$Z_{\uplus} := \mathrm{H}_{\mathtt{reg}}(+\infty), \quad \text{where} \quad \mathrm{H}_{\mathtt{reg}} := \prod_{l\in\mathcal{L}ynY-\{y_1\}}^{\searrow} e^{\mathrm{H}_{\Sigma_l}\Pi_l}.$$

Theorem 21 (first Abel like theorem)

$$\lim_{z\to 1} e^{y_1 \log(1-z)}\pi_Y \mathrm{L}(z) = \lim_{n\to\infty} e^{\sum_{k\geq 1}\mathrm{H}_{y_k}(n)(-y_1)^k/k}\mathrm{H}(n) = \pi_Y Z_{\sqcup\!\sqcup}.$$

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24. The $\{\mathrm{H}_u\}_{u\in Y^*}$'s, so-called harmonic sums, are arithmetical functions.

Back to polyzetas

The polymorphism ζ is defined by

$$\zeta : \begin{array}{c}(\mathbb{Q}[\mathcal{L}ynX - X], ш, 1_{X^*})\\ (\mathbb{Q}[\mathcal{L}ynY - \{y_1\}], ⧢, 1_{Y^*})\end{array} \twoheadrightarrow (\mathcal{Z}, ., 1),$$

$$\begin{array}{c} x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \in \mathcal{L}ynX - X \\ y_{s_1}\ldots y_{s_r} \in \mathcal{L}ynY - \{y_1\}\end{array} \mapsto \zeta(s_1,\ldots,s_r) = \sum_{n_1>\ldots>n_r} n_1^{-s_1}\ldots n_r^{-s_r}$$

($\mathcal{Z} := \mathrm{span}_{\mathbb{Q}}\{\zeta(s_1,\ldots,s_r)\}_{s_1>2,s_2\ldots,s_r\geq 1}$). It can be extended as characters :

$$\zeta_{ш} : (\mathbb{Q}[\mathcal{L}ynX], ш, 1_{X^*}) \twoheadrightarrow (\mathcal{Z}, ., 1),$$
$$\zeta_{⧢}, \gamma_{\bullet} : (\mathbb{Q}[\mathcal{L}ynY], ⧢, 1_{Y^*}) \twoheadrightarrow (\mathcal{Z}, ., 1),$$

$$\begin{array}{rcll}
\zeta_{ш}(x_0) = 0 = & \log(1), & \\
\zeta_{ш}(x_1) = 0 = & \mathrm{f.p.}_{z\to 1}\log(1-z), & \{(1-z)^a\log^b(1-z)\}_{a\in\mathbb{Z},b\in\mathbb{N}},\\
\zeta_{⧢}(y_1) = 0 = & \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_1(n), & \{n^a\mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}},\\
\gamma_{y_1} = \gamma = & \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_1(n), & \{n^a\log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}.
\end{array}$$

Because, for any $l \in \mathcal{L}yn\mathcal{X}, l \notin \{x_0\}$, one has (see a theorem by Radford)

$$\begin{array}{rcll}
\gamma_l & = & \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_l(n), & \{n^a\log^b(n)\}_{a\in\mathbb{Z}_{\leq -1},b\in\mathbb{N}},\\
\zeta_{⧢}(l) & = & \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_l(n), & \{n^a\mathrm{H}_1^b(n)\}_{a\in\mathbb{Z}_{\leq -1},b\in\mathbb{N}},\\
\zeta_{ш}(l) & = & \mathrm{f.p.}_{z\to 1}\mathrm{Li}_l(z), & \{(1-z)^a\log^b(1-z)\}_{a\in\mathbb{Z}_{\leq -1},b\in\mathbb{N}}.
\end{array}$$

Hence, their graphs, viewed as noncommutative generating series, are

$$\sum_{w\in Y^*}\gamma_w w =: Z_\gamma = e^{\gamma y_1}Z_{⧢}, \quad \sum_{w\in Y^*}\zeta_{⧢}(w)w = Z_{⧢}, \quad \sum_{w\in X^*}\zeta_{ш}(w)w = Z_{ш}.$$

Generalized Euler's gamma constant

Theorem 22 (bridge equations)

Let $B(y_1) = e^{\gamma y_1 - \sum_{k\geq 2}\zeta(k)(-y_1)^k/k}$ *and* $\mathtt{Mono}(y_1) = e^{-\sum_{k\geq 2}\zeta(k)(-y_1)^k/k}$.
Then, by cancellation, $Z_\gamma = B(y_1)\pi_Y Z_{ш} \iff Z_{⊞} = \mathtt{Mono}(y_1)\pi_Y Z_{ш}$.
Identifying the coefficients of $y_1^k w$ in $Z_\gamma = B(y_1)\pi_Y Z_{ш}$, one has

1. $\gamma_{y_1^k} = \sum_{\substack{s_1,\ldots,s_k>0\\ s_1+\ldots+ks_k=k}} \frac{(-1)^k}{s_1!\ldots s_k!}(-\gamma)^{s_1}\left(-\frac{\zeta(2)}{2}\right)^{s_2}\ldots\left(-\frac{\zeta(k)}{k}\right)^{s_k}$.
2. $\gamma_{y_1^k w} = \sum_{i=0}^{k}\frac{\zeta(x_0(-x_1)^{k-i} ш \pi_X w])}{i!}\left(\sum_{j=1}^{i} b_{i,j}(\gamma,-\zeta(2),2\zeta(3),\ldots)\right)$,

where $k\in\mathbb{N}_+$, $w\in Y^+$ and $b_{n,k}(t_1,\ldots,t_k)$ are Bell polynomials.

