Combinatoire de Catalan et Polynômes Harmoniques Diagonaux

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ℓ sets of n variables

$$X = (x_{ij})_{\substack{1 \le i \le \ell \\ 1 \le j \le n}}$$

$$A = (a_{ij})_{\substack{1 \le i \le \ell \\ 1 \le j \le n}} \qquad a_{ij} \in \mathbb{N}$$

$$X^A := \prod_{\substack{1 \le i \le \ell \\ 1 \le j \le n}} x_{ij}^{a_{ij}}$$

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DEGREE (ROW SUM)

$$\mathrm{DEG}\left(X^{A} \right) := \left(\sum_{1 \leq j \leq m} a_{ij} \right)_{1 \leq i \leq \ell}$$

Π_d PROJECTION ON HOMOGENOUS COMPONENT OF DEGREE d.

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SÉRIE DE HILBERT

$k[x]$ ANNEAU DES POLYNÔMES EN LES VARIABLES X

$\mathcal{V} \subseteq k[x]$ GRADUÉ

$$\mathcal{V} = \bigoplus_{d \in \mathbb{N}^\ell} \mathcal{V}_d \qquad \mathcal{V}_d := \pi_d(\mathcal{V})$$

$$\mathcal{V}(q) := \sum_{d \in \mathbb{N}^\ell} q^d \dim(\mathcal{V}_d)$$

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2 COMMUTING ACTIONS
(GL_ℓ AND $\mathfrak{S}_n$)

$$(\sigma \cdot f)(x) := f(x \cdot \sigma)$$

$$\sigma \in \mathfrak{S}_n$$

PERMUTING VARIABLES
IN EACH SET

2 COMMUTING ACTIONS
(GL_ℓ AND $\mathfrak{S}_n$)

$$(f\cdot\tau)(X) := f(\tau\cdot X)$$

$\tau \in GL_\ell$

PERMUTING SETS OF VARIABLES

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2 COMMUTING ACTIONS
(GL_ℓ AND $\mathfrak{S}_n$)

$\boxed{\mathcal{V} \text{ INVARIANT FOR BOTH ACTIONS}}$

$\forall f \in \mathcal{V} \quad \sigma \cdot f \in \mathcal{V}$ AND $f \cdot \tau \in \mathcal{V}$

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DIAGONAL INVARIANT POLYNOMIALS

$$\sigma \cdot f = f \qquad \forall \sigma \in S_n$$

$$x_{i_1}^k x_{j_1}^l + \ldots + x_{i_m}^k x_{j_m}^l$$

FONCTIONS SYMÉTRIQUES

$$e_k(x_1, \ldots, x_m) \qquad h_k(x_1, \ldots, x_m)$$

$$\sum_{k \geq 0} e_k t^k = \prod_i (1 + x_i t)$$

$$\sum_{k \geq 0} h_k t^k = \prod_i \frac{1}{1 - x_i t}$$

$$p_{3211} = p_3 p_2 p_1^2 \qquad p_k = x_1^k + \ldots + x_m^k$$

$$m_{321} = \cdots + x_i^3 x_j^2 x_k + \cdots$$

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$$h_k(1^\ell) = h_k(\underbrace{1, 1, \ldots, 1}_{\ell \text{ copies}})$$

$$= \binom{\ell+k-1}{k}$$

Diagonal Harmonic Polynomials

$$I := \langle f(x) \mid f(x) \text{ invariant}, f(0)=0 \rangle$$

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Diagonal Harmonic Polynomials

$$\mathcal{D} \simeq k[x]/I$$

$\boxed{\text{Case} \quad \ell = 1}$

$$\dim(\mathcal{D}) = n!$$

DIAGONAL HARMONIC POLYNOMIALS

$$\mathcal{D} \simeq k[x]/I$$

$\boxed{\text{CAS} \quad \ell = 2}$

$$\dim(\mathcal{D}) = (n+1)^{n-1}$$

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$$m = 2$$

$$\mathcal{D} = k\{1, x_{11} - x_{12}, \ldots, x_{\ell 1} - x_{\ell 2}\}$$

$$\mathcal{D}(q_1, q_2, \ldots, q_\ell) = 1 + h_1(q)$$

SÉRIE DE HILBERT

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SÉRIE DE HILBERT GÉNÉRIQUE

$$\mathfrak{D}_n(q) = \sum_{\sigma \in \mathfrak{S}_n} h_{\mu(\sigma)}(q)$$

où

$\mu(\sigma)$ PARTAGE DE $inv(\sigma)$

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Série de Hilbert générique

$$\mathcal{D}_n = \sum_{\sigma \in \mathfrak{S}_n} h_{\mu(\sigma)}$$

où

$\mu(\sigma)$ partage de $\mathrm{inv}(\sigma)$

$$\mathcal{D}_1 = 1$$

$$\mathcal{D}_2 = 1 + h_1$$

$$\mathcal{D}_3 = 1 + 2h_1 + h_1^2 + h_2 + h_3$$

$$\begin{aligned}\mathcal{D}_4 = 1 &+ 3h_1 + 3h_1^2 + 2h_2 \\ &+ h_1^3 + 3h_1h_2 + 2h_3 \\ &+ 4h_1h_3 + h_4 \\ &+ h_1h_4 + 2h_5 + h_6\end{aligned}$$

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CAS $\ell = 3$

$$\mathcal{D}_1(q_1, q_2, q_3) = 1$$

$$\mathcal{D}_2(q_1, q_2, q_3) = 1 + (q_1 + q_2 + q_3)$$

$$\mathcal{D}_3(q_1, q_2, q_3) = 1 + 2(q_1 + q_2 + q_3) + (q_1 + q_2 + q_3)^2 + (q_1^2 + q_2^2 + q_3^2 + q_1 q_2 + q_1 q_3 + q_2 q_3) + (q_1^3 + \cdots + q_1 q_2 q_3)$$