Example 23

$$\begin{aligned}
\gamma_{1,1} &= \frac{1}{2}(\gamma^2-\zeta(2)),\\
\gamma_{1,1,1} &= \frac{1}{6}(\gamma^3-3\zeta(2)\gamma+2\zeta(3)),\\
\gamma_{1,7} &= \zeta(7)\gamma+\zeta(3)\zeta(5)-\tfrac{54}{175}\zeta(2)^4,\\
\gamma_{1,1,6} &= \frac{4}{35}\zeta(2)^3\gamma^2+(\zeta(2)\zeta(5)+\frac{2}{5}\zeta(3)\zeta(2)^2-4\zeta(7))\gamma\\
&\quad+\zeta(6,2)+\frac{19}{35}\zeta(2)^4+\frac{1}{2}\zeta(2)\zeta(3)^2-4\zeta(3)\zeta(5).
\end{aligned}$$

Homogenous polynomials relations[25] on local coordinates

Identifying the local coordinates in $Z_\gamma = B(y_1)\pi_Y Z_{\sqcup\sqcup}$, one has

	Polynomial relations on $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Polynomial relations on $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2}) = \zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4}) = \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1}) = \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2}) = \frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2}) = \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2}) = 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1}) = -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1}) = -\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2}) = \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2}) = \frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3}) = \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4}) = \zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6}) = \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2}) = \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1}) = \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1}) = \frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2}) = -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3}) = \frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1}) = 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2}) = \frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2}) = \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1}) = -\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2}) = \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3}) = \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3}) = \frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4}) = \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$

25. These polynomials relations are independent from γ and similarly for the case where the ring of their coefficients is the commutative ring A containing $\mathbb{Q}$.

Cloned Abel like results and cloned bridge equations

Let $e^C \in \mathrm{Gal}_{\mathbb{C}}(DE) = \{e^C\}_{C\in\mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle}$ and $\overline{\mathrm{L}} := \mathrm{L}e^C, \overline{Z}_{\sqcup\!\sqcup} := Z_{\sqcup\!\sqcup}\, e^C$. Let

$$\mathtt{Const} := \sum_{k\geq 0} \mathrm{H}_{y_1^k} y_1^k = \exp\left(-\sum_{k\geq 0} \mathrm{H}_{y_k} \frac{(-y_1)^k}{k}\right).$$

Then $\overline{\mathrm{L}}(z) \sim_1 e^{-x_1\log(1-z)}\overline{Z}_{\sqcup\!\sqcup}$ and then $\overline{\mathrm{H}}(n) \sim_{+\infty} \mathtt{Const(n)}\pi_Y\overline{Z}_{\sqcup\!\sqcup}$.

Theorem 24 (cloned first Abel like theorem)

$$\lim_{z\to 1} e^{y_1\log(1-z)}\pi_Y\overline{\mathrm{L}}(z) = \pi_Y\overline{Z}_{\sqcup\!\sqcup} = \lim_{n\to\infty} \mathtt{Const(n)}^{-1}\overline{\mathrm{H}}(n).$$

If[26] $\overline{Z}_{\sqcup\!\sqcup} \in dm(A) := \{Z_{\sqcup\!\sqcup}\, e^C \mid C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle, \langle e^C|x_0\rangle = \langle e^C|x_1\rangle = 0\}$ then $\overline{Z}_\gamma = e^{\gamma y_1}\overline{Z}_{\uplus}$ and recall also that $\langle\overline{Z}_{\sqcup\!\sqcup}|x_0\rangle = \langle\overline{Z}_{\sqcup\!\sqcup}|x_1\rangle = 0, \langle\overline{Z}_\gamma|y_1\rangle = \gamma$ and (for $l \in \mathcal{L}yn\mathcal{X}, l \notin \{x_0, x_1, y_1\}$)

$$\begin{aligned}
\langle\overline{Z}_{\sqcup\!\sqcup}|l\rangle &= \mathrm{f.p.}_{z\to 1}\overline{\mathrm{Li}}_l(z), & \{(1-z)^a\log^b(1-z)\}_{a\in\mathbb{Z}_{\leq -1}, b\in\mathbb{N}},\\
\langle\overline{Z}_{\uplus}|l\rangle &= \mathrm{f.p.}_{n\to+\infty}\overline{\mathrm{H}}_l(n), & \{n^a\mathrm{H}_1^b(n)\}_{a\in\mathbb{Z}_{\leq -1}, b\in\mathbb{N}},\\
\langle\overline{Z}_\gamma|l\rangle &= \mathrm{f.p.}_{n\to+\infty}\overline{\mathrm{H}}_l(n), & \{n^a\log^b(n)\}_{a\in\mathbb{Z}_{\leq -1}, b\in\mathbb{N}}.
\end{aligned}$$

Corollary 25 (cloned bridge equations)

If $\overline{Z}_{\sqcup\!\sqcup} \in dm(A)$ then ($\overline{Z}_\gamma = B(y_1)\pi_Y\overline{Z}_{\sqcup\!\sqcup} \iff \overline{Z}_{\uplus} = \mathtt{Mono}(y_1)\pi_Y\overline{Z}_{\sqcup\!\sqcup}$).

26. $dm(A)$ contains $DM(A)$, introduced by Cartier and Racinet, and is a strict normal subgroup of $\mathrm{Gal}_A(DE)$.

$\mathrm{Dom}(\mathrm{Li}_\bullet)$, $\mathrm{Dom}_R(\mathrm{Li}_\bullet)$ and $\mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet)$

Let $\mathcal{C} := \mathbb{C}[z^a, (1-z)^b]_{a,b\in\mathbb{C}}$. Let $[S]_n = \sum_{w\in X^*, |w|=n} \langle S|w\rangle w$ denotes the homogeneous components of S (of degree n). Then $\mathrm{Dom}(\mathrm{Li}_\bullet)$ is the set of $S = \sum_{n\geq 0}[S]_n$ s.t. $\sum_{n\geq 0} \mathrm{Li}_{[S]_n}$ is unconditionally convergent for the standard topology on $\mathcal{H}(\Omega)$.

Denoting the open disk by $D_{<R}$ $(0 < R \leq 1)$, let

$$\mathrm{Dom}_R(\mathrm{Li}_\bullet) := \{S \in \mathbb{C}\langle\langle X\rangle\rangle x_1 \oplus \mathbb{C}1_{X^*} \,|\, \sum_{n\geq 0} \mathrm{Li}_{[S]_n} \text{ is unconditionally convergent for the standard topology on } \mathcal{H}(D_{<R})\}.$$

$$\mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet) := \bigcup_{0<R\leq 1} \mathrm{Dom}_R(\mathrm{Li}_\bullet).$$

Proposition 7 ($\mathrm{L}(z) = C_{z_0 \rightsquigarrow z}\mathrm{L}(z_0)$)

Let $\rho := \langle R\|\mathrm{L}\rangle$ ($R \in \mathrm{Dom}(\mathrm{Li}_\bullet)$). Then $\partial^n \rho = \langle R\|\mathbf{d}^n\mathrm{L}\rangle$ and $\mathbf{d}^n\mathrm{L} = p_n\mathrm{L}$, where $\{p_n\}_{n\geq 0}$ are given previously, using

$$\tau_r(x_0) = -r!(-z)^{-(r+1)}x_0 \text{ and } \tau_r(x_1) = r!(1-z)^{-(r+1)}x_1.$$

The following assertions are equivalent :

1. *ρ satisfies a differential equation with coefficients in $(\mathcal{C}, \partial)$.*
2. *There exists $P \in \mathcal{C}\langle X\rangle$ such that $\langle R\|P\mathrm{L}\rangle = \langle R \triangleleft P\|\mathrm{L}\rangle = 0$.*

$\mathrm{Dom}(\mathrm{H}_\bullet)$

Proposition 8

1. $\mathrm{Dom}(\mathrm{Li}_\bullet)$, *containing* $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle \sqcup\!\sqcup \mathbb{C}\langle X\rangle$, *is closed by* $\sqcup\!\sqcup$ *and then* $\mathrm{Li}_{S \sqcup\!\sqcup T} = \mathrm{Li}_S \mathrm{Li}_T$, *for* $S, T \in \mathrm{Dom}(\mathrm{Li}_\bullet)$.
2. *Let* $S \in \mathbb{C}\langle\langle X\rangle\rangle x_1 \oplus \mathbb{C}1_{X^*}$ *and* $0 < R \leq 1$ *s.t.* $\sum_{n\geq 0} \mathrm{Li}_{[S]_n}$ *is unconditionally convergent, for the standard topology, on* $\mathcal{H}(D_{<R})$. *Then* $\sum_{N\geq 0} a_N z^N = (1-z)^{-1} \sum_{n\geq 0} \mathrm{Li}_{[S]_n}(z)$ *is unconditionally convergent in the same domain and* $a_N = \sum_{n\geq 0} \mathrm{H}_{\pi_Y([S]_n)}(N)$.
3. $S \sqcup\!\sqcup T \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet)$ *and* $\pi_X(\pi_Y(S) \uplus \pi_Y(T)) \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet)$, *for* $S, T \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet)$. *Moreover,*
$$\begin{aligned}
\mathrm{Li}_{S \sqcup\!\sqcup T} &= \mathrm{Li}_S \mathrm{Li}_T\,.\\
\mathrm{H}_{\pi_Y(S) \uplus \pi_Y(T)}(N) &= \mathrm{H}_{\pi_Y(S)}(N)\mathrm{H}_{\pi_Y(T)}(N), \qquad N \geq 0.\\
\frac{\mathrm{Li}_S(z)}{1-z} \odot \frac{\mathrm{Li}_T(z)}{1-z} &= \frac{\mathrm{Li}_{\pi_X(\pi_Y(S) \uplus \pi_Y(T))}(z)}{1-z}.
\end{aligned}$$
4. *If* $S \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet)$ *then* $\mathrm{H}_{\pi_Y(S)} \in \mathrm{Dom}(\mathrm{H}_\bullet) := \pi_Y \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet)$. *The last contains* $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle Y\rangle\rangle \uplus \mathbb{C}\langle Y\rangle$ *and is closed by* $\uplus$. *Hence,* $\mathrm{H}_{S \uplus T} = \mathrm{H}_S \mathrm{H}_T$, *for* $S, T \in \mathrm{Dom}(\mathrm{H}_\bullet)$.

Extensions of $\mathrm{Li}_\bullet$ and of $\mathrm{H}_\bullet$ ($\mathcal{C} = \mathbb{C}\{z^a, (1-z)^b\}_{a,b\in\mathbb{C}}$)

Theorem 26 (indexing by noncommutative rational series)

1. $\{\mathrm{Li}_w\}_{w\in X^*}$ *is* $\mathcal{C}$*-linearly independent*[27]*. Moreover, the* kernel *of the following map is the* $\sqcup\!\sqcup$ *-ideal is generated by* $x_0^* \sqcup\!\sqcup\, x_1^* - x_1^* + 1$

$$\begin{array}{rcl}\mathrm{Li}_\bullet : (\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle \sqcup\!\sqcup\, \mathbb{C}\langle X\rangle, \sqcup\!\sqcup\,, 1_{X^*}) & \to & (\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}, ., 1_\Omega),\\ R & \mapsto & \mathrm{Li}_R\,.\end{array}$$