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Dimensions

$$\mathcal{D}_1(1^\ell) = 1$$

$$\mathcal{D}_2(1^\ell) = 1 + \binom{\ell}{1}$$

$$\mathcal{D}_3(1^\ell) = 1 + 2\binom{\ell}{1} + \binom{\ell}{1}^2 + \binom{\ell+1}{2} + \binom{\ell+2}{3}$$

$$\begin{aligned}\mathcal{D}_4(1^\ell) = {} & 1 + 3\binom{\ell}{1} + 3\binom{\ell}{1}^2 + 2\binom{\ell+1}{2} + \binom{\ell}{1}^3 \\ & +3\binom{\ell}{1}\binom{\ell+1}{2} + 2\binom{\ell+2}{3} + 4\binom{\ell}{1}\binom{\ell+2}{3} \\ & +\binom{\ell+3}{4} + \binom{\ell}{1}\binom{\ell+3}{4} + 2\binom{\ell+4}{5} + \binom{\ell+5}{6}\end{aligned}$$

$$\mathcal{D}_n(1) = n!$$

$$\mathcal{D}_n(1,1) = (n+1)^{n-1}$$

$$\mathcal{D}_n(1,1,1) \stackrel{?}{=} 2^n (n+1)^{n-2}$$

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$$\mathcal{D}_n(q) = \prod_{i=1}^{n} (1+\ldots+q^{i-1})$$

$$q^{\binom{n}{2}} \mathcal{D}_n(q, 1/q) = [n+1]_q^{n-1}$$

Graded Hilbert Series of Alternants of $\mathcal{D}$

$$\mathcal{A}_1 = 1$$

$$\mathcal{A}_2 = s_1$$

$$\mathcal{A}_3 = s_{11} + s_3$$

$$\mathcal{A}_4 = s_{111} + s_{31} + s_{41} + s_6$$

$$\mathcal{A}_5 = s_{1111} + s_{311} + s_{411} + s_{42} + s_{43}$$
$$+ s_{511} + s_{61} + s_{62} + s_{71} + s_{81} + s_{10}$$

Alternants Dimensions

$$A_1(1^\ell) = 1$$

$$A_2(1^\ell) = 1 + \binom{\ell-1}{1}$$

$$A_3(1^\ell) = 1 + 2\binom{\ell-1}{1} + \binom{\ell-1}{1}^2 + \binom{\ell+1}{3}$$

$$A_4(1^\ell) = 1 + 3\binom{\ell-1}{1} + 3\binom{\ell-1}{1}^2 + \binom{\ell-1}{1}^3 + 2\binom{\ell+1}{3}$$
$$+2\binom{\ell-1}{1}\binom{\ell+1}{3} + \binom{\ell-1}{1}\binom{\ell+2}{4} + \binom{\ell+4}{6}$$

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$$A_n(1,1) = \frac{1}{n+1}\binom{2n}{n}$$

$$q^{\binom{n}{2}} A_n\left(q, \frac{1}{q}\right) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

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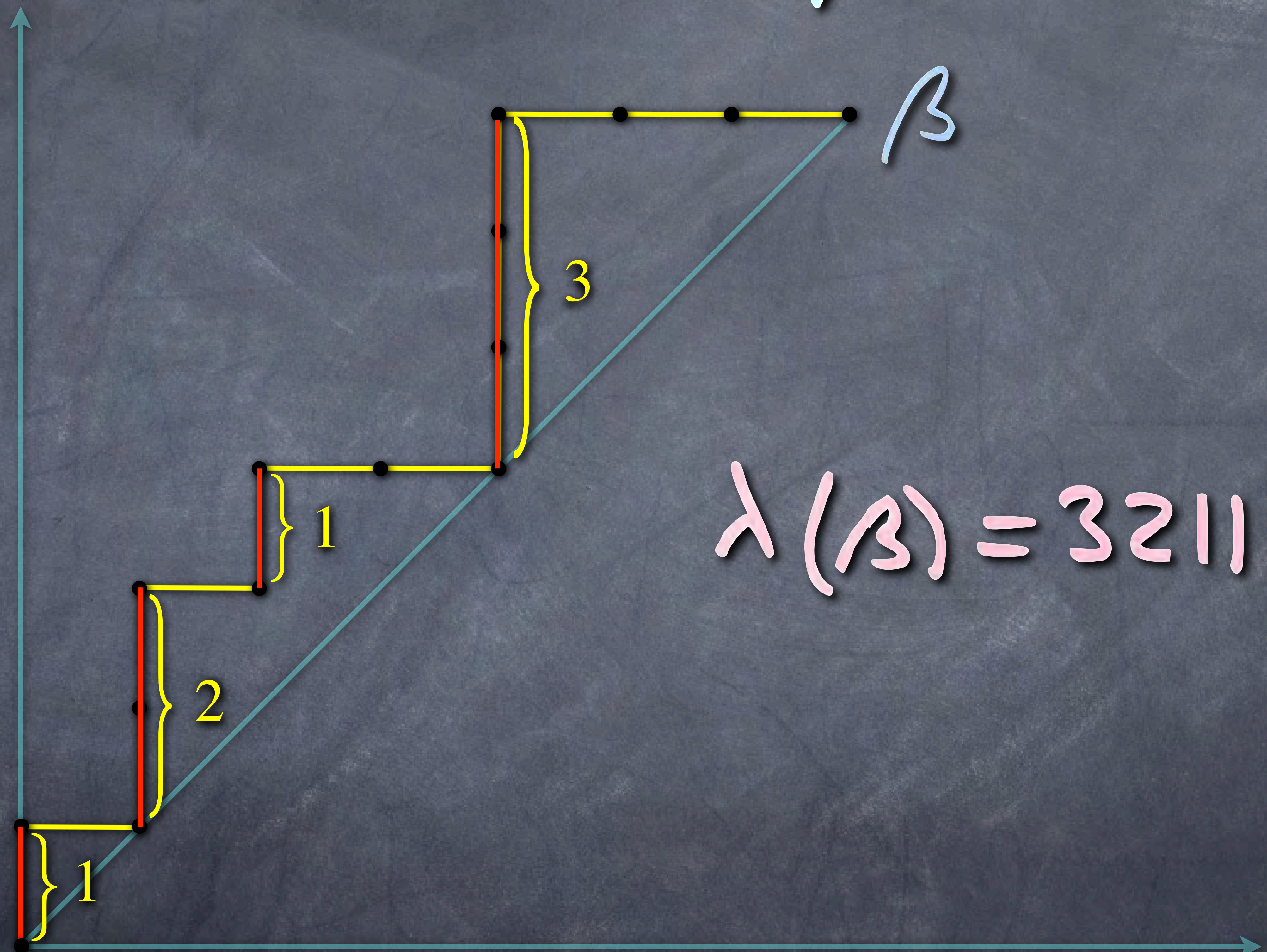
$$A_n(1,1,1) \stackrel{?}{=} \frac{2}{n(n+1)} \binom{4n+1}{n-1}$$

λ(β): Forme de β



$\lambda(\beta) = 3211$

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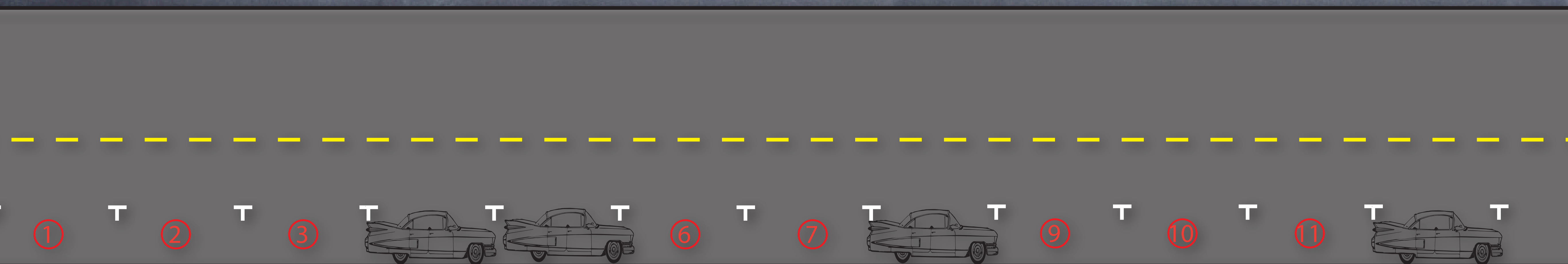
Fonctions de stationnement.



$$\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$$

$$\# \pi^{-1}(\{1,..,k\}) \geq k$$

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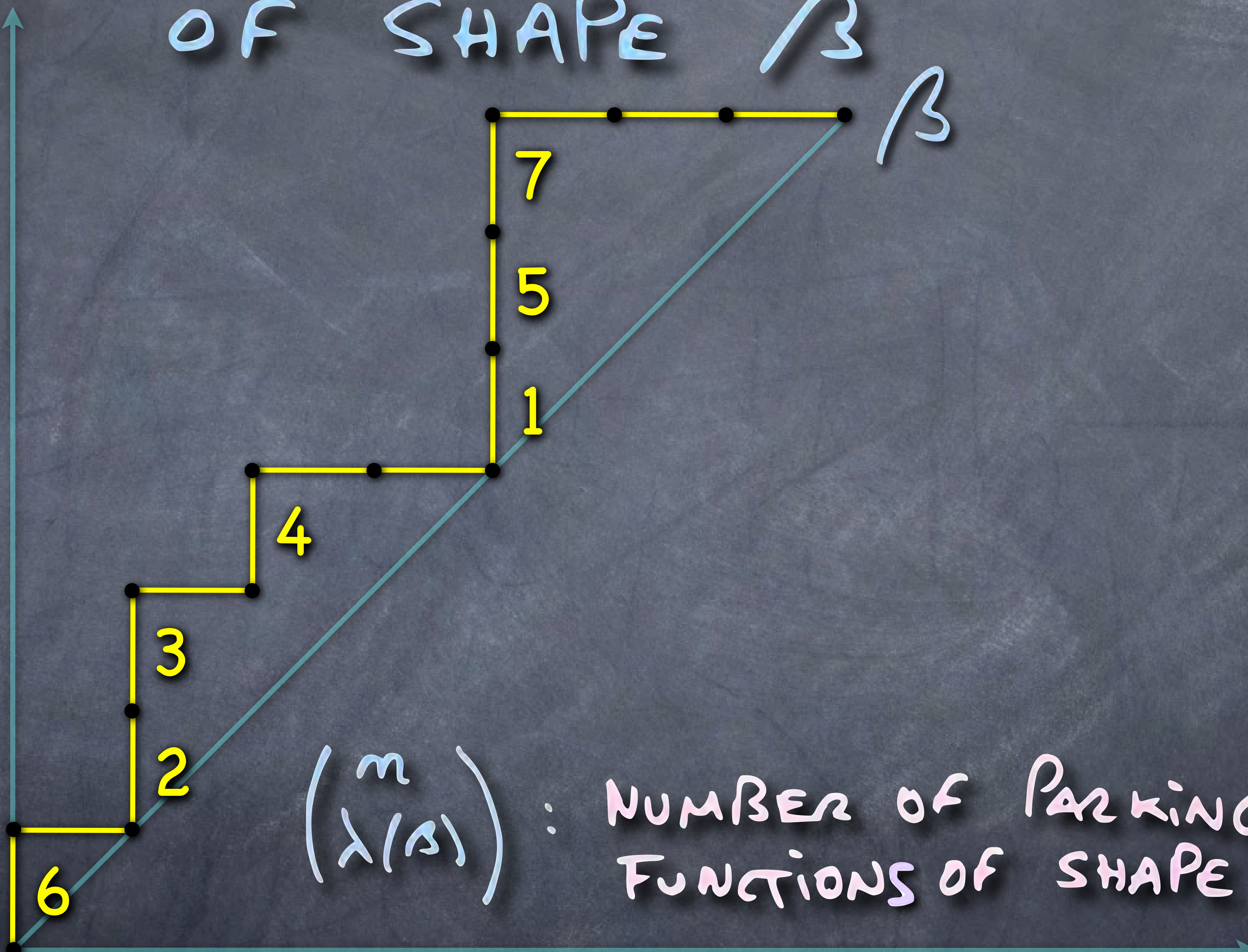
Parking Function of Shape β



$\binom{n}{\lambda(\beta)}$: Number of parking functions of shape β

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Fonctions de Stationnement.