2. *The algebra* $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ *is closed under the differential operators* $\theta_0 = z\partial$ *and* $\theta_1 = (1-z)\partial$*, and under their sections*[28] ι_0 *and* ι_1*.*

Corollary 27

The arithmetic function $\mathrm{H}_{(zy_r)^*}$ *is given, for* $r \geq 1, z \in \mathbb{C}, |z| < 1$*, by*[29]

$$\mathrm{H}_{(z^r y_r)^*} = \sum_{k\geq 0} \mathrm{H}_{y_r^k} z^{kr} = \exp\left(-\sum_{k\geq 1} \mathrm{H}_{y_{kr}} \frac{(-z^r)^k}{k}\right)$$

and, for $a_s, b_s \in \mathbb{C}, |a_s|, |b_s| < 1$ $(s \geq 1)$*,*

$$\mathrm{H}_{(\sum_{s\geq 1} a_s y_s)^*} \mathrm{H}_{(\sum_{s\geq 1} b_s y_s)^*} = \mathrm{H}_{(\sum_{s\geq 1}(a_s+b_s)y_s + \sum_{r,s\geq 1} a_s b_r y_{s+r})^*}\,.$$

27. The proof uses also BTT.
28. *i.e.* $\theta_0\iota_0 = \theta_1\iota_1 = \mathrm{Id}$.
29. $-\sum_{k\geq 1} \mathrm{H}_{kr}(-z^r)^k/k$ is termwise dominated by $\|\ell_r\|_\infty$ and then $\mathrm{H}_{(z^r y_r)^*}$ is dominated in norm by $e^{\ell_r(z)} = \Gamma_{y_r}^{-1}(1+z)$, using Newton-Girard formula.

Domain of ($\sqcup\!\sqcup$ or $\uplus$) characters

Any ($\sqcup\!\sqcup$ or $\uplus$) character χ classically extends $\mathcal{H}(\Omega)\langle\mathcal{X}\rangle$ by

$$\chi(P) = \sum_{w\in\mathcal{X}^*} \langle P|w\rangle\langle\chi|w\rangle$$

as a character from $\mathcal{H}(\Omega)\langle\mathcal{X}\rangle$ with values in $\mathcal{H}(\Omega)$.

Theorem 28 (Extended characters)

Let $\chi : \mathbb{C}\langle\mathcal{X}\rangle \to \mathbb{C}$ *be a character*[30]. *For any* $T \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle$, *let*

$$\mathrm{Dom}(\chi,\Omega) := \{T \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle | (\chi([T]_n))_{n\in\mathbb{N}} \text{ is summable in } \mathcal{H}(\Omega)\}$$

The result, $\sum_{n\geq 0}\chi([T]_n)$, *will be still noted* $\chi(T)$. *One has*

1. $\mathcal{H}(\Omega)\langle\mathcal{X}\rangle \subset \mathrm{Dom}(\chi,\Omega)$.
2. $\mathrm{Dom}(\chi,\Omega)$ *is a subalgebra of* $\mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle$ *(for* $\sqcup\!\sqcup$ *or* $\uplus$ *).*
3. *Let* $S \in \mathrm{Dom}(\chi,\Omega)$. $\exp_{\sqcup\!\sqcup}(S)$ *and* $\exp_{\uplus}(S) \in \mathrm{Dom}(\chi,\Omega)$. *Moreover,* $\chi(\exp_{\sqcup\!\sqcup}(S)) = e^{\chi(S)}$ *and* $\chi(\exp_{\uplus}(S)) = e^{\chi(S)}$.

Example 29

For any $z \in \mathbb{C}, |z| < 1, x \in X = \{x_0, x_1\}, y_r \in Y = \{y_k\}_{k\geq 1}$, since $(zx)^* = \exp_{\sqcup\!\sqcup}(z)$ and $(zy_r)^* = \exp_{\uplus}(\sum_{k\geq 1} y_{kr}(-z)^{k-1}/k)$ then $\zeta_{\sqcup\!\sqcup}((zx)^*) = e^{z\zeta_{\sqcup\!\sqcup}(x)}$ and $\gamma_{(zy_r)^*} = e^{\sum_{k\geq 1}\zeta_{\uplus}(y_{kr})(-z)^{k-1}/k}$.

30. We will still note its extension to $\mathcal{H}(\Omega)\langle\mathcal{X}\rangle$ by χ.

Extended polymorphism ζ

With the notations in Example 13, we have

Theorem 30 (Regularization by Newton-Girard formula)

The characters $\zeta_{\shuffle}, \gamma_{\bullet}$ *can be extended as follows*

$$\begin{aligned}
\zeta_{\shuffle} : (\mathbb{C}\langle X\rangle \shuffle \mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle, \shuffle, 1_{X^*}) &\to (\mathbb{C}, ., 1),\\
\forall z \in \mathbb{C}, |z| < 1, (zx_0)^*, (zx_1)^* &\mapsto 1_{\mathbb{C}}.\\
\gamma_{\bullet} : (\mathbb{C}\langle Y\rangle \stuffle \mathbb{C}^{\mathrm{rat}}\langle\langle Y\rangle\rangle, \stuffle, 1_{Y^*}) &\to (\mathbb{C}, ., 1),\\
\forall z \in \mathbb{C}, |z| < 1, (z^r y_r)^* &\mapsto \Gamma^{-1}_{y_r}(1+z), r \geq 1.
\end{aligned}$$

Moreover, with $\omega_r = \partial \ell_r, r \geq 1$, *and for* $z \in \mathbb{C}, |z| < 1$, *the following morphism is injective*

$$\begin{aligned}
\alpha_0^z : (\mathbb{C}[\{y_r^*\}_{r\geq 1}], \stuffle, 1_{Y^*}) &\to (\mathbb{C}[\{e^{\ell_r}\}_{r\geq 1}], \times, 1),\\
\forall z \in \mathbb{C}, |z| < 1, y_r^* &\mapsto \Gamma^{-1}_{y_r}(1+z), r \geq 1,
\end{aligned}$$

and $\Gamma_{y_{2r}}(1 + \sqrt[2r]{-1}z) = \Gamma_{y_r}(1+z)\Gamma_{y_r}(1+\sqrt[r]{-1}z)$.

Corollary 31

1. $\gamma_{\stuffle_{r\geq 1}(z^r y_r)^*} = \prod_{r\geq 1} \gamma_{(z^r y_r)^*} = \prod_{r\geq 1} e^{\ell_r(z)} = \prod_{r\geq 1} \Gamma^{-1}_{y_r}(1+z) = \alpha_0^z(\stuffle_{r\geq 1} y_r^*)$.
2. *One has, for* $|a_s| < 1, |b_s| < 1$ *and* $|a_s + b_s| < 1$,
$\gamma_{(\sum_{s\geq 1}(a_s+b_s)y_s + \sum_{r,s\geq 1} a_s b_r y_{s+r})^*} = \gamma_{(\sum_{s\geq 1} a_s y_s)^*} \gamma_{(\sum_{s\geq 1} b_s y_s)^*}$. *Hence,*
$\gamma_{(a_s y_s + a_r y_r + a_s a_r y_{s+r})^*} = \gamma_{(a_s y_s)^*}\gamma_{(a_r y_r)^*}, \gamma_{(-a_s^2 y_{2s})^*} = \gamma_{(a_s y_s)^*}\gamma_{(-a_s y_s)^*}$.

$\{\gamma_{-s_1,\ldots,-s_r}\}_{s_1,\ldots,s_r\in\mathbb{N}_{\geq 1}}$ by computer

By Example 15, since

$$\begin{aligned}
\mathrm{Li}_{-1,-1} &= -\mathrm{Li}_{x_1^*} + 5\,\mathrm{Li}_{(2x_1)^*} - 7\,\mathrm{Li}_{(3x_1)^*} + 3\,\mathrm{Li}_{(4x_1)^*},\\
\mathrm{Li}_{-2,-1} &= \mathrm{Li}_{x_1^*} - 11\,\mathrm{Li}_{(2x_1)^*} + 31\,\mathrm{Li}_{(3x_1)^*} - 33\,\mathrm{Li}_{(4x_1)^*} + 12\,\mathrm{Li}_{(5x_1)^*},\\
\mathrm{Li}_{-1,-2} &= \mathrm{Li}_{x_1^*} - 9\,\mathrm{Li}_{(2x_1)^*} + 23\,\mathrm{Li}_{(3x_1)^*} - 23\,\mathrm{Li}_{(4x_1)^*} + 8\,\mathrm{Li}_{(5x_1)^*},
\end{aligned}$$

then

$$\begin{aligned}
\mathrm{H}_{-1,-1} &= -\mathrm{H}_{y_1^*} + 5\mathrm{H}_{(2y_1)^*} - 7\mathrm{H}_{(3y_1)^*} + 3\mathrm{H}_{(4y_1)^*},\\
\mathrm{H}_{-2,-1} &= \mathrm{H}_{y_1^*} - 11\mathrm{H}_{(2y_1)^*} + 31\mathrm{H}_{(3y_1)^*} - 33\mathrm{H}_{(4y_1)^*} + 12\mathrm{H}_{(5y_1)^*},\\
\mathrm{H}_{-1,-2} &= \mathrm{H}_{y_1^*} - 9\mathrm{H}_{(2y_1)^*} + 23\mathrm{H}_{(3y_1)^*} - 23\mathrm{H}_{(4y_1)^*} + 8\mathrm{H}_{(5y_1)^*}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\zeta_{\sqcup\!\sqcup}(-1,-1) &= 0,\\
\zeta_{\sqcup\!\sqcup}(-2,-1) &= -1,\\
\zeta_{\sqcup\!\sqcup}(-1,-2) &= 0,
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{-1,-1} &= -\Gamma^{-1}(2) + 5\Gamma^{-1}(3) - 7\Gamma^{-1}(4) + 3\Gamma^{-1}(5) &= \frac{11}{24},\\
\gamma_{-2,-1} &= \Gamma^{-1}(2) - 11\Gamma^{-1}(3) + 31\Gamma^{-1}(4) - 33\Gamma^{-1}(5) + 12\Gamma^{-1}(6) &= -\frac{73}{120},\\
\gamma_{-1,-2} &= \Gamma^{-1}(2) - 9\Gamma^{-1}(3) + 23\Gamma^{-1}(4) - 23\Gamma^{-1}(5) + 8\Gamma^{-1}(6) &= -\frac{67}{120}.
\end{aligned}$$

Zetas and eulerian functions

For $v = -u$ ($|u| < 1$), one gets

$$\frac{1}{\Gamma_{y_1}(1-u)\Gamma_{y_1}(1+u)} = \exp\left(-\sum_{k\geq 1}\zeta(2k)\frac{u^{2k}}{k}\right) = \frac{\sin(u\pi)}{u\pi}.$$

Taking the logarithms and then taking the Taylor expansions, one obtains

$$\begin{aligned} -\sum_{k\geq 1}\zeta(2k)\frac{u^{2k}}{k} &= \log\left(1+\sum_{n\geq 1}\frac{(u\mathrm{i}\pi)^{2n}}{\Gamma_{y_1}(2n)}\right) \\ &= \sum_{l\geq 1}\frac{(-1)^{l-1}}{l}\sum_{k\geq 1}(u\mathrm{i}\pi)^{2k}\sum_{\substack{n_1,\ldots,n_l\geq 1\\ n_1+\ldots+n_l=k}}\prod_{i=1}^{l}\frac{1}{\Gamma_{y_1}(2n_i)} \\ &= \sum_{k\geq 1}(u\mathrm{i}\pi)^{2k}\sum_{l\geq 1}\frac{(-1)^{l-1}}{l}\sum_{\substack{n_1,\ldots,n_l\geq 1\\ n_1+\ldots+n_l=k}}\prod_{i=1}^{l}\frac{1}{\Gamma_{y_1}(2n_i)}. \end{aligned}$$