$$(n+1)^{n-1} = \sum_{\beta} \binom{n}{\lambda(\beta)}$$

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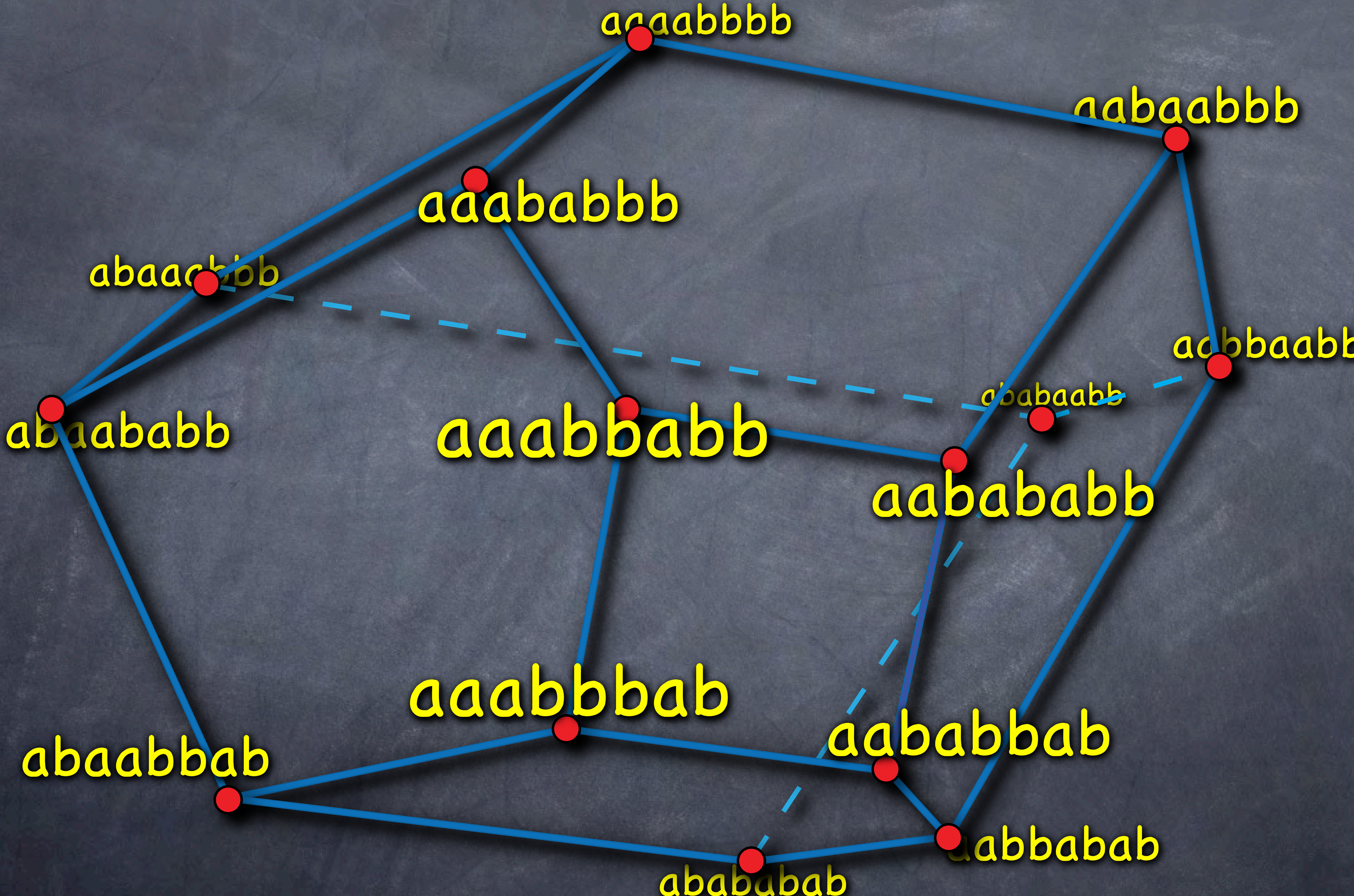

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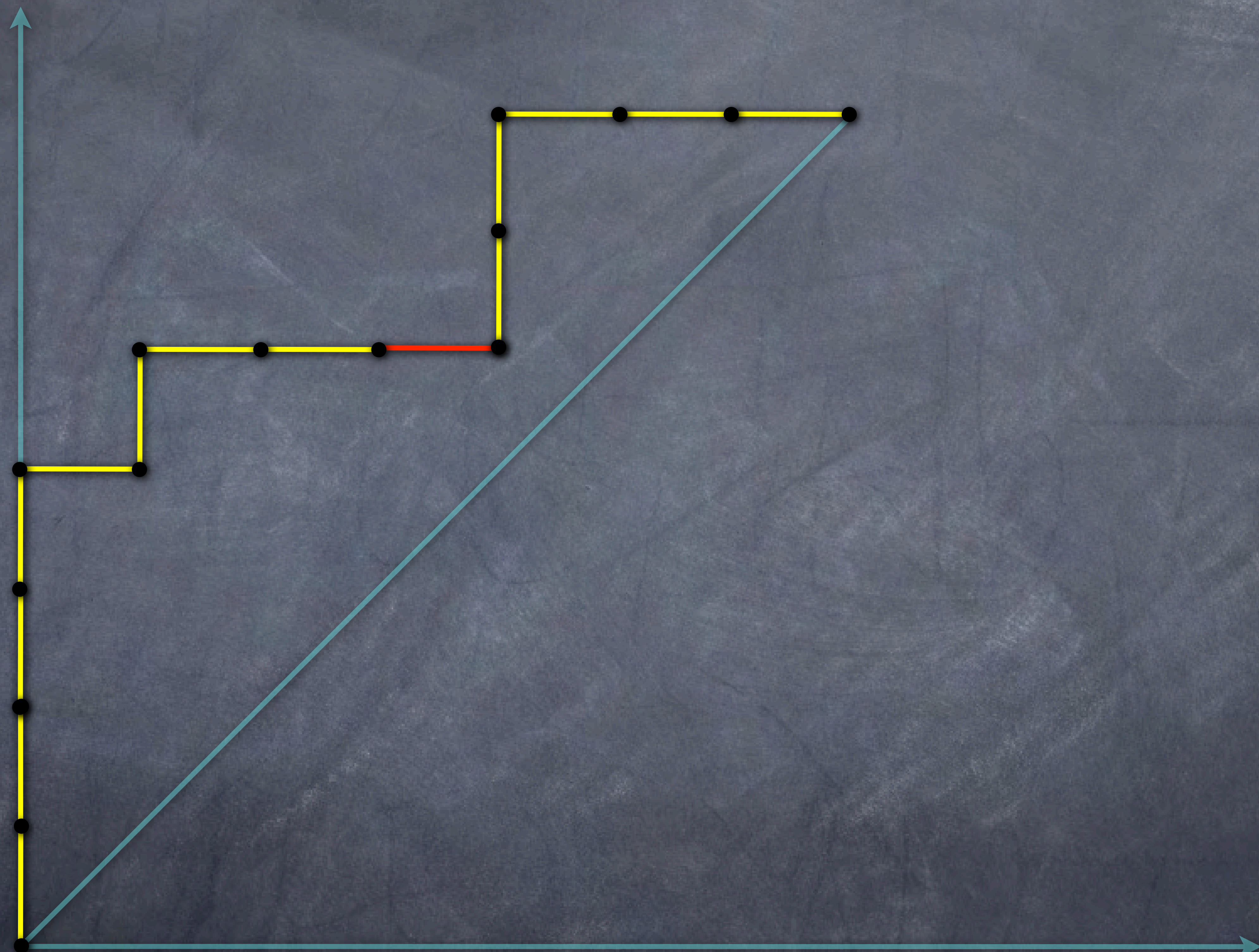
TREILLIS DE TAMARI