One can deduce then the following expression for $\zeta(2k)$:

$$\frac{\zeta(2k)}{\pi^{2k}} = k\sum_{l=1}^{k}\frac{(-1)^{k+l}}{l}\sum_{\substack{n_1,\ldots,n_l\geq 1\\ n_1+\ldots+n_l=k}}\prod_{i=1}^{l}\frac{1}{\Gamma_{y_1}(2n_i)} \in \mathbb{Q}.$$

Euler gave an other explicit formula using Bernoulli numbers $\{b_k\}_{k\in\mathbb{N}}$:

$$\zeta(2k)/(2\mathrm{i}\pi)^{2k} = -b_{2k}/2(2k)! \in \mathbb{Q}.$$

More about polyzetas and extended eulerian functions

$$\begin{array}{rclcl}
 & \gamma_{(-t^2y_2)^*} & = & \gamma_{(ty_1)^*}\gamma_{(-ty_1)^*} & \\
\Leftrightarrow & \Gamma_{y_2}^{-1}(1+\mathrm{i}t) & = & \Gamma_{y_1}^{-1}(1+t)\Gamma_{y_1}^{-1}(1-t) & \\
\Leftrightarrow & e^{-\sum_{k\ge 2}\zeta(2k)t^{2k}/k} & = & \dfrac{\sin(t\pi)}{t\pi} & = \displaystyle\sum_{k\ge 1}\frac{(\mathrm{i}t\pi)^{2k}}{(2k)!}.\\
 & \gamma_{(-t^4y_4)^*} & = & \gamma_{(t^2y_2)^*}\gamma_{(-t^2y_2)^*} & \\
\Leftrightarrow & \Gamma_{y_4}^{-1}(1+\sqrt[4]{-1}t) & = & \Gamma_{y_2}^{-1}(1+t)\Gamma_{y_2}^{-1}(1+\mathrm{i}t) & \\
\Leftrightarrow & e^{-\sum_{k\ge 1}\zeta(4k)t^{4k}/k} & = & \dfrac{\sin(\mathrm{i}t\pi)}{\mathrm{i}t\pi}\,\dfrac{\sin(t\pi)}{t\pi} & = \displaystyle\sum_{k\ge 1}\frac{2(-4t\pi)^{4k}}{(4k+2)!}.
\end{array}$$

Since $\gamma_{(-t^4y_4)^*} = \zeta((-t^4y_4)^*), \gamma_{(-t^2y_2)^*} = \zeta((-t^2y_2)^*), \gamma_{(t^2y_2)^*} = \zeta((t^2y_2)^*)$ then, using the poly-morphism ζ, one deduces

$$\begin{aligned}
\zeta((-t^4y_4)^*) = \quad & \zeta((-t^2y_2)^*)\zeta((t^2y_2)^*) && = \zeta((-t^2x_0x_1)^*)\zeta((t^2x_0x_1)^*)) \\
= \ & \zeta((-t^2x_0x_1)^* \sqcup\!\sqcup (t^2x_0x_1)^*) && = \zeta((-4t^4x_0^2x_1^2)^*).
\end{aligned}$$

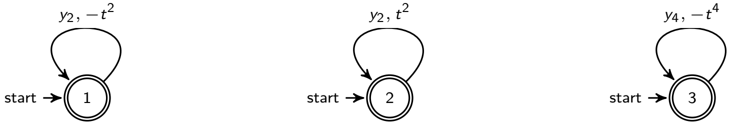
It follows then, by identification the coeffients of t^{2k} and t^{4k} :

$$\zeta(\overbrace{2,\ldots,2}^{k\mathtt{times}})/\pi^{2k} = 1/(2k+1)! \in \mathbb{Q},$$

$$\zeta(\overbrace{3,1,\ldots,3,1}^{k\mathtt{times}})/\pi^{4k} = 4^k\zeta(\overbrace{4,\ldots,4}^{k\mathtt{times}})/\pi^{4k} = 2/(4k+2)! \in \mathbb{Q}.$$

More about extended polymorphism ζ

Example 32 (Identity $(-t^2y_2)^* \uplus (t^2y_2)^* = (-4t^4y_4)^*$)



$(-t^2y_2)^* \leftrightarrow (\nu_2, \mu_2(y_2), \eta_2) = (1, -t^2, 1)$, $(t^2y_2)^* \leftrightarrow (\nu_1, \mu_1(y_2), \eta_1) = (1, t^2, 1)$, $(-t^4y_4)^* \leftrightarrow (\nu, \mu(y_4), \eta) = (1, -t^4, 1)$.

Corollary 33 (comparison formula)

For any $z, a, b \in \mathbb{C}$ such that $|z| < 1$ and $\Re(a) > 0, \Re(b) > 0$, we have

$$\mathrm{B}(z; a, b) = \mathrm{Li}_{x_0[(ax_0)^* \shuffle ((1-b)x_1)^*]}(z) = \mathrm{Li}_{x_1[((a-1)x_0)^* \shuffle (-bx_1)^*]}(z).$$

Hence, on the one hand[31]

$$\mathrm{B}(a, b) = \zeta_{\shuffle}(x_0[(ax_0)^* \shuffle ((1-b)x_1)^*]) = \zeta_{\shuffle}(x_1[((a-1)x_0)^* \shuffle (-bx_1)^*])$$

and, on the other hand

$$\mathrm{B}(a, b) = \frac{\gamma_{((a+b-1)y_1)^*}}{\gamma_{((a-1)y_1)^* \uplus ((b-1)y_1)^*}} = \frac{\gamma_{((a+b-1)y_1)^*}}{\gamma_{((a+b-2)y_1+(a-1)(b-1)y_2)^*}}.$$

---

31. $x_0[(ax_0)^* \shuffle ((1-b)x_1)^*]$ and $x_1[((a-1)x_0)^* \shuffle (-bx_1)^*]$ are of the form (F_2).
What is $\zeta_{\shuffle}(S)$, for S of the form (F_2) ?
What is $\Gamma_{y_r}(a)\Gamma_{y_r}(b)/\Gamma_{y_r}(a+b)$, for $a, b \in \mathbb{C}$ and $r \geq 2$?

Polyzetas and extended eulerian functions

Let $R := t_0^2 t_1 x_0[(t_0x_0)^* ⧢ (t_1x_1)^*]x_1$ $(t_0, t_1 \in \mathbb{C}, |\, t_0 \,| < 1, |\, t_1 \,| < 1)$.

With $\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1-z)^{-1}dz$, we get

$$\begin{aligned}
\mathrm{Li}_R(1) &= t_0^2 t_1 \int_0^1 \frac{ds}{s} \int_0^s \left(\frac{s}{r}\right)^{t_0} \left(\frac{1-r}{1-s}\right)^{t_1} \frac{dr}{1-r} \\
&= t_0^2 t_1 \int_0^1 (1-s)^{t_0 t_1} s^{t_0-1} \int_0^s (1-r)^{t_0-1} r^{-t_0} dsdr.
\end{aligned}$$

By changes of variables, $r = st$ and then $y = (1-s)/(1-st)$, we obtain

$$\begin{aligned}
\zeta(R) &= t_0^2 t_1 \int_0^1 \int_0^1 (1-s)^{t_0 t_1} (1-st)^{t_0-1} t^{-t_0} dtds \\
&= t_0^2 t_1 \int_0^1 \int_0^1 (1-ty)^{-1} t^{-t_0} y^{t_0 t_1} dtdy.
\end{aligned}$$

By expending $(1-ty)^{-1}$ and then by integrating, we get on the one hand

$$\zeta(R) = \sum_{n \geq 1} \frac{t_0}{n - t_0} \frac{t_0 t_1}{n - t_0^2 t_1} = \sum_{k > l > 0} \zeta(k) t_0^k t_1^l.$$

Since $R = t_0 x_0 (t_0 x_0 + t_1 x_1)^* t_0 t_1 x_1$ then we get also on the other hand

$$\zeta(R) = \sum_{k>0} \sum_{l>0} \sum_{s_1+\ldots+s_l = k, s_1 \geq 2, s_2 \ldots, s_l \geq 1} \zeta(s_1, \ldots, s_l) t_0^k t_1^l.$$

Identifying the coeffients of $\langle \zeta(R) | t_0^k t_1^l \rangle$, we deduce the sum formula

$$\zeta(k) = \sum_{s_1+\ldots+s_l = k, s_1 \geq 2, s_2 \ldots, s_l \geq 1} \zeta(s_1, \ldots, s_l).$$

Bibliography I

J. Berstel & C. Reutenauer.– *Rational series and their languages*, Spr.-Ver., 1988.

V.C. Bui, G.H.E. Duchamp, V. Hoang Ngoc Minh, L. Kane, C. Tollu.– *Dual bases for non commutative symmetric and quasi-symmetric functions via monoidal factorization*, J. of Symbolic Computation (2015).

V.C. Bui, G.H.E. Duchamp, V. Hoang Ngoc Minh, H. Nguyen, C. Tollu.– *Combinatorics of φ-deformed stuffle Hopf algebras*, https ://hal.archives-ouvertes.fr/hal-00793118 (2014).

V.C. Bui, G.H.E. Duchamp, V. Hoang Ngoc Minh, Q.H. Ngo, C. Tollu.– *(Pure) transcendence bases in φ-deformed shuffle bialgebras*, Journal électronique du Sém. Lotharingien de Combinatoire B74f (2018).

V.C. Bui, V. Hoang Ngoc Minh, Q.H. Ngo.– *Families of eulerian functions involved in regularization of divergent polyzetas*, arXiv :2009.03931.

V.C. Bui, G.H.E. Duchamp, V. Hoang Ngoc Minh, Q.H. Ngo, K. Penson.– *A local Theory of Domains and its (Noncommutative) Symbolic Counterpart*, arXiv :2009.05125.

P. Cartier.– *Jacobiennes généralisées, monodromie unipotente et intégrales itérées*, Sém. Bourbaki, 687 (1987), 31–52.

P. Cartier.– *Fonctions polylogarithmes, nombres polyzetas et groupes pro-unipotents*– Sém. Bourbaki, 53, (2000).

K.T. Chen.– *Iterated integrals and exponential homomorphisms*, Proc. Lon. Math. S. 3-4 (1954) 502–512.

C. Costermans, J.Y. Enjalbert and V. Hoang Ngoc Minh.– *Algorithmic and combinatoric aspects of multiple harmonic sums*, Discrete Mathematics & Theoretical Computer Science Proceedings, 2005.

C. Costermans, Hoang Ngoc Minh.– *Some Results à l'Abel Obtained by Use of Techniques à la Hopf*, Workshop on Global Integrability of Field Theories and Applications, Daresbury (UK), 1-3, November 2006.