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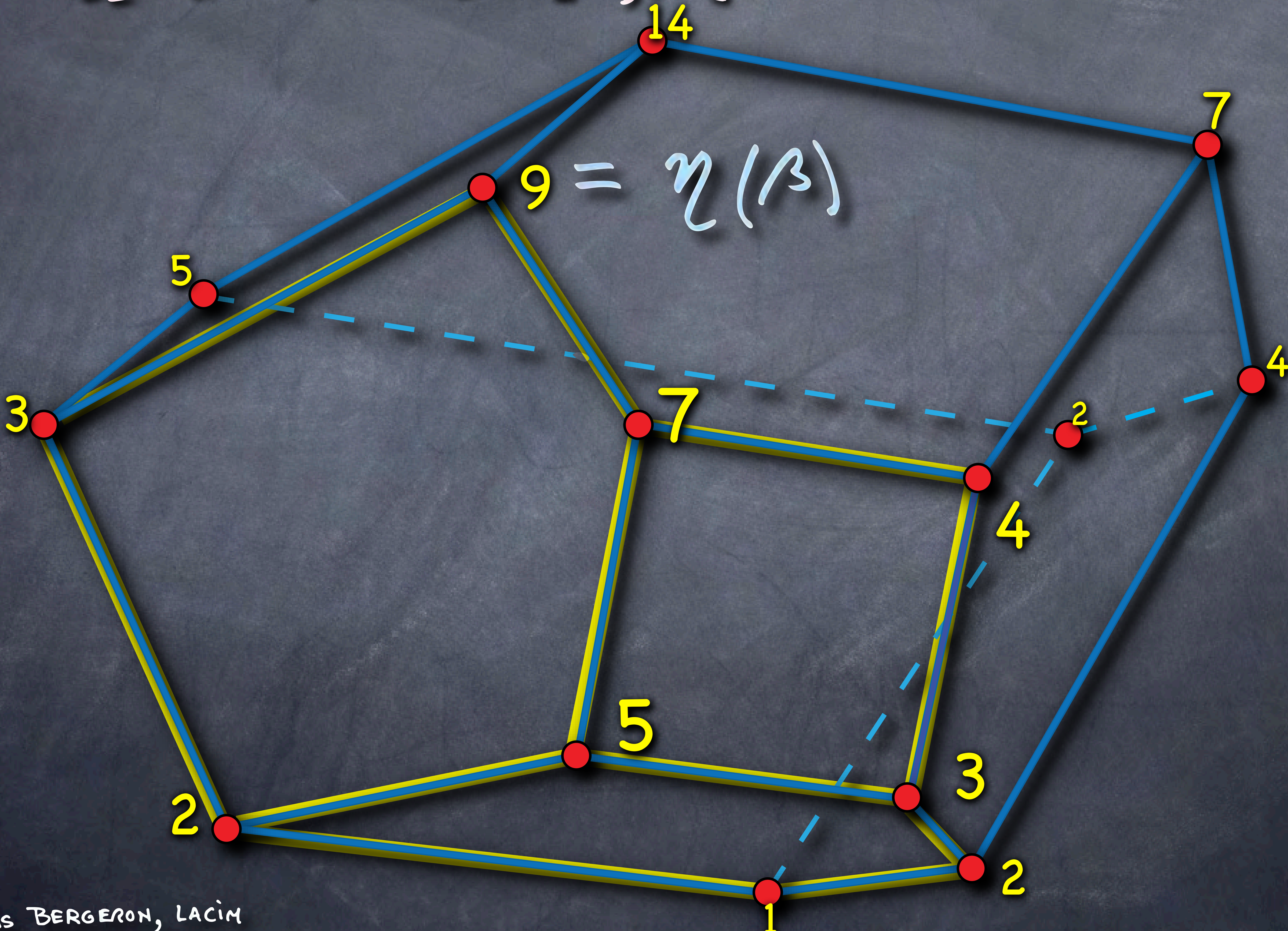
$\eta(\beta)$: NOMBRE D'INTERVALLES DE LA FORME $[\alpha, \beta]$



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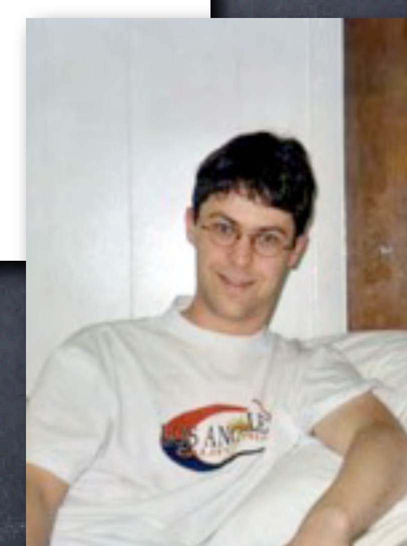
Séminaire Lotharingien de Combinatoire **55** (2006), Article B55f

SUR LE NOMBRE D'INTERVALLES DANS LES TREILLIS DE TAMARI

F. CHAPOTON

RÉSUMÉ. On compte le nombre d'intervalles dans les treillis de Tamari. On utilise pour cela une description récursive de l'ensemble des intervalles. On introduit ensuite une notion d'intervalle nouveau dans les treillis de Tamari et on compte les intervalles nouveaux. On obtient aussi l'inverse de deux séries particulières dans un groupe de séries formelles en arbres.

ABSTRACT. We enumerate the intervals in the Tamari lattices. For this, we introduce an inductive description of the intervals. Then a notion of "new interval" is defined and these are also enumerated. As a side result, the inverse of two special series is computed in a group of tree-indexed series.