Bibliography II

C. Costermans, Hoang Ngoc Minh.– *Noncommutative algebra, multiple harmonic sums and applications in discrete probability*, J. of Sym. Comp. (2009), 801-817.

M. Deneufchâtel, G.H.E. Duchamp, V. Hoang Ngoc Minh, A.I. Solomon.– *Independence of hyperlogarithms over function fields via algebraic combinatorics*, in LNCS (2011), 6742.

G.H.E. Duchamp, V. Hoang Ngoc Minh, Q.H. Ngo, K. Penson, P. Simonnet.– *Mathematical renormalization in quantum electrodynamics via noncommutative generating series*, in Applications of Computer Algebra, Springer Proceedings in Mathematics and Statistics, pp. 59-100 (2017).

G.H.E. Duchamp, V. Hoang Ngoc Minh, Q.H. Ngo.– *Kleene stars of the plane, polylogarithms and symmetries*, Theoretical Computer Science, Volume 800, 31 December 2019, Pages 52-72

G.H.E. Duchamp, V. Hoang Ngoc Minh, K.A. Penson.– *About Some Drinfel'd Associators*, International Workshop on Computer Algebra in Scientific Computing CASC 2018 - Lille, 17-21 September 2018.

G.H.E. Duchamp, V. Hoang Ngoc Minh, V. Nguyen Dinh.– *Towards a noncommutative Picard-Vessiot theory*, arXiv :arXiv :2008.10872.

V. Drinfel'd– On quasitriangular quasi-hopf algebra and a group closely connected with gal$(\bar{q}/q)$, *Leningrad Math. J.*, 4, 829-860, 1991.

J. Ecalle.– ARI/GARI, la dimorphie et l'arithmétique des multizêtas : un premier bilan, J. Th. des nombres de Bordeaux, 15, (2003), pp. 411-478.

Philippe Flajolet & Robert Sedgewick– *Analytic combinatorics*, Cambridge University Press, 2009.

Furusho, H.– *The Grothendieck-Teichmüller group, the double shuffle group and the motivic Galois group*, RIMS-kokyuroku No.1714, Quantum Groups and quantum Topology (2010), 63-79.

Bibliography III

J.A. Lappo-Danilevsky.– *Théorie des systèmes des équations différentielles linéaires*, Chelsea, NY, 1953.

A. Lascoux.– *Fonctions symétriques, J. élec. du Sém. Lotharingien de Combinatoire*, B08e, (1983).

T.Q.T. Lê & J. Murakami.– *Kontsevich's integral for Kauffman polynomial*, Nagoya Math., pp 39-65, 1996.

A. M. Legendre.– *Exercices de calcul intgral sur divers ordres de transcendantes et sur les quadratures*, Volume 1, Courcier, 1811" (from p.298)

M. Lothaire.– *Combinatorics on Words*, Enc. of Math. and its App., Addison-Wesley, 1983.

K. Ihara, M. Kaneko & D. Zagier.– *Derivation and double shuffle relations for multiple zetas values*, Compositio Math. 142, pp. 307-338, 2006.

W.Magnus.– *On the exponential solution of differential equations for a linear operator.*, AC on Pure and App. Math., VII :649673, 1954.

J. Gonzalez-Lorca.– *Série de Drinfel'd, monodromie et algèbres de Hecke*, Ph. D., Ecole Normale Supérieure, Paris, 1998.

Hoang Ngoc Minh, Jacob G., N.E. Oussous, M. Petitot.– *Aspects combinatoires des polylogarithmes et des sommes d'Euler-Zagier, journal électronique du Sém. Lotharingien de Combinatoire*, B43e, (2000).

Hoang Ngoc Minh, G. Jacob.– *Symbolic Integration of meromorphic differential equation via Dirichlet functions*, Disc. Math. 210, pp. 87-116, 2000.

Hoang Ngoc Minh, M. Petitot.– *Lyndon words, polylogarithmic functions and the Riemann ζ function*, Discrete Math., 217, 2000, pp. 273-292.

Bibliography IV

- Hoang Ngoc Minh, M. Petitot, J. Van der Hoeven.– *Polylogarithms and Shuffle Algebra*, *Proceedings of FPSAC'98*, 1998.
- V. Hoang Ngoc Minh.– *Differential Galois groups and noncommutative generating series of polylogarithms*, Automata, Combinatorics & Geometry, W. Mul. Conf. Systemics, Cybernetics & Informatics, Florida, 2003.
- V. Hoang Ngoc Minh.– *On the solutions of the universal differential equation with three regular singularities (On solutions of KZ_3)*, Confluentes Mathematici (2019).
- M. Hoffman.– *Multiple harmonic series*, Pacific J. Math. 152 (1992), pp. 275-290.
- M. Hoffman.– *Quasi-shuffle products*, J. Alg. Combin. **11** (1) (2000) 49–68.
- G. Racinet.– *Séries génératrices non-commutatives de polyzêtas et associateurs de Drinfel'd*, Ph. D., Amiens, 2000.
- D.E. Radford.– *A natural ring basis for shuffle algebra and an application to group schemes Journal of Algebra*, 58, pp. 432-454, 1979.
- Ree R.,– *Lie elements and an algebra associated with shuffles Ann. Math* **68** 210–220, 1958.
- C. Reutenauer.– *Free Lie Algebras*, London Math. Soc. Monographs (1993).
- G. Viennot.– *Algèbres de Lie libres et monoïdes libres*, Lect. N. in Math., Springer-Verlag, 691, 1978.
- M. Waldschmidt– *Hopf Algebra and Transcendental numbers, Zeta-functions*, Topology and Quantum Physics 2003, Kinki : Japan, 2003.
- D. Zagier.– Values of zeta functions and their applications, in "First European Congress of Mathematics", vol. 2, Birkhäuser (1994), pp. 497-512.

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