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$$\frac{2}{n(n+1)}\binom{4n+1}{n-1} = \sum_{\substack{\beta \\ \text{DYCK}}} \eta(\beta)$$

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Preuve combinatoire?

$$2^n (n+1)^{n-2} \overset{?}{=} \sum_{\substack{\beta \\ \text{Dyck}}} \eta(\beta) \binom{n}{\lambda(\beta)}$$

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Frobenius transform of the graded character of $\mathcal{D}$

$$\mathcal{D}_n(w;q) := \sum_{d\in\mathbb{N}^\ell} q^d \frac{1}{n!}\sum_{\sigma\in S_n} x^{\gamma_d(\sigma)} p_{\lambda(\sigma)}$$

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Frobenius transform of the graded character of $\mathcal{D}$

$$\mathcal{D}_2(w;q) = m_2(w) + (1+h_1(q))\, m_{11}(w)$$

$$\mathcal{D}_3(w;q) = m_3 + (1+h_1+h_2)\, m_{21}$$
$$+ (1+2h_1+h_1^2+h_2+h_3)\, m_{111}$$

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$$\mathcal{D}_n(w;q) \stackrel{?}{=} \sum_{\lambda \vdash n} m_\lambda \sum_{\mathrm{DESC}(\sigma) \subseteq S(\lambda)} h_{\mu_\lambda(\sigma)}$$

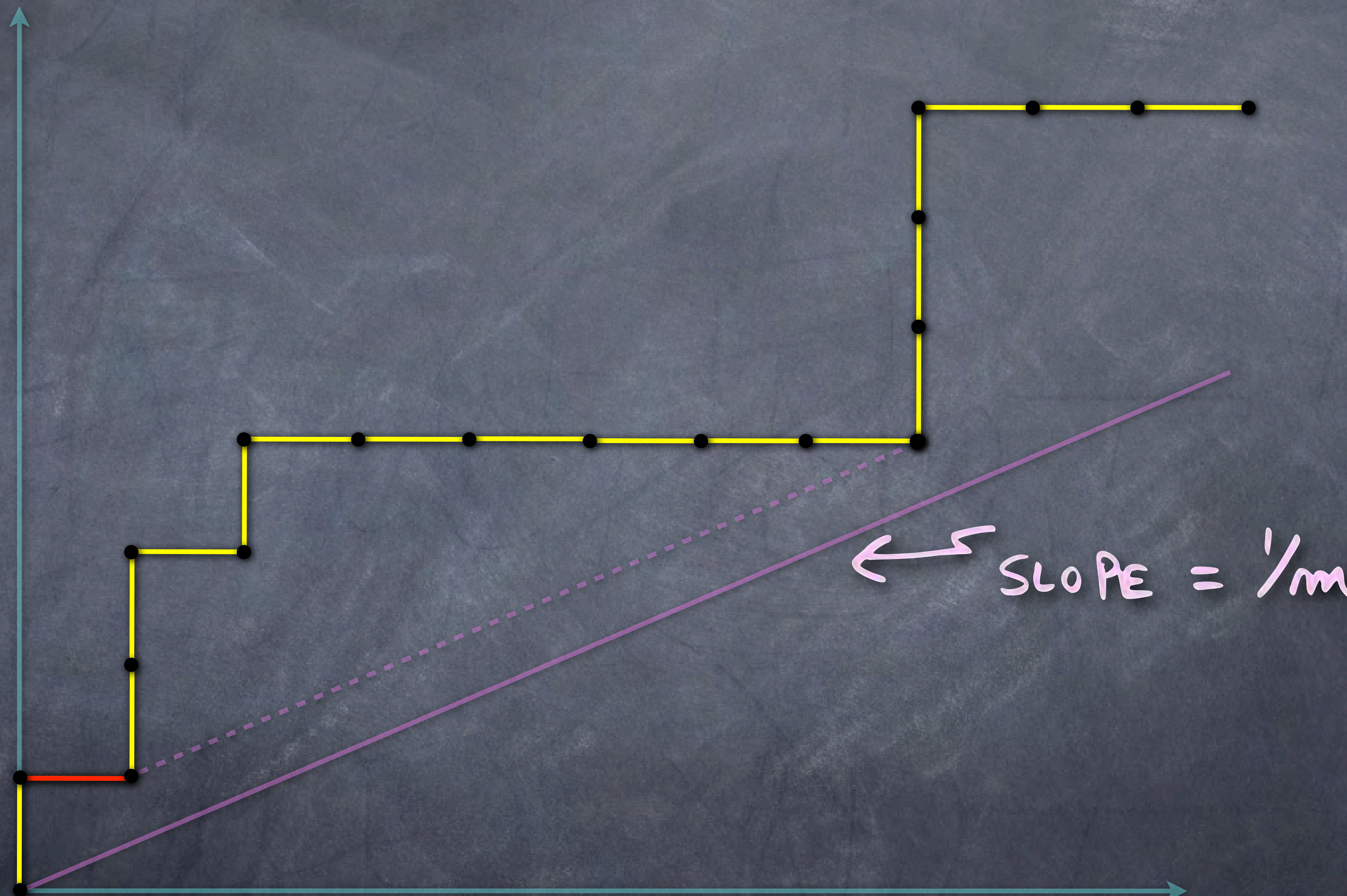
$$S(\lambda) = \{\lambda_1, \lambda_1 + \lambda_2, \dots\}$$

$$\mathcal{D}_n(w;1) = h_1^n$$

$$\mathcal{D}_n(w;1,1) = \sum_{\beta} e_{\lambda(\beta)}$$

$$\mathcal{D}_n(w;1,1,1) \overset{?}{=} \sum_{\beta} \eta(\beta)\, e_{\lambda(\beta)}$$

TREILLIS DE m-TAMARI



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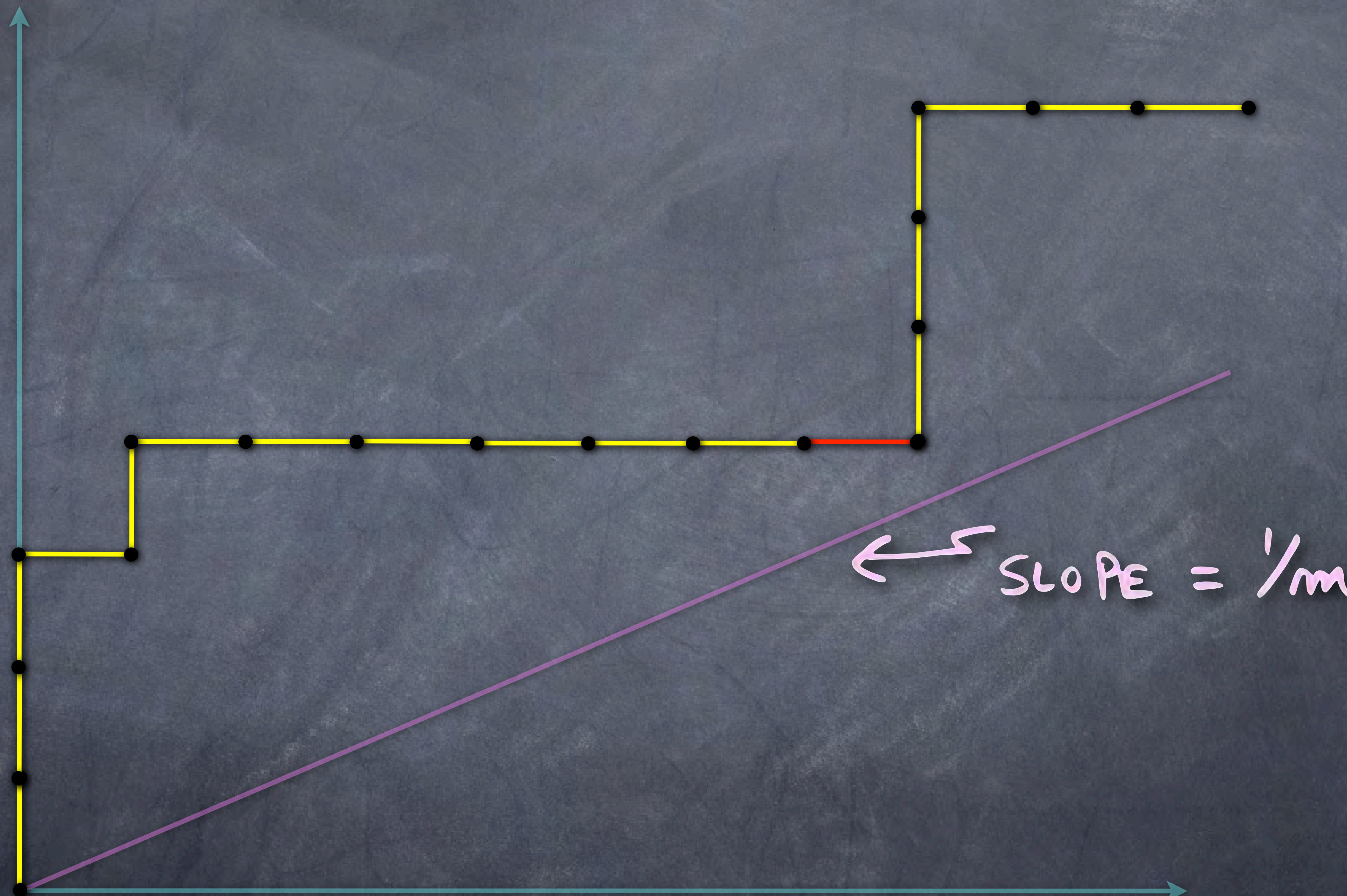
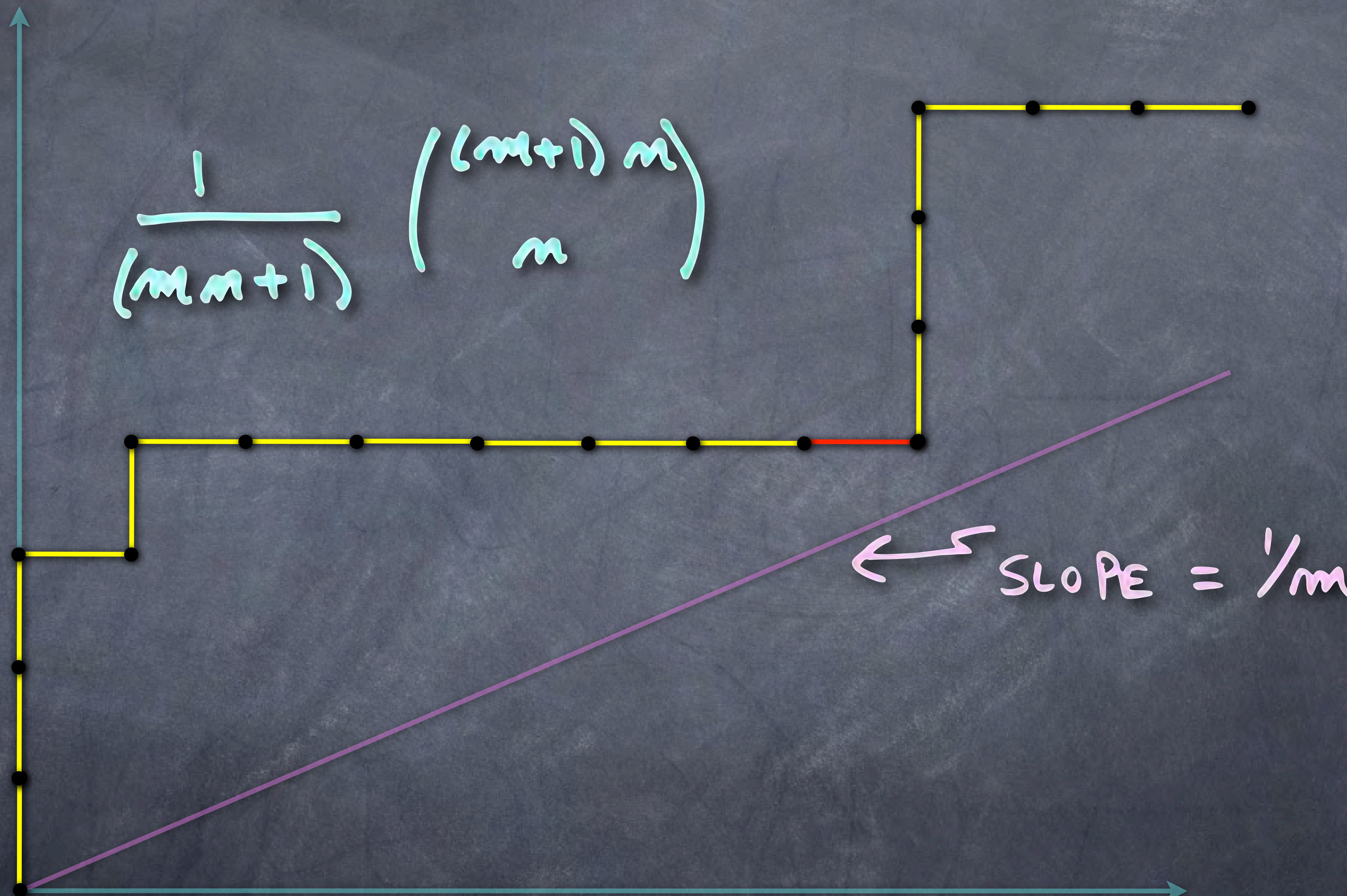
TREILLIS DE m-TAMARI



TREILLIS DE m-TAMARI

$$\frac{1}{(mn+1)}\binom{(m+1)n}{n}$$



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$\eta^{(m)}(\beta)$: NOMBRE D'INTERVALLES DE LA FORME $[\alpha, \beta]$

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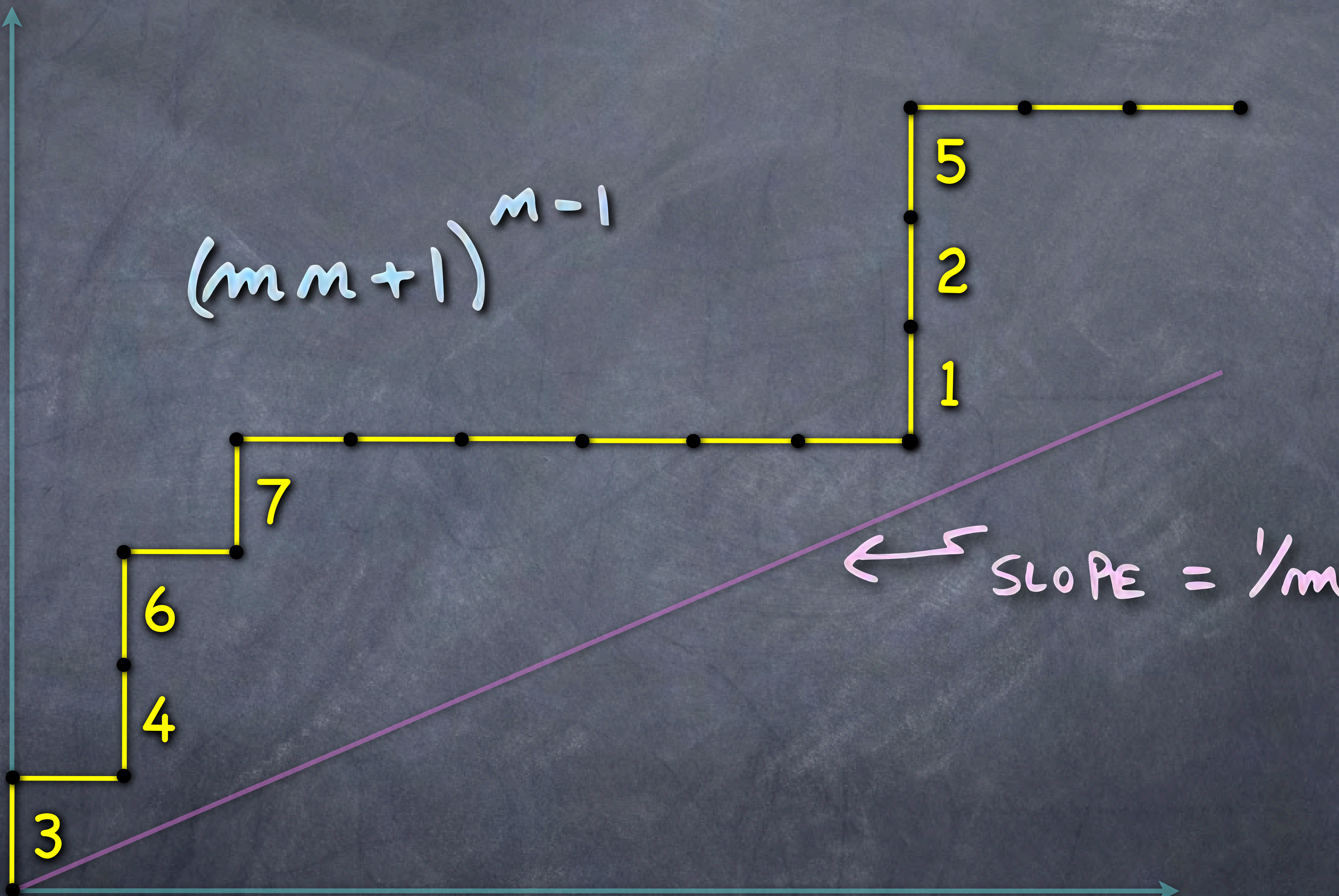
m-Parking Function



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m-Parking Function

$$\pi : \{1, \dots, n\} \to \{1, \dots, m \cdot n\}$$

$$\# \pi^{-1}(\{1, \dots, km\}) \geq k$$

m-PARKING FUNCTION OF SHAPE β

$$(mn+1)^{n-1} = \sum_{\substack{\beta \\ m\text{-DYCK}}} \binom{n}{\lambda(\beta)}$$

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$\mathcal{D}^m$ A CERTAIN ANALOGOUS SPACE FOR EACH m.

$\mathbb{A}$ IDEAL GENERATED BY DIAGONAL ALTERNANTS

$$\mathcal{D}^m := \mathbb{A}^{m-1} / I\mathbb{A}^{m-1}$$

$$\sigma * f := \mathrm{sign}(\sigma)^{m-1} \sigma \cdot f$$

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$\mathcal{D}^m$ A CERTAIN ANALOGOUS SPACE FOR EACH m.

$$\mathcal{D}_n^{(m)}(w;1,1) = \sum_{\substack{\beta \\ m\text{-DYCK}}} e_{\lambda(\beta)}$$

$$\mathcal{D}_n^{(m)}(1,1) = (mn+1)^{n-1}$$

$$A_n^{(m)}(1,1) = \frac{1}{(mn+1)} \binom{(m+1)n}{n}$$

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$$\mathcal{D}_n^{(m)}(\omega;1,1,1) \overset{?}{=} \sum_{\substack{\beta \\ m\text{-DYCK}}} \eta^{(m)}(\beta)\, e_{\lambda(\beta)}$$

$$\mathcal{D}_n^{(m)}(1,1,1) \overset{?}{=} (m+1)^n\,(mn+1)^{n-2}$$

$$A_n^{(m)}(1,1,1) \overset{?}{=} \frac{(m+1)}{n\,(mn+1)} \binom{(m+1)^2 n + m}{n-1}$$

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$$\mathcal{D}_n^{(m)}(w;1,1,1) \stackrel{?}{=}$$

$$\sum_{\mu \vdash n} \frac{(-1)^{n-\ell(\mu)}}{z_\mu} p_\mu(w)$$

$$(mn+1)^{\ell(\mu)-2} \prod_{k \in \mu} \binom{k(m+1)}{k}$$

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$$\frac{(m+1)}{n\,(mn+1)}\binom{(m+1)^2 n+m}{n-1} \overset{?}{=} \sum_{\substack{\beta\\ m\text{-DYCK}}} \eta^{(m)}(\beta)$$

$$(m+1)^n\,(mn+1)^{n-2} \overset{?}{=} \sum_{\substack{\beta\\ m\text{-DYCK}}} \eta^{(m)}(\beta)\binom{n}{\lambda(\beta)}$$

Purely combinatorial statements

